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MODAL TARSKI ALGEBRAS

A b s t r a c t. In this paper we shall study the representation theory for Tarski algebras with a modal operator. In particular, we shall give a representation for finite Tarski algebras and finite Tarski algebras with a modal operator by means of the so-called Tarski sets and Tarski relational sets, respectively. We will also consider some varieties of Tarski modal algebras.

1. Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [2] (see also [3], [8], [7] and [11]). These algebras are an algebraic counterpart of the $\{\vee, \rightarrow\}$-fragment of the propositional classical calculus. The variety of monadic Tarski algebras was introduced by A. Monteiro and L. Iturrioz

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[10] as the algebraic models of the $\{\vee, \to, \Box\}$-fragment of the **S5** modal classical calculus. But, as far as we know, there are no findings on the algebrization of others $\{\vee, \to, \Box\}$-fragments of the classical modal logics. In this paper, we shall begin the research of these fragments of the algebraic viewpoint. We shall introduce Tarski algebras with a modal operator as a generalization of the concept of Boolean algebra with a modal operator.

In Section 3 we will introduce the notion of Tarski set as a pair $\langle X, \mathcal{K} \rangle$ where X is a non-empty set and $\mathcal{K}$ is a non-empty subset of $\mathcal{P}(X)$. We will prove that every finite Tarski algebra A can be represented as a Tarski algebra of sets $D_{\mathcal{K}}(X) \subset \mathcal{P}(X)$ for some finite set Tarski set $\langle X, \mathcal{K} \rangle$. In Section 4 we introduce the modal Tarski algebras. We will prove a representation theorem for modal Tarski algebras by means of the so-called Tarski relational sets, and we shall prove that certain formulas correspond to additional properties defined in the associated Tarski relational sets. We will also prove that the varieties considered have the finite model property. Although most of the paper results are written in algebraic form, they have their logical significance.

2. Tarski algebras

Definition 2.1. *An algebra $\langle A, \to, 1 \rangle$ of type $(2, 0)$ is a Tarski algebra if it satisfies the following identities:*

T1. $1 \to a \approx a$,

T2. $a \to a \approx 1$,

T3. $a \to (b \to c) \approx (a \to b) \to (a \to c)$,

T4. $(a \to b) \to a \approx (b \to a) \to a$, *for all* $a, b, c \in A$.

We note that the conditions T1 to T3 are an axiomatization of the variety of Hilbert algebras (see [8], [8], [4]). We denote by $\mathcal{T}$ the variety of Tarski algebras and by $\mathcal{B}$ the variety of Boolean algebras. All Boolean algebra $\langle A, \vee, \wedge, \neg, 0, 1 \rangle$ is a Tarski algebra where the implication $\to$ is defined by $a \to b = \neg a \vee b$.

In a Tarski algebra A we can define an order relation $\leq$ by setting $a \leq b$ if and only if $a \to b = 1$. It is well known that $\langle A, \leq \rangle$ is an ordered set and that A is a join-semilattice where the supremum of two elements $a, b \in A$ is defined by $a \vee b = (a \to b) \to b$ (see [3]).

Let A be a Tarski algebra and let $p_1, p_2, ..., p_n \in A$. We shall use the following notation introduced in [4]:

$$(p_1, p_2, ..., p_{n-1}; p_n) = \begin{cases} p_n & \text{if } n = 1 \\ p_1 \to (p_2, ..., p_{n-1}; p_n) & \text{if } n > 1 \end{cases}$$

If α is any permutation of the set $\{1, .., n\}$, then by the properties of the implication $\to$ we have

1. $(p_1, p_2, ..., p_n; p_{n+1}) = \left(p_{\alpha(1)}, p_{\alpha(2)}, ..., p_{\alpha(n)}; p_{n+1}\right)$,
2. $(p_1, p_2, ..., p_n; p \to q) = (p_1, p_2, ..., p_n; p) \to (p_1, p_2, ..., p_n; q)$,
3. If $(p_1, p_2, ..., p_n; q) = 1$ then

$$\begin{aligned} &(p_1, p_2, ..., (q_1, q_2, ..., q_k, p_n); (q_1, q_2, ..., q_k, q)) \\ = \ &(p_1, p_2, ..., (q_1, q_2, ..., q_k, p_n), q_1, q_2, ..., q_k; q) = 1. \end{aligned}$$

A non-empty subset D in a Tarski algebra A is a called *filter* if $1 \in D$, and for any $a, b \in A$, if $a, a \to b \in D$, then $b \in D$. The set of all filters of A is denoted by $Fi(A)$. It is known that $Fi(A)$ is a distributive lattice (see [7] or [8]). If $X \subseteq A$, then the filter generated by X, in symbols $\langle X \rangle$, is the set

$$\langle X \rangle = \{a \in A : \exists \{x_0, x_1, ..., x_n\} \subseteq X \text{ such that } (x_0, x_1, x_n; a) = 1\}.$$

In particular, the filter generated by $\{x\}$, is $\langle x \rangle = \{a \in A : x \to a = 1\} = \{a \in A : x \leq a\}$. A proper filter $D \subseteq A$ is *maximal* if and only if for any $H \in Fi(A)$ such that $D \subseteq H$, we have $D = H$ or $H = A$. The set of all maximal filters is denoted by $Sp(A)$. The following results are known (see [2], [3] or [5]).

Theorem 2.2. *Let A be a Tarski algebra. Then the following conditions are equivalent:*

1. $P \in Sp(A)$.

2. *P is prime, i.e., for all $a, b \in A$ if $a \vee b \in P$, then $a \in P$ or $b \in P$.*

3. *For every $D_1, D_2 \in Fi(A)$, if $P = D_1 \cap D_2$, then $P = D_1$ or $P = D_2$.*

Let A be a Tarski algebra. A subset I of A is called an *ideal* of A if (1) for every $a, b \in A$, if $b \in I$ and $a \leq b$, then $a \in I$, and for all $a, b \in I$, $a \vee b \in I$. The following result can be found in [6] for Hilbert algebras.

Theorem 2.3. *Let A be a Tarski algebra. Let $D \in Fi(A)$ and let I be an ideal of A such that $D \cap I = \emptyset$. Then there exists $P \in Sp(A)$ such that $D \subseteq P$ and $P \cap I = \emptyset$.*

Theorem 2.4. *Let A be a Tarski algebra. Then*

1. *For all $a, b \in A$, if $a \nleq b$ there exists $P \in Sp(A)$ such that $a \in P$ and $b \notin P$.*

2. *If $P \in Sp(A)$, then $a \to b \notin P$ if and only if $a \in P$ and $b \notin P$.*

Let X be a non-empty set. It is known that $\langle \mathcal{P}(X), \Rightarrow, X \rangle$ is a Tarski algebra, where $U \Rightarrow V = (X - U) \cup V$. All subalgebras of $\mathcal{P}(X)$ will be called *Tarski algebras of sets.*

Theorem 2.5. *Let A be a Tarski algebra. Then the map $\sigma_A : A \to \mathcal{P}(Sp(A))$ defined by $\sigma_A(a) = \{P \in Sp(A) : a \in P\}$ is an injective homomorphism of Tarski algebras.*

Proof. It is clear that the map $\sigma_A : A \to \mathcal{P}(Sp(A))$ is injective. The identity $\sigma_A(a \to b) = \sigma_A(a) \Rightarrow \sigma_A(b)$ follows by Theorem 2.4. $\square$

3. Representation for finite Tarski algebras

It is known that for every finite Boolean algebra A there exists a finite set X such that $A \cong \mathcal{P}(X)$ (for example, the set $X = Sp(A)$). The converse is also valid: if X is a finite non-empty set, then $X \cong Sp(\mathcal{P}(X))$. But in general, these good correspondences are not held for finite Tarski algebras. In fact it is not difficult to find a finite Tarski algebra A such that neither $A \cong \mathcal{P}(X)$ nor $A \cong \mathcal{P}(X) - \{\emptyset\}$ for any set X. For instance, the figure

Fig. 1 shows a finite Tarski algebra $A = \{a, b, c, d, 1\}$ where the operation $\rightarrow$ is defined by:

$$x \rightarrow y = \begin{cases} 1 & if \quad x \leq y \\ y & if \quad x \nleq y \end{cases},$$

for $x, y \in A$, and where $Sp\,(A) = \{\{c, d, 1\}, \{b, d, 1\}, \{a, b, c, 1\}\}$, such that A neither isomorphic to $\mathcal{P}\,(Sp\,(A))$ nor $\mathcal{P}\,(Sp\,(A)) - \{\emptyset\}$.

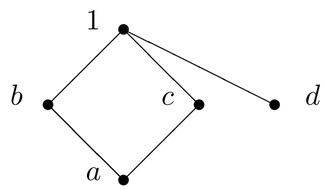

Fig.1

We shall show that there exists a good representation for finite Tarski algebras. The key of this representation is to give an adequate characterization of the image of the mapping $\sigma_A : A \rightarrow \mathcal{P}\,(Sp\,(A))$.

Definition 3.1. Let A be a Tarski algebra. We shall say that $a \in A - \{1\}$ is a *dual atom or an antiatom*, if for all $x \in A$ such that $a \leq x \leq 1$, then $a = x$ or $x = 1$.

The set of all dual atoms of a finite Tarski algebra A will be denoted by $Ant\,(A)$. We note that in finite Tarski algebra every element different from 1 is an infimum of a non-empty set of dual atoms. We can also prove that a filter P is maximal if and only if there exists $a \in Ant\,(A)$ such that $P = A - (a] = (a]^c$. Thus, if A is a finite Tarski algebra, $Sp\,(A) \cong Ant\,(A)$ (see [1]).

Definition 3.2. A *Tarski set* is a pair $\langle X, \mathcal{K} \rangle$ where X is a non-empty set and $\mathcal{K}$ is a non-empty subset of $\mathcal{P}\,(X)$.

The *dual* of a Tarski set $\langle X, \mathcal{K} \rangle$ is the subset $D\,(X)$ of $\mathcal{P}\,(X)$ defined by:

$$D\,(X) = \{U \in \mathcal{P}\,(X) : \exists W \in \mathcal{K} \,\&\, \exists S \subseteq W \text{ such that } U = W^c \cup S\}.$$

Proposition 3.3. *Let $\langle X, \mathcal{K}\rangle$ be a Tarski set. Then $\langle D(X), \Rightarrow, X\rangle$ is a Tarski subalgebra of $\mathcal{P}(X)$.*

Proof. Since $\mathcal{K}$ is non-empty, there exists $W \in \mathcal{K}$. Thus, $X = W^c \cup W \in D(X)$. Let $U, V \in D(X)$. Let us prove that $U \Rightarrow V = U^c \cup V \in D(X)$. As there exists $W \in \mathcal{K}$ and $S \subseteq W$ such that $V = W^c \cup S$, then $W^c \cup (W \cap (U^c \cup V)) = W^c \cup U^c \cup V = U^c \cup V$. Thus, $U^c \cup V \in D(X)$ and $\langle D(X), \Rightarrow, X\rangle$ is a Tarski algebra of $\mathcal{P}(X)$ □

Let A be a finite Tarski algebra. Let us consider the mapping $\sigma_A : A \rightarrow \mathcal{P}(Sp(A))$ and let us consider the family of subsets $\mathcal{K}_A = \{\sigma_A(a)^c : a \in A\}$. The pair $\langle Sp(A), \mathcal{K}_A\rangle$ is a Tarski set called the *dual* Tarski set of A or the *associated* Tarski set of A.

Theorem 3.4. *Let A be a finite Tarski algebra. Then,*

$$\sigma_A(A) = D(Sp(A)).$$

Thus, $A \cong D(Sp(A))$.

Proof. We first note that for every $a \in A$, $\sigma_A(a) \in D(Sp(A))$, because $\sigma_A(a) = \sigma_A(a) \cup \emptyset$. Let $U \in D(Sp(A))$. We prove that there exists $b \in A$ such that $U = \sigma_A(b)$. By definition of $D(Sp(A))$ we get that there exists $a \in A$ and there exists $S \subseteq \sigma_A(a)^c$ such that $U = \sigma_A(a) \cup S$. Since A is finite, S is a finite subset of $Sp(A)$. Let $S = \{P_1, ..., P_n\}$. Let us recall that for each $P_i \in S$ there exists $a_i \in Ant(A)$ such that $P_i = (a_i]^c$. It is easy to see that $\sigma_A(a_i)^c = \{(a_i]^c\}$. Thus, $S = \sigma_A(a_1)^c \cup \ldots \cup \sigma_A(a_n)^c$. So,

$$U = \sigma_A(a) \cup \sigma_A(a_1)^c \cup \ldots \cup \sigma_A(a_n)^c = \sigma_A((a_1, a_2, ..., a_n; a)) = \sigma_A(b).$$

Therefore, $U \in \sigma_A(A)$, and consequently $\sigma_A(A) = D(Sp(A))$. □

Lemma 3.5. *Let $\langle X, \mathcal{K}\rangle$ be a Tarski set. If $U \in Ant(D(X))$, then there exists $x \in X$ such that $U = \{x\}^c$.*

Proof. Let $U \in Ant(D(X))$. As $U \neq X$, then there exists $x \notin U$. So, $U \subseteq \{x\}^c$. Since there exists $W \in \mathcal{K}$ and $S \subseteq W$ such that $U = W^c \cup S$, then $x \in W$ and $x \notin S$. Let $W_x = W - \{x\}$. So, as $S \cup \{x\} \subseteq W_x$, then the set $U_x = U \cup \{x\} = W_x \cup (S \cup \{x\}) \in D(X)$. Since $U \subset U_x$ and U is a dual atom, $U_x = X$. Thus, for every $y \in X$ such that $y \neq x$, we get $y \in U$, i.e., $U = \{x\}^c$. □

It is known that for every finite and non-empty set X, $X \cong Sp(\mathcal{P}(X))$. For the case of a finite Tarski set $\langle X, \mathcal{K} \rangle$ we shall need an additional condition on X.

Definition 3.6. We shall say that a Tarski set $\langle X, \mathcal{K} \rangle$ is *dense* if for every $x \in X$ there exists $W \in \mathcal{K}$ such that $x \in W$.

Lemma 3.7. *Let* $\langle X, \mathcal{K} \rangle$ *be a dense Tarski set. For all* $x \in X$, $\{x\}^c \in Ant(D(X))$.

Proof. Let $x \in X$. As $\langle X, \mathcal{K} \rangle$ is dense, there exists $W \in \mathcal{K}$ such that $x \in W$. Let $S_x = W - \{x\}$. So, $S_x \subset W$ and $W^c \cup S^x = \{x\}^c \in D(X)$. It is clear that $\{x\}^c \in Ant(D(X))$. □

The dual frame $\langle Sp(A), \mathcal{K}_A \rangle$ of a finite Tarski algebra A is always dense, because any maximal filter P is proper and thus there exists $a \in A$ such that $P \in \sigma_A(a)^c \in \mathcal{K}_A$.

Theorem 3.8. *Let* $\langle X, \mathcal{K} \rangle$ *be a finite dense Tarski set. Then the mapping* $\varepsilon_X : X \to Sp(D(X))$ *defined by* $\varepsilon_X(x) = \{U \in D(X) : x \in U\}$ *is injective and onto.*

Proof. It is clear that $\varepsilon_X(x) \in Sp(D(X))$. If $x, y \in X$ and $x \neq y$, then $y \in \{x\}^c \in Ant(D(X))$. Thus, ε_X is injective.

Let $P \in Sp(D(X))$. Since $D(X)$ is finite, there exists $U \in Ant(D(X))$ such that $P = (U]^c$. As $\langle X, \mathcal{K} \rangle$ is dense, there exists $x \in X$ such that $U = \{x\}^c$. Now, it is easy to check that $\varepsilon_X(x) = P$. Thus, $X \cong Sp(D(X))$. □

By Theorem 3.4 and Theorem 3.8 we can identify a finite Tarski algebra A with the Tarski algebra $D(Sp(A))$. This means we need no longer consider abstract finite Tarski algebras, but only those of the form $D(X)$ for a finite Tarski set $\langle X, \mathcal{K} \rangle$.

4. Modal Tarski algebras

Let A and B be two Boolean algebras. Let us recall that a map $\Box : A \to B$ is called a *hemimorphism* if $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$, for all

$a, b \in A$. A *modal algebra* is a pair $\langle A, \Box \rangle$ where A is a Boolean algebra and $\Box : A \to A$ is a hemimorphism. We note that a modal algebra can also be defined as a Boolean algebra with a unary operation $\Box$ satisfying $\Box 1 = 1$ and $\Box(a \to b) \leq \Box a \to \Box b$, for all $a, b \in A$. giving place to the following definition.

Definition 4.1. *A modal Tarski algebra is an algebra $\langle A, \Box \rangle$ where A is a Tarski algebra and $\Box$ is a unary operator defined in A such that it verifies the following conditions:*

MT1 $\Box 1 = 1$,

MT2 $\Box(a \to b) \leq \Box a \to \Box b$, *for all $a, b \in A$.*

The class of modal Tarski algebra is a variety denote by $\mathcal{MT}$.

Now we shall study the representation by means of sets of modal Tarski algebras. Let A be a modal Tarski algebra. Let us define a binary relation $R_A \subseteq Sp(A)^2$ by:

$$(P, Q) \in R_A \Leftrightarrow \Box^{-1}(P) \subseteq Q,$$

where $\Box^{-1}(P) = \{a \in A : \Box a \in P\}$.

Lemma 4.2. *Let $A \in \mathcal{MT}$. Let $P \in Sp(A)$ and $a \in A$. Then, $\Box a \notin P$ if and only if there exists $Q \in Sp(A)$ such that $(P, Q) \in R_A$ and $a \notin Q$.*

Proof. Let us suppose that $\Box a \notin P$. It is clear that $\Box^{-1}(P)$ is a filter. Since $a \notin \Box^{-1}(P)$, then there exists $Q \in Sp(A)$ such that $\Box^{-1}(P) \subseteq Q$ and $a \notin Q$. The other direction is immediate. $\Box$

A *relational frame or frame* is a pair $\langle X, R \rangle$ where X is a set and let $R \subseteq X^2$. Given a frame $\langle X, R \rangle$ let us define a modal operator $\Box_R$ in the Tarski algebra $\langle \mathcal{P}(X), \Rightarrow, X \rangle$ by

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\}.$$

Then $\langle \mathcal{P}(X), \Box_R, \Rightarrow, X \rangle$ is a modal Tarski algebra. Any modal subalgebra of a modal Tarski algebra of the form $\langle \mathcal{P}(X), \Box_R, \Rightarrow, X \rangle$ will be called a *modal Tarski algebra of sets.*

A *Tarski relational set* is a structure $\langle X, R, \mathcal{K} \rangle$ where $\langle X, R \rangle$ is a relational frame, $\langle X, \mathcal{K} \rangle$ is a Tarski set, and $\Box_R(U) \in D(X)$ for each

$U \in D(X)$. So it is easy to see that the structure $\langle D(X), \Box_R, \Rightarrow, X\rangle$ is a modal Tarski algebra of sets.

Let A be a modal Tarski algebra. The structure $\langle Sp(A), R_A\rangle$ is called the *frame associated to* A and $\langle Sp(A), R_A, \mathcal{K}_A\rangle$ is called the *relational Tarski frame* associated to A.

Theorem 4.3. *Any modal Tarski algebra A is isomorphic to a modal Tarski algebra of sets.*

Proof. Let us consider the Tarski algebra $\sigma_A(A) = \{\sigma_A(a) : a \in A\}$. Let us consider the modal operator $\Box_{R_A}$ defined in $\mathcal{P}(Sp(A))$ by $\Box_{R_a}(U) = \{P \in Sp(A) : R_A(P) \subseteq U\}$. By Lemma 4.2 we have $\Box_{R_A}\sigma_A(a) = \sigma_A(\Box a)$. So, σ_A is an injective homomorphism of modal Tarski algebras. Thus, A is isomorphic to the modal Tarski algebra of sets $\sigma_A(A)$. $\Box$

For finite Tarski modal algebras we have the following result.

Theorem 4.4. *Any finite modal Tarski algebra is isomorphic to some modal algebra of sets for some finite Tarski relational set.*

Proof. Let A be a finite modal Tarski algebra. By Theorem 3.4 we have that $A \cong D(Sp(A))$ and by Lemma 4.2 we get that σ_A is a modal isomorphism between A and $D(Sp(A))$. $\Box$

An *isomorphism* between two Tarski relational sets $\langle X_1, R_1, \mathcal{K}_1\rangle$ and $\langle X_2, R_2, \mathcal{K}_2\rangle$ is a bijective mapping $f : X_1 \to X_2$ such that

$$(x, y) \in R_1 \text{ if and only if } (f(x), f(y)) \in R_2,$$

for every $x, y \in X_1$.

Theorem 4.5. *Let $\langle X, R, \mathcal{K}\rangle$ be a finite Tarski dense relational set. Then $\varepsilon_X : X \to Sp(D(X))$ is an isomorphism.*

Proof. Let $x, y \in X$. If $(x, y) \in R$, then it is clear that $(\varepsilon_X(x), \varepsilon_X(y)) \in R_{D(X)}$. Let us suppose that $(\varepsilon_X(x), \varepsilon_X(y)) \in R_{D(X)}$. If $(x, y) \notin R$, then $R(x) \subseteq \{y\}^c = U_y$. By Lemma 3.7, $U_y \in Ant(D(X))$. Then, $x \in \Box_R(U_y)$, and thus $U_y \in \varepsilon_X(y)$, i.e., $y \in U_y$, which is a contradiction. Thus, $(x, y) \in R$. $\Box$

By the above results we can identify a finite modal Tarski algebra A, with the modal Tarski algebra $D(Sp(A))$.

Let $A \in \mathcal{MT}$. Let $a \in A$. For each $n \geq 0$ we define inductively the formula $\Box^n a$ as $\Box^0 a = a$ and $\Box^{n+1} a = \Box\Box^n a$. We also define the symbol $(\alpha_n(a); y) = (a, \Box a, \ldots, \Box^n a; y)$.

The variety of modal Tarski algebras generated by a finite set of identities Γ will be denoted by $\mathcal{MT} + \{\Gamma\}$. Now we shall consider some particular varieties of modal Tarski algebras. Let us consider the following identities:

$$\begin{array}{ll} \mathbf{T} & \Box x \rightarrow x \approx 1 \\ \mathbf{4} & \Box x \rightarrow \Box^2 x \approx 1 \\ \mathbf{4}_n & \Box^n x \rightarrow \Box^{n+1} x \approx 1 \\ \mathbf{4}_{wn} & \left(\alpha_n(x); \Box^{n+1} x\right) \approx 1, \\ \mathbf{5} & (\Box x \rightarrow \Box y) \rightarrow \Box(\Box x \rightarrow \Box y) \approx 1. \end{array}$$

Remarks. 1. We note that in the variety $\mathcal{MT} + \{\mathbf{5}\}$ the identity $\Box x \rightarrow \Box^2 x \approx 1$ is held, because $1 \approx (\Box 1 \rightarrow \Box x) \rightarrow \Box(\Box 1 \rightarrow \Box x) \approx \Box x \rightarrow \Box\Box x$.
2. In the variety $\mathcal{MT} + \{\mathbf{4}_{wn}\}$ the identity $(\alpha_n(x); y) \approx (\alpha_{n+k}(x); y)$ is valid for all $k \geq 0$. Indeed:

$$\begin{aligned} (\alpha_{n+1}(x); y) &= \left(x, \Box x, ..., \Box^n x, \Box^{n+1} x; y\right) \\ &= \left(x, \Box x, ..., \Box^n x; \Box^{n+1} x\right) \rightarrow (x, \Box x, ..., \Box^n x; y) \\ &= \left(\alpha_n(x); \Box^{n+1} x\right) \rightarrow (x, \Box x, ..., \Box^n x; y) \\ &= 1 \rightarrow (x, \Box x, ..., \Box^n x; y) \\ &= (\alpha_n(x); y). \end{aligned}$$

Thus, $(\alpha_n(x); y) = (\alpha_{n+1}(x); y) = (\alpha_{n+2}(x); y) .... = (\alpha_{n+k}(x); y)$.
3. It is easy to see that the variety $\mathcal{MT} + \{\mathbf{T}, \mathbf{5}\}$ is equivalent to the variety of monadic Tarski algebras $\mathcal{MT} + \{\mathbf{T}, \Box(x \vee \Box y) \approx \Box x \vee \Box y\}$ (see [1]).

In order to extend Lemma 4.2 we will need some notations. Let X be a set and let $R \subseteq X^2$. Define in X the binary relation R^n, for $n \geq 0$, and the binary relation R^*, as follows:

$$\begin{array}{lcl} (x, y) \in R^0 & \Leftrightarrow & x = y \\ (x, y) \in R^{n+1} & \Leftrightarrow & \exists z \in X \ \ ((x, z) \in R^n \text{ and } (z, y) \in R) \\ R^* & = & \bigcup_{n \geq 0} R^n. \end{array}$$

It is clear that R^* is reflexive and transitive.

Lemma 4.6. *Let $A \in \mathcal{MT}$. Let $P, Q \in Sp(A)$. Then*

$$(P, Q) \in R_A^n \Leftrightarrow \{a \in A : \Box^n a \in P\} \subseteq Q.$$

Proof. The proof is by induction on n. The case $n = 1$ is the definition of the binary relation R_A. Suppose that the result holds for n. If $(P, Q) \in R_A^{n+1}$, then it is easy to see that $\{a \in A : \Box^{n+1} a \in P\} \subseteq Q$. Let us suppose that $\{a \in A : \Box^{n+1} a \in P\} \subseteq Q$ and let us consider the following subset of $Ds(A)$:

$$\mathcal{F} = \left\{ D \in Ds(A) : \Box^{-1}(P) \subseteq D \text{ and } D \cap \Box^n (A - Q) = \emptyset \right\}.$$

We note that $\Box^{-1}(P) \in \mathcal{F}$, because if $\Box^{-1}(P) \cap \Box^n (A - Q) \neq \emptyset$, then there exists $a \in \Box^{-1}(P)$ and $q \notin Q$ such that $a = \Box^n q$. So, $\Box a = \Box^{n+1} q \in P$. Thus, $q \in Q$, which is a contradiction. By Zorn's lemma, there exists a filter D maximal in $\mathcal{F}$. We prove that $D \in Sp(A)$. Let $D_1, D_2 \in Ds(A)$ such that $D = D_1 \cap D_2$ and let us suppose that $D \subset D_1$ and $D \subset D_2$. Thus, $D_1, D_2 \notin \mathcal{F}$. Then $D_1 \cap \Box^n (A - Q) \neq \emptyset$ and $D_2 \cap \Box^n (A - Q) \neq \emptyset$, i.e., there exist $a \in D_1$, $b \in D_2$, $q_1, q_2 \notin Q$ such that $a = \Box^n q_1$ and $b = \Box^n q_2$. Since, $q = q_1 \vee q_2 \notin Q$, $\Box^n q_1 \leq \Box^n q$ and $\Box^n q_2 \leq \Box^n q$, then $\Box^n q \in D_1 \cap D_2 = D$. Hence, $D \cap \Box^n (A - Q) \neq \emptyset$, which is a contradiction. Thus, $D \in Sp(A)$, $(P, D) \in R_A$ and $D \cap \Box^n (A - Q) = \emptyset$. By inductive hypothesis, we have $(D, Q) \in R_A^n$. Therefore, $(P, Q) \in R_A^{n+1}$. $\Box$

Now we prove that the identities **T**, $\mathbf{4}_n$, $\mathbf{4}_{wn}$ and **5** correspond to additional first-order properties defined in the associated frame (see [9] for the logic interpretation of the identity $\mathbf{4}_{wn}$).

Theorem 4.7. *Let $A \in \mathcal{MT}$. Then:*

1. $A \in \mathcal{MT} + \{\mathbf{T}\} \Leftrightarrow R_A$ *is reflexive.*

2. $A \in \mathcal{MT} + \{\mathbf{4}_n\} \Leftrightarrow R_A$ *is n-transitive, i.e.,* $R_A^{n+1} \subseteq R_A^n$.

3. $A \in \mathcal{MT} + \{\mathbf{4}_{wn}\} \Leftrightarrow \forall P, Q \in Sp(A)$: *if* $(P, Q) \in R_A^{n+1}$ *then there exists* $0 \leq j \leq n$ *such that* $(P, Q) \in R_A^J$.

4. $A \in \mathcal{MT} + \{\mathbf{5}\} \Leftrightarrow \forall P, Q, D \in Sp(A)$: *if* $(P, Q) \in R_A$ *and* $(Q, D) \in R_A$ *then* $(P, D) \in R_A$ *and for every* $Z \in Sp(A)$ *such that* $(P, Z) \in R_A$ *implies that* $(Q, Z) \in R_A$.

5. $A \in \mathcal{MT} + \{\mathbf{T}, \mathbf{5}\} \Leftrightarrow R_A$ *is an equivalence relation.*

Proof. We prove only 3. and 4.

3. $\Rightarrow$ Let us suppose that there exists $P, Q \in Sp(A)$ such that $(P,Q) \in R_A^{n+1}$ and $(P,Q) \notin R_A^J$ for all $0 \leq j \leq n$ Then there exists $a_0 \in P - Q$ and for each $1 \leq i \leq n$ there exists $a_i \in A$ such that $\Box^i a_i \in P$ and $a_i \notin Q$ .So, $a = a_0 \vee a_1 \vee ... \vee a_n \notin Q$ and as $\Box^i a_i \leq \Box^i a$, then $\Box^i a \in P$ for all $0 \leq i \leq n$. Since, $\left(\alpha_{n+1} a, \Box^{n+1} a\right) = 1 \in P$, then $\Box^{n+1} a \in P$, and taking into account that $(P,Q) \in R_A^{n+1}$, we have $a \in Q$, which is a contradiction.

$\Leftarrow$ Let us suppose that there exists $a \in A$ such that $\left(\alpha_{n+1} a, \Box^{n+1} a\right) \neq 1$. Then there exists $P \in Sp(A)$ such that $\Box^i a \in P$ for all $0 \leq i \leq n$ and $\Box^{n+1} a \notin P$. From Lemma 4.6 there exists $Q \in Sp(A)$ such that $(P,Q)\, R_A^{n+1}$ and $a \notin Q$. By assumption, there exists $0 \leq j \leq n$ such that $(P,Q) \in R_A^j$, and thus $a \in Q$, which is a contradiction.

4. $\Rightarrow$ Let $P, Q, D \in Sp(A)$:such that $(P,Q) \in R_A$ and $(Q,D) \in R_A$. Since

$$\Box a = 1 \to \Box a = \Box 1 \to \Box a \leq \Box\left(\Box 1 \to \Box a\right) = \Box\Box a,$$

then R_A is transitive. So, $(P,D) \in R_A$. Let us suppose that there exits $Z \in Sp(A)$ such that $(P,Z) \in R_A$ and $(Q,Z) \notin R_A$. Then there exists $a \in A$ such that $\Box a \in Q$ and $a \notin Z$, and consequently $\Box a \notin P$. Let us consider the filters $F = \langle P \cup \{\Box a\}\rangle$. Since P is maximal, $F = A$. On the other hand, as D is proper there exists $d \notin D$. Then $\Box d \in F$, i.e., $\Box a \to \Box d \in P$. Thus, $\Box\left(\Box a \to \Box d\right) \in P$, and as $(P,Q) \in R_A$ and $\Box a \in Q$, $\Box d \in Q.d$. It follows, $d \in D$, which is a contradiction.

$\Leftarrow$ Suppose that there exist $a, b \in A$ such that $\Box a \to \Box b \nleq \Box(\Box a \to \Box b)$. Then there exists P and $Q \in Sp(A)$ such that $\Box a \to \Box b \in P$, $(P,Q) \in R_A$ and $\Box a \to \Box b \notin Q$. Hence, Q is maximal, $\Box a \in Q$ and $\Box b \notin Q$. Thus, there exists $D \in Sp(A)$ such that $(Q,D) \in R_A$ and $b \notin D$. As $(P,D) \in R_A$, $\Box b \notin P$, and since $\Box a \to \Box b \in P$, we have $\Box a \notin P$. Therefore, $(P,Z) \in R_A$ and $a \notin Z$ for some $Z \in Sp(A)$. But as $(Q,Z) \in R_A$ and $\Box a \in Q$, $a \in Z$, which is a contradiction. Therefore, $\Box a \to \Box b \leq \Box\left(\Box a \to \Box b\right)$. $\Box$

We shall say that a variety of modal Tarski algebras $\mathcal{V}$ have the *finite model property* if for each identity

$$\varphi \approx \psi \notin Eq(\mathcal{V}) = \{\phi \approx \alpha : A \models \phi \approx \alpha, \text{ for any } A \in \mathcal{V}\}$$

there exists a finite algebra $A \in \mathcal{V}$ such that $A \not\models \varphi \approx \psi$. It is clear that a variety $\mathcal{V}$ has the finite model property if and only if $\mathcal{V}$ is generated by the class of the finite algebras of the variety $\mathcal{V}$. Now we shall prove that the variety $\mathcal{MT}$ and the subvarieties considered above are generated by finite algebras. The proof of this result was suggested by the referee.

Theorem 4.8. *Let $\mathcal{V}$ be one of the varieties defined in Theorem 4.7, and let $\mathcal{V}_M$ be the variety of modal algebras defined by the same identities. Then $\mathcal{V}$ consists of subreducts of members of $\mathcal{V}_M$, and $\mathcal{V}$ has the finite model property.*

Proof. As each algebra A of $\mathcal{V}$ is isomorphic to a modal subalgebra of the modal algebra of sets $\langle \mathcal{P}(Sp(A)), \cap, \Box_R, \Rightarrow, \emptyset, X \rangle$, then $\mathcal{V}$ consists of subreducts of members of $\mathcal{V}_M$.

Let φ and ψ be terms in the language of modal Tarski algebras. Let us suppose that $\mathcal{V} \nvDash \varphi \approx \psi$. Then, there exists $A \in \mathcal{V}$ and there exists a valuation $\vec{a} = (a_1, ..., a_n)$ with $a_i \in A$ such that $\varphi(\vec{a}) \neq \psi(\vec{a})$ in A. Now, as A is a subreduct of some $B \in \mathcal{V}_M$, then $\varphi(\vec{a}) \neq \psi(\vec{a})$ in B. Since each variety $\mathcal{V}_M$ has the finite model property, there exists a finite modal algebra $C \in \mathcal{V}_M$ and a valuation $\vec{c} = (c_1, ..., c_n)$ with $c_i \in C$ such that $\varphi(\vec{c}) \neq \psi(\vec{c})$ in C. But the $\{\rightarrow, 1\}$-reduct of C belongs to $\mathcal{V}$, yielding a finite countermodel. Therefore, $\mathcal{V}$ has the finite model property. $\Box$

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