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Kamila BENDOVÁ

INTERPOLATION AND THREE-VALUED LOGICS

1. Three-valued logics

We consider propositional logic. Three-valued logics are old: the first one is Lukasiewicz three valued logic from 1920 [8]. Gödel in [5] from 1932 studied a hierarchy of finite-valued logics, containing Gödel three-valued logic. Our main interest pays to Kleene three-valued logic [6]. Other threevalued logics will not be considered here. Let us agree that the three truth values are $0, \frac{1}{2}, 1$ in the natural ordering. All three logics have connectives \wedge, \vee interpreted as minimum and maximum; Kleene and Łukasiewicz have Lukasiewicz negation \neg_L , whose truth function is 1 - x involutive negation. Lukasiewicz has his implication \rightarrow_L (truth function $\min(1, 1 - x + y)$); it is the residuum of strong Lukasiewicz conjunction $\max(0, x + y - 1)$. Kleene's implication \rightarrow_K may be omitted since it is definable as $\neg_L x \vee$ y. Gödel's implication \rightarrow_G has the truth function equal 1 if $x \leq y$ and equal y otherwise; Gödel's negation is $\neg_G 0 = 1, \ \neg_G x = 0$ otherwise. We shall not introduce truth constants (\perp, \times, \top) for our truth values. Call the investigated logics K_3, L_3, G_3 .

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Trivially, Kleene logic has no tautologies; we shall work with two commonly known consequence relations \models_D (preservation of the designated value) and \models_C (comparative). For formulas $\varphi, \psi, \varphi \models_D \psi$ iff for each evaluation M of variables, $M(\varphi) = 1$ implies $M(\psi) = 1$; and $\varphi \models_C \psi$ iff for each $M, M(\varphi) \leq M(\psi)$. (This is meaningful in each of our logics and of course dependent on the truth functions of the logic chosen.) Note that for logics with residuated implication (our L_3 and G_3) $\varphi \models_C \psi$ is equivalent to $\varphi \rightarrow \psi$ being a tautology; and for $G_3, \varphi \models_D \psi$ is equivalent to $\varphi \models_C \psi$ since G_3 has classical deduction theorem. An axiom system complete for (tautologies of) L_3 was presented by Wajsberg [11] in 1931; for infinite-valued Gödel logic by Dummett [4]. From these one easily gets known complete axiomatizations for \models_D, \models_C in L_3, G_3 .

A complete (Gentzen style) axiomatization of \models_C for K_3 was presented by Cleave [3] and of \models_D for K_3 by Urquhart [9, 10].

2. Interpolation

An interpolation theorem says that if ψ is a consequence of φ then (under some other conditions) there is a formula χ whose propositional variables occur both in φ and in ψ such that χ is a consequence of φ and ψ is a consequence of χ ($\varphi \models \chi$ and $\chi \models \psi$). If we do not allow any truth constants and state no "other conditions" then such (inappropriately formulated) interpolation trivially fails even for the classical propositional calculus, $p \land$ $\neg p$ and $q \lor \neg q$ being a counterexample. This counterexample works also for our three logics (the value of $p \land \neg p$ being 0 in G_3 and $\leq \frac{1}{2}$ in L_3 and K_3). If we add the truth constants \top and \bot then interpolation holds for G_3 (see [1, 2]) but not for L_3 (see [7]) and evidently also not for K_3 (for L_3 and K_3 , the only interpolant of the example above is the constant $\frac{1}{2}$ which is evidently not definable from 1 and 0). But if we allow three truth constants then we can prove interpolation just imitating the classical proof.

This leads us to the notion of *satisfiable interpolation*: our "other condition" is that φ is satisfiable, i.e. $M(\varphi) = 1$ for some evaluation M. Our main result reads as follows:

Theorem 1 (satisfiable interpolation for K_3 and \models_D). Let φ, ψ be formulas, φ satisfiable, and let $\varphi \models_D \psi$. Then there is an interpolant χ

such that $\varphi \models_D \chi$ and $\chi \models_D \psi$.

The theorem is proved in the next section; then we also discuss the satisfiable interpolation for \models_C and for Lukasiewicz and Gödel.

3. Proving the theorem

We work with K_3 . Let $\varphi \equiv_3 \psi$ stand for $\varphi \models_C \psi$ and $\psi \models_C \varphi$ (φ, ψ are semantically equivalent). Commutativity and associativity of \land, \lor , distributivity, de Morgan rules etc. are valid semantical equivalences. A *literal* is a propositional variable or a negated propositional variable. A *fundamental conjunction* is a conjunction of finitely many pairwise disjoint literals (at least one). Evidently, each formula is semantically equivalent to a conjunction of fundamental disjunctions (note that e.g. $p_1 \lor \neg p_1 \lor p_3$ is a fundamental disjunction). A *language* is a (finite) set of propositional variables; an evaluation N of elements of a language L by truth values $0, \frac{1}{2}, 1$ is called a *model* of L. The extension of N to all formulas with variables from L using truth functions is also denoted N. $N \models \varphi$ stands for $N(\varphi) = 1$. Let $L \subseteq L'$ be languages. A model N' of L' is L-reducible if $N'(p) = \frac{1}{2}$ for all $p \in L' - L$. If $N = N' \upharpoonright L$ is the restriction of N' to L and N' is L-reducible then N' is called a *trivial extension* of N.

Lemma 1. If a formula φ of a language L is satisfied in an L-reducible model N_0 then there is a formula $\bar{\varphi}$ of \bar{L} such that $\bar{\varphi} \models_D \varphi$ and, for each \bar{L} -reducible model $N, N \models \varphi$ iff $N \models \bar{\varphi}$.

Proof. Let $N_0 \models \varphi$. Our φ is semantically equivalent to a formula

$$\bigwedge_{i\in I}\bigvee_{j\in J_i}\varepsilon_{ij}p_{ij},$$

briefly $\bigwedge_{i \in I} D_i$, D_i being fundamental disjunctions. We show that for each $i \in I$ there is a $j \in J_i$ such that $p_{ij} \in \overline{L}$. Assume the contrary; if $k \in I$ is such that $p_{kj} \notin \overline{L}$ for each $j \in J_k$ then $N_0(\varphi) \leq N_0(D_k) = \frac{1}{2}$, a contratiction. Let

$$\bar{\varphi} = \bigwedge_{i \in I} \bigvee_{j \in J_i} \{ \varepsilon_{ij} p_{ij} | j \in J_i \text{ and } p_{ij} \in \bar{L} \}.$$

Evidently $N(\bar{\varphi}) \leq N(\varphi)$ for each N, thus $\bar{\varphi} \models_D \varphi$.

Let now N be \overline{L} -reducible and such that $N \models \varphi$. Then for each $i \in I$ there is a $j \in J_i$ such that $N \models \varepsilon_{ij_i} p_{ij_i}$, hence $p_{ij_i} \in \overline{L}$. This gives immediately $N(\overline{\varphi}) = 1$.

Proof of the main theorem. Let φ be a formula of L_1, ψ a formula of $L_2, \varphi \models_D \psi, \varphi$ satisfiable, let N be a model of L_1 with $N \models \varphi$. Let N' be the trivial $L_1 \cup L_2$ -extension of N; then $N' \models \varphi$ and hence $N' \models \psi$. Consequently, ψ is satisfied in the model $N' \upharpoonright L_2$, which is evidently $(L_1 \cap L_2)$ -reducible. Thus, by the lemma above, there is a formula χ of $L_1 \cap L_2$ with $\chi \models_D \psi$; we prove that this χ (from the lemma) satisfies $\varphi \models_D \chi$. Let now M be any model of L_1 with $M \models \varphi$. Then φ is satisfied in its trivial extension M' to $L_1 \cup L_2$, hence $M' \models \psi, M' \upharpoonright L_2 \models \psi, M' \upharpoonright L_2$ is $(L_1 \cap L_2)$ reducible, thus $M' \upharpoonright L_2 \models \chi$ by our Lemma. Moreover,

$$M \models \chi \text{ iff } M' \models \chi \text{ iff } N' \upharpoonright L_2 \models \chi \text{ iff } M' \upharpoonright (L_1 \cap L_2) \models \chi,$$

since χ is a formula of the language $L_1 \cap L_2$. Thus χ is satisfied in each model satisfying φ . This completes the proof.

Remark 1. Note that K_3 (and hence L_3) does not have satisfiable interpolation for \models_C . Observe

$$q \lor (p \land \neg p) \models_C q \lor (r \lor \neg r).$$

Evidently, $q \lor (p \land \neg p)$ is satisfiable, but no formula built from q interpolates. It seems to be an open *problem* whether L_3 (without truth constants) has satisfiable interpolation. (Similarly for G_3 .)

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UOOU

Pplk.Sochora 27 170 00 Praha 7 Czech Republic

kamila.bendova@centrum.cz