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Piotr WOJTYLAK

ON A PROBLEM OF H. FRIEDMAN AND ITS SOLUTION BY T. PRUCNAL

A b s t r a c t T. Prucnal proved, see [8] and [9], that Medvedev's logic **ML** of finite problems is structurally complete. So far, **ML** is the only known structurally complete intermediate logic with the disjunction property. Another proof of Prucnal's theorem was given by D. Skvortsov [10]. We refresh Prucnal's original argument and prove that all structurally complete intermediate logics with the disjunction property coincide on monadic formulas. The logics are also provided with appropriate Kripke frames.

0. Introduction. T. Prucnal solved (see [8] and [9]) the Problem 41 of H. Friedman [2] whether there exist sets H of propositional formulas such that the following conditions are satisfied for every formulas α, β

(F1) $\quad \alpha \wedge \beta \in H \;\Leftrightarrow \alpha \in H$ and $\beta \in H$;

(F2) $\quad \alpha \vee \beta \in H \;\Leftrightarrow \alpha \in H$ or $\beta \in H$;

(F3) $\quad \alpha \to \beta \in H \Leftrightarrow \big(e(\alpha) \in H \Rightarrow e(\beta) \in H,$ for every substitution $e\big)$;

(F4) $\quad \neg\alpha \in H \quad \Leftrightarrow \big(e(\alpha) \notin H,$ for every substitution $e\big)$.

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He noticed that (F1) and (F4) are satisfied if H is an intermediate logics. Hence any intermediate logic H is a solution of Friedman's problem iff H is structurally complete[1] and enjoys the disjunction property. Then he showed that Medvedev's logic *ML* enjoys these properties. The uniqueness of the solution remains still open.

Prucnal's proof is purely syntactical. Another proof of the same theorem was given by D. Skvortsov [10]. The paper [9] contains also an answer to a similar Problem 42 of H. Friedman [2] concerning modal logic. This problem, however, is beside the scope of the present paper.

1. Intermediate logics. We consider the standard propositional language $\{\wedge, \vee \rightarrow, \neg\}$. Formulas of our language $\alpha, \beta, \ldots$ are built up from the variables $p_1, p_2, \ldots$ (we also write p instead of p_1). The set of all formulas is denoted by F. As usual, $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$ is abbreviated to $\alpha \equiv \beta$ and we use $\bigwedge X$ and $\bigvee X$ for generalized (finite) conjunctions and disjunctions. Substitutions are endomorphisms of the algebra of the language.

Let *INT* and *CL* be the sets of all intuitionistically and classically valid formulas, respectively. Any intermediate logic is a set H of formulas[2] closed under the substitution and the modus ponens rule and $INT \subseteq H \subseteq CL$. The logic *KP* of Kreisel and Putnam is the least intermediate logic with

$$(\neg\alpha \rightarrow \beta \vee \gamma) \rightarrow (\neg\alpha \rightarrow \beta) \vee (\neg\alpha \rightarrow \gamma)$$

Let B be the set of formulas defined as follows:

(i) $\neg\alpha \in B$, for every $\alpha \in F$;

(ii) $\alpha, \beta \in B \quad \Rightarrow \quad \alpha \vee \beta,\ \alpha \wedge \beta,\ \alpha \rightarrow \beta \in B$, for every $\alpha, \beta \in F$.

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[1] A logic is structurally complete, the notion is due to W.A.Pogorzelski [7], iff all its admissible and structural rules are derivable. In the case of intermediate logics (for which the modus ponens rule and the deduction theorem are valid) the structural completeness is equivalent to (F3)

[2] Intermediate logics are identified with the sets of their theorems as it is assumed that the modus ponens rule is the only basic inferential rule there.

Note that $e(\alpha) \in B$ for every substitution e if $\alpha \in B$. Moreover, $e(\alpha) \in B$ if $e(p_i) \in B$ for every p_i occurring in α. Medvedev's logic ML of finite problems is not finitely axiomatizable, see L.L. Maximova, D.P. Skvortsov, V.B. Šehtman [5]. The following characterization of ML:

$$\alpha \in ML \quad \Leftrightarrow \quad \big(e(\alpha) \in KP, \text{ for every substitution } e : F \to B\big)$$

is due to L.A. Levin [3]. Clearly, $KP \subseteq ML$ and $KP \cap B = ML \cap B$. It is less clear that $KP \neq ML$. Let us prove that for every $\alpha, \beta \in F$:

Lemma 1. $((\neg\neg\alpha \to \alpha) \to \alpha \vee \beta) \to \neg\neg\alpha \vee ((\neg\neg\alpha \to \alpha) \to \beta) \in ML$.

Proof. Let $\phi = e(\alpha)$ and $\psi = e(\beta)$ for a substitution $e : F \to B$. Since $\phi \in B$, there are $\beta_1, \cdots, \beta_k \in F$ such that

$$\phi \equiv \neg\beta_1 \vee \cdots \vee \neg\beta_k \in KP\ .$$

Then $\neg\phi$ is equivalent (in KP) to $\neg\neg(\beta_1 \wedge \ldots \wedge \beta_k)$ and $\neg\neg\phi \to \phi$ is equivalent to $\bigvee_i(\neg\neg\phi \to \neg\beta_i)$. Next, let us note that the following equivalences belong to KP, for each $i, j \leq k$:

$$(\neg\neg\phi \to \neg\beta_i) \to \neg\beta_j \equiv (\neg\phi \vee \neg\beta_i) \to \neg\beta_j \equiv$$

$$\equiv (\neg\phi \to \neg\beta_j) \wedge (\neg\beta_i \to \neg\beta_j) \equiv (\beta_1 \wedge \ldots \wedge \beta_k \to \neg\beta_j) \wedge (\neg\beta_i \to \neg\beta_j) \equiv$$
$$\equiv \neg(\beta_1 \wedge \ldots \wedge \beta_k) \wedge (\neg\beta_i \to \neg\beta_j) \equiv \neg\neg\phi \wedge (\neg\beta_i \to \neg\beta_j)$$

Thus, we get $\bigvee_j((\neg\neg\phi \to \neg\beta_i) \to \neg\beta_j) \equiv \neg\neg\phi \in KP$ and hence we obtain the following equivalences in KP :

$$(\neg\neg\phi \to \phi) \to (\phi \vee \psi) \equiv \bigvee_i(\neg\neg\phi \to \neg\beta_i) \to (\phi \vee \psi) \equiv$$

$$\equiv \bigwedge_i[(\neg\neg\phi \to \neg\beta_i) \to (\neg\beta_1 \vee \cdots \vee \neg\beta_k \vee \psi)] \equiv$$

$$\equiv \bigwedge_i[\bigvee_j((\neg\neg\phi \to \neg\beta_i) \to \neg\beta_j) \vee ((\neg\neg\phi) \to \neg\beta_i) \to \psi)] \equiv$$

$$\equiv \bigwedge_i [\neg\neg\phi \vee ((\neg\neg\phi \to \neg\beta_i) \to \psi)] \equiv$$

$$\equiv \neg\neg\phi \vee \bigwedge_i ((\neg\neg\phi \to \neg\beta_i) \to \psi) \equiv \neg\neg\phi \vee ((\neg\neg\phi \to \bigvee_i \neg\beta_i) \to \psi) \equiv$$

$$\equiv \neg\neg\phi \vee ((\neg\neg\phi \to \phi) \to \psi) \qquad \blacksquare$$

¿From the above lemma it follows that the formula (schema)

$$((\neg\neg\alpha \to \alpha) \to \alpha \vee \neg\alpha) \to (\neg\neg\alpha \vee \neg\alpha),$$

called Scott's law, belongs of *ML*. Since it does not belong to *KP* (use the Rieger-Nishimura lattice as a model for *KP*), we get *KP* $\neq$*ML*. It also follows from the above lemma that there is only finitely many (up to equivalence) formulas built up from a single variable, say p, in *ML*. The Lindenbaum algebra for the monadic fragment of *ML* is given by:

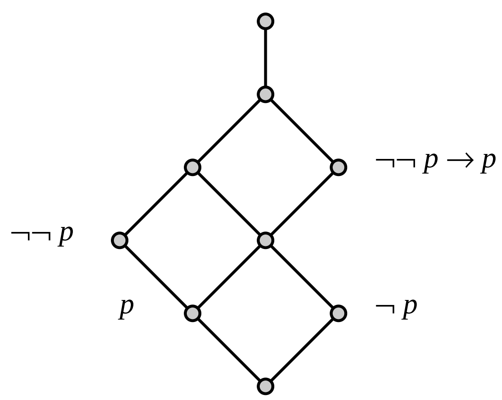

Figure 1.

2. Prucnal's trick. For any formula α let us define a substitution e_α putting $e_\alpha(p_i) = \alpha \to p_i$ for every variable p_i. The following can be shown by an easy induction on the length of the formula ϕ:

Lemma 2(i). *$\alpha \to (\phi \equiv e_\alpha(\phi)) \in INT$, for every ϕ in $\{\to, \wedge, \vee, \neg\}$;*
(ii). *$(\alpha \to \phi) \equiv e_\alpha(\phi) \in INT$, for every ϕ in $\{\to, \wedge\}$.*

¿From the above lemma it follows, for instance, that any intermediate logic in the fragment $\{\to, \wedge\}$ is structurally complete (an earlier result by T. Prucnal). The problem of structural completeness becomes more complicated if one admits $\vee$ and $\neg$.[3] Lemma 2(ii) does not extend to $\{\to, \wedge, \vee, \neg\}$. Nevertheless, Prucnal managed, see [9], to use his trick to show:

Theorem 3. *Medvedev's logic ML is structurally complete.*

Proof. Suppose that $\gamma \to \beta \notin ML$. We need to find a substitution e such that $e(\gamma) \in ML$ and $e(\beta) \notin ML$. Using the definition of ML, we reduce our argument to the case where $\beta \in B$ and $\gamma = \neg\alpha$ for some α.

Since $\neg\gamma \notin ML$, we have $\neg\gamma \notin CL$ and hence $\gamma_1 \wedge \ldots \wedge \gamma_k \to \gamma \in ML$ for some consistent set $\{\gamma_1, \ldots, \gamma_k\}$ of literals (i.e. variables or their negations). We can assume that $\gamma_1, \ldots, \gamma_k$ are variables as if we replace a literal by its negation nothing essential will change (recall $\beta, \gamma \in B$).

By Lemma 2(i), we get $e_\gamma(\beta) \notin ML$ as $\gamma \to \beta \notin ML$. We need to prove that $e_\gamma(\gamma) \in ML$. Let us note that $\gamma \to e_\gamma(\gamma) \in ML$ by Lemma 2(i) and $\neg\gamma \to e_\gamma(\gamma) \in ML$ by the assumption that $\gamma_1 \wedge \ldots \wedge \gamma_k \to \gamma \in ML$. Hence $e_\gamma(\gamma)$ is classically valid which gives $e_\gamma(\gamma) \in ML$. ∎

[3] I owe to professor Andrzej Wroński information about his unpublished results:
(i) each intermediate logic in $\{\to, \neg, \wedge\}$ is structurally complete – this result is implicitly included in Minari–Wroński [6];
(ii) the fragment $\{\to, \neg\}$ of intuitionistic logic is not (!) structurally complete;
(iii) the fragment $\{\vee, \to\}$ of intuitionistic logic, and any other fragment with $\to$ and $\vee$, is not structurally complete.

Though $(\alpha \to \beta \vee \gamma) \to (\alpha \to \beta) \vee (\alpha \to \gamma)$ is not intutitionistically valid, it is sometimes possible to derive the validity of $(\alpha \to \beta) \vee (\alpha \to \gamma)$ from the validity of $\alpha \to (\beta \vee \gamma)$. Prucnal's trick is very useful for this kind of results. In particular, one can prove, see [9]:

Theorem 4. *For every intermediate logic H and every formulas α, β, γ:*

$$\neg\alpha \to \beta \vee \gamma \in H \quad \Leftrightarrow \quad (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma) \in H.$$

Proof. One implication is clear. Let $\neg\alpha \to \beta \vee \gamma \in H$. Similarly as above, one reduces considerations to the case where $\gamma_1 \wedge \ldots \wedge \gamma_k \to \gamma \in H$ for some variables $\gamma_1, \ldots, \gamma_k$. Then we get $e_{\neg\alpha}(\neg\alpha) \in H$ and hence $e_{\neg\alpha}(\beta) \vee e_{\neg\alpha}(\gamma) \in H$ which gives $(\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma) \in H$ by use of Lemma 2(i) . ∎

The above theorem has been substantially strengthened by P. Minari and A. Wroński [6] with a slight modification of Prucnal's trick:

Theorem 5. *For every intermediate logic H, every formulas $\beta, \gamma \in F$ and every Harrop formula* [4] *α:*

$$\alpha \to \beta \vee \gamma \in H \quad \Leftrightarrow \quad (\alpha \to \beta) \vee (\alpha \to \gamma) \in H.$$

Proof. Argue as above but instead of e_α take the substitution

$$f_\alpha(p_i) = \begin{cases} \alpha \to p_i & \text{if } \ p_i \in X \\ \neg\neg\alpha \wedge (\alpha \to p_i) & \text{otherwise} \end{cases}$$

where X is a maximal consistent set closed under the modus ponens (but not under the substitution) rule, containing α and CL. ∎

In a similar way we may prove

[4] A Harrop formula (in propositional logic) is any formula in which every occurrence of disjunction lays inside the antecedent of some implication or inside the scope of some negation.

Theorem 6. *For every intermediate logic H, every $\alpha, \beta, \gamma \in F$ and every formulas $\phi_1, \ldots, \phi_k$ which do not contain the variables $p_1, \ldots, p_k$*

$$\bigwedge_{i \leq k} (p_i \equiv \phi_i) \to \beta \vee \gamma \in H \;\Leftrightarrow\; (\bigwedge_{i \leq k} (p_i \equiv \phi_i) \to \beta) \vee (\bigwedge_{i \leq k} (p_i \equiv \phi_i) \to \gamma) \in H.$$

Proof. Let e be a substitution such that $e(p_i) = \phi_i$ for $i \leq k$ and $e(p_i) = p_i$ for $i > k$. By induction on the length of ϕ one shows

$$\bigwedge_i (p_i \equiv \phi_i) \to (\phi \equiv e(\phi)) \in INT \tag{1}$$

Since $e(p_i \equiv \phi_i) \in INT$ for each i, we get

$$e(\phi) \in H \quad \text{iff} \quad \bigwedge_i (p_i \equiv \phi_i) \to \phi \in H \tag{2}$$

The implication ($\Leftarrow$) of our theorem is clear. Now, let us assume that $\bigwedge_i (p_i \equiv \phi_i) \to \beta \vee \gamma \in H$. Then, by (2), we get $e(\beta) \vee e(\gamma) \in H$ and hence, using (1), we obtain

$$(\bigwedge_i (p_i \equiv \phi_i) \to \beta) \vee (\bigwedge_i (p_i \equiv \phi_i) \to \gamma) \in H. \qquad \blacksquare$$

Note that the formulas $\phi_1, \ldots, \phi_k$ may contain disjunctions. Thus, we get a negative answer to a problem posed by Minari, Wroński [6]. They asked whether any formula α with the considered property (i. e. such that for every intermediate logic H and every formulas β, γ: if $\alpha \to \beta \vee \gamma \in H$, then $(\alpha \to \beta) \vee (\alpha \to \gamma) \in H$ is equivalent to a Harrop formula.

3. Bounds. Let H be a structurally complete intermediate logic with the disjunction property. It follows from Theorem 4 that $KP \subseteq H$ and, thus, we get a lower bound on the considered family of logics. We can also find an upper bound using (see e.g. L. Maximova [4]):

Theorem 7. *Medvedev's logic ML is the greatest intermediate logic among those which have the disjunction property and contain KP.*

Proof. To show that ML has the disjunction property it is sufficient to know that KP does. We apply Levin's definition of ML. Let L be a logic with the disjunction property and $KP \subseteq L$. If $\alpha \in L \setminus ML$, we can assume that $\alpha \in B \setminus KP$ and easy get to a contradiction with Glivienko's theorem. ∎

We conclude that $KP \subseteq H \subseteq ML$. The upper bound is optimal as ML is a structurally complete intermediate logic with the disjunction property. The lower bound can be improved. Prucnal said in [9], without giving a detailed proof, that H must contain all formulas of the form $(\alpha \to \beta) \to \beta$ where $\alpha \in ML$ and $\beta \in B$. Let us try to reconstruct his argument.

Suppose that B' is the extension of the set B with the following condition (in addition to (i) and (ii) from the definition of B)

(iii) if $\beta \in B'$ and $\alpha \in F$, then $\alpha \to \beta \in B'$.

We prove that $H \cap B' = ML \cap B'$. The inclusion $(\subseteq)$ is obvious as $H \subseteq ML$. Assume that $e(\phi) \in ML$, and prove (by induction on the length of ϕ) that $e(\phi) \in H$ for every substitution e and every $\phi \in B'$. If $\phi = \neg\alpha$ for some α, the above obviously holds. If ϕ is $\alpha \wedge \beta$ or $\alpha \vee \beta$, it suffices only to make use of the inductive hypothesis (and the disjunction property for ML). Suppose that $\phi = \alpha \to \beta$ for some α and some $\beta \in B'$. Let e, f be substitutions and $e(\alpha \to \beta) \in ML$. If $f(e(\alpha)) \in H$, then $f(e(\alpha)) \in ML$ and hence $f(e(\beta)) \in H$ by the inductive hypothesis. Then by structural completeness of H, we get $e(\alpha) \to e(\beta) \in H$.

Now, we can improve the lower bound. Note, for instance, that Scott's law (see Lemma 1.) belongs to B'. Hence H must be an extension of Scott's logic, as well. It means, in particular, that $KP = H$ is not allowed. Since KP has the disjunction property, it means that KP is not structurally complete.[5]

Another conclusion which can be drawn is that the logic H and ML coincide on monadic formulas, that is formulas built up from a single variable p. The problem arises if one can show they coincide on arbitrary formulas.

4. Kripke models. We set out with an observation that the conditions (F1)-(F4) resemble very much forcing conditions in a Kripke model. So, let us assume that H fulfills (F1)-(F4) (we do not assume at the moment that H is an intermediate logic), and define a Kripke model for H. A closer inspection of Friedman's conditions reveals that, if we want to incorporate them in a Kripke model, we must take substitutions as 'possible worlds'. So, let E be the set of all substitutions in F and define a binary relation $\preceq$ on E by

$$e \preceq f \quad \Leftrightarrow \quad \exists_{g \in E}\ f = g \circ e.$$

Note that $\preceq$ is transitive and reflexive. The model $\mathcal{E}(H) = \langle E, \preceq, \Vdash_H \rangle$ is given if the forcing relation $\Vdash_H$ is defined, for every p_i:

$$e \Vdash_H p_i \quad \Leftrightarrow \quad e(p_i) \in H.$$

Note that the frame $\langle E, \preceq \rangle$ of our model does not depend on H, only the relation $\Vdash_H$ does. Moreover, $\mathcal{E}(H)$ is an intuitionistic model, that is

$$\text{if } e \preceq f \text{ and } e \Vdash_H p_i, \text{ then } f \Vdash_H p_i,$$

[5] An unpublished result of Andrzej Wroński. Note that we manage to show that KP is not structurally complete without producing a structural and admissible rule which is not derivable there. It does not follow from the above that the rule

$$(\neg\neg\alpha \to \alpha) \to (\alpha \vee \neg\alpha)\ /\neg\neg\alpha \vee \neg\alpha,$$

given by Scott's law is an admissible rule of KP.

if (and only if) H is closed under substitutions. Let us prove that the forcing relation extends in a natural way to arbitrary formulas:

Theorem 8. *For every $e \in E$ and every formula $\alpha \in F$*

$$e \Vdash_H \alpha \quad \Leftrightarrow \quad e(\alpha) \in H.$$

Proof. We prove by induction on the length of the formula α. If α is a variable the above holds by definition. Let us assume the equivalence holds for α and β.

If $e \Vdash_H \alpha \wedge \beta$, then $e \Vdash_H \alpha$ and $e \Vdash_H \beta$, hence $e(\alpha \wedge \beta) \in H$ by our inductive hypothesis and $(F1)$. The reverse is also clear. A similar argument can also be used for $\alpha \vee \beta$ using (F2) instead of (F1).

Suppose that $e \Vdash_H \alpha \to \beta$ and let $g(e(\alpha)) \in H$ for a substitution $g \in E$. Note that $e \preceq g \circ e$ and $g \circ e \Vdash_H \alpha$ by inductive hypothesis. Hence $g \circ e \Vdash_H \beta$ by the definition of the forcing relation. Using inductive hypothesis, we obtain $g(e(\beta)) \in H$. Thus, it has been shown that, for every substitution g, if $g(e(\alpha)) \in H$, then $g(e(\beta)) \in H$. By (F3), we get $e(\alpha) \to e(\beta) \in H$.

Suppose, on the other hand, that $e(\alpha \to \beta) \in H$ and let $f \Vdash_H \alpha$ for a substitution $f \succeq e$. Then $f = g \circ e$ for some g and, by inductive hypothesis, we get $g(e(\alpha)) \in H$. Thus, by (F3), we have $g(e(\beta)) \in H$ and hence $f \Vdash_H \beta$. We have shown that $f \Vdash_H \alpha$ yields $f \Vdash_H \beta$, for every $f \succeq e$. It means that $e \Vdash_H \alpha \to \beta$ by the definition of the forcing relation in any Kripke model.

The formula $\neg\alpha$ is handled in a similar way. If $e(\neg\alpha) \in H$, then by the condition (F4) we get $g(e(\alpha)) \notin H$ and hence $g \circ e \nVdash_H \alpha$ for every substitution g. This gives $e \Vdash_H \neg\alpha$. If, on the other hand, $e \Vdash_H \neg\alpha$, then $g \circ e \nVdash_H \alpha$ and hence $g(e(\alpha)) \notin H$ for every g. It means that $e(\neg\alpha) \in H$ by the condition (F4). ∎

There is always a plenty of substitutions and hence the received set E of all possible worlds is rather huge. On the other hand substitutions are very much alike and hence possible words can be reduced by certain similarities.

In the first place, we can restrict ourselves to finitely generated sub-languages. Suppose that F_n is the set of all formulas in $\{\rightarrow, \vee, \wedge, \neg\}$ built up from the variables $p_1, \ldots, p_n$ and let $\mathcal{E}_n(L)$ be the reduct of the model $\mathcal{E}(L)$ to the language F_n. Practically, there is no much difference between the two models, except for the forcing relation in $\mathcal{E}_n(L)$ is restricted to the variables $p_1, \ldots, p_n$. Since it does not play any role for $e \Vdash_H \alpha$ how the relation is defined on variables which do not occur in α, for every formula $\alpha \in F_n$ we get:

$$\mathcal{E}(L) \Vdash \alpha \quad \Leftrightarrow \quad \mathcal{E}_n(L) \Vdash \alpha.$$

In the second place, we can reduce $\mathcal{E}_n(L)$ by use of some $p-$morhpisms. Let us recall that given two Kripke models, say $\mathcal{A} = \langle A, \preceq_A, \Vdash_A\rangle$ and $\mathcal{B} = \langle B, \preceq_B, \Vdash_B\rangle$, a mapping M from A onto B is called a $p-$morphism iff

(i) $a \preceq_A b \quad \Rightarrow \quad M(a) \preceq_B M(b)$;

(ii) $M(a) \preceq_B y \quad \Rightarrow \quad \exists_b(y = M(b)$ and $a \preceq_A b)$;

(iii) $a \Vdash_A p_i \quad \Leftrightarrow \quad M(a) \Vdash_B p_i$;

for every $a, b \in A$, every $y \in B$, and every variable p_i. One easily shows that, if M is a $p-$morhpism, then for every $a \in A$ and every formula α:

$$a \Vdash_A \alpha \quad \Leftrightarrow \quad M(a) \Vdash_B \alpha$$

Let us consider the case $n = 1$, assuming this time that H is an intermediate logic. Each possible world e in $\mathcal{E}_1(L)$, which is a substitution in F_1, can be identified with the formula $e(p)$. What is more, the only information required is whether $e(p) \in H$, or not. We get the following possibilities

(1) $e(p) \in H$. Then $f(p) \in H$ and hence $f \Vdash_H p$ for each $f \succeq e$. We can clearly identify all such worlds e that $e(p) \in H$ as they force exactly the same formulas.

(2) $\neg e(p) \in H$ and hence $e(p)$ is inconsistent. Then $f(p)$ is inconsistent for each $f \succeq e$ and hence $f \Vdash_H \neg p$. Similarly as above, the identification of all such substitutions e is possible.

(3) $e(p) \in CL \setminus H$. Then $f(p)$ can be valid for some $f \succeq e$, but cannot be inconsistent. Hence $e \Vdash_H \neg\neg p$ and $e \nVdash_H p$. Again, all such worlds e force the same formulas and we can identify them.

(4) $e(p)$ is regular, that is equivalent in H to $\neg\alpha$ for some α, but neither valid, nor inconsistent. Substitutions of $e(p)$ can be valid or inconsistent but cannot belong to $CL \setminus H$. So, we have $e \Vdash_H \neg\neg p \to p$ but $e \nVdash_H \neg\neg p$. Again, the identification of all such worlds e is possible.

All remaining worlds can be put into the five class. The received Kripke model $\mathcal{P} = \langle P, \leq, \Vdash \rangle$ is given in Figure 2. where instead of the forcing relation we label nodes with forced formulas. It means, in particular, that the variable p is forced at one node only – the one labeled with p.

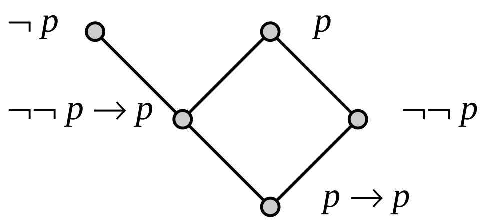

Figure 2.

We also get a mapping $M : E_1 \to P$ and it is clear that the mapping fulfills the conditions (i) and (iii) from the definition of a $p-$morphism. There is a little problem with the condition (ii). To show this we should consider all possibilities. Almost all of them are trivial. For instance, if $M(e) = \neg\neg p \to p$, then $e(p)$ is regular, but neither valid, nor inconsistent. Hence $f(e(p)) \in INT$ and $\neg g(e(p)) \in INT$ for some substitutions f and g. Thus, we get $M(f \circ e) = p$ and $M(g \circ e) = \neg p$. Similarly, if $M(e) = \neg\neg p$, then $e(p)$ is classically valid and hence there is a substitution f such that $f(e(p)) \in H$. Thus, $M(f \circ e) = p$ and $f \circ e \succeq e$.

But let us consider the worst case with $M(e) = p \to p$. If we take $y = p$ or $y = \neg p$, there will be no problem to find $f \succeq e$ such that $M(f) = y$ as $e(p)$ is neither valid, nor inconsistent. Let $y = \neg\neg p$. If $\neg\neg f(e(p)) \in H$ yields $f(e(p)) \in H$ for every substitution f, then we get $\neg\neg e(p) \to e(p) \in H$ by (F3) and hence $e(p)$ is regular which contradicts our assumptions. Thus, there exists a substitution f such that $\neg\neg f(e(p)) \in H$ and $f(e(p)) \notin H$ and it means that $M(f \circ e) = y$. There remains to consider the possibility that $y = \neg\neg p \to p$. If $\neg\neg f(e(p)) \to f(e(p)) \in H$ yields $f(e(\neg p)) \vee f(e(p)) \in H$ for every substitution f, then $(\neg\neg e(p) \to e(p)) \to \neg e(p) \vee e(p) \in H$ by (F3) and hence $\neg e(p) \vee \neg\neg e(p) \in H$ by Scott's law. Then, by (F2), either $\neg e(p) \in H$ or $\neg\neg e(p) \in H$ which is impossible as $M(e) = p \to p$. Thus there must exist a substitution f such that $\neg\neg f(e(p)) \to f(e(p)) \in H$ and $\neg f(e(p)) \vee f(e(p)) \notin H$ which gives $M(f \circ e) = y$.

We conclude that $\mathcal{E}_1(H)$ reduces to a much simpler (finite) model $\mathcal{P}$, the same model for each logic H. Note that the mapping M might depend on the logic H. It might happen that $e(p)$ were valid in one logic and invalid in another, hence $M(e)$ would be p in one logic and something else in the other.

5. The fragment $\{\to, \wedge, \neg\}$. Simple examples show that the situation become more complicated when we deal with arbitrary (not only monadic) formulas. In particular, let us note that the product of two copies of the p−morphism used for $\mathcal{E}_1(H)$ is not a p−morphism from $\mathcal{E}_2(H)$ onto $\mathcal{P}^2$. Indeed, let e be a substitution such that $e(p_1) = e(p_2) = p$. We have $f(p_1) = f(p_2)$ for every $f \succeq e$. Thus, the submodel of $\mathcal{E}_2(H)$ determined by e (that is the submodel on $\{f : e \preceq f\}$) is mapped by the product of the p−morphisms onto a proper submodel of $\mathcal{P}^2$ (isomorphic with $\mathcal{P}$), not onto $\mathcal{P}^2$. On the other hand, the substitution e is identified in the product with the identity substitution. This would not be possible if $\mathcal{E}_2(H)$ and $\mathcal{P}^2$ were p−morphic.

One sees that to describe (in terms of $p-$morphisms) $\mathcal{P}_2(H)$ one needs not only $\mathcal{P}^2$, but its certain submodels as well. Submodels occur as substitutions may introduce certain dependencies between variables. It is not clear, however, how to construct a model from these submodels of $\mathcal{P}^2$. A fresh start is required.

Let us note that the model $\mathcal{P}$ coincides with the frame of all (up to equivalence) consistent monadic formulas in $\{\rightarrow, \neg\}$ ordered by the (inverse of the) usual Lindenbaum order relation. Though connections between the model $\mathcal{E}_n(H)$ and formulas in $\{\rightarrow, \neg\}$ are not clear, let us try to develop this idea and use a similar labeling of frame nodes in a general case. Thus, we correlate with each substitution e, the set of all $\{\rightarrow, \neg\}$-formulas (built up from the variables $\{p_1, \ldots, p_n\}$) which are satisfied by e, i. e. formulas α such that $e(\alpha) \in H$. In fact, we correlate a single $\{\rightarrow, \neg, \wedge\}$-formula with each e as the following theorem is well-known (see Diego [1]:)

Theorem 9. *There is only finitely many (up to equivalence) formulas in the intuitionistic fragment* $\{\rightarrow, \neg, \wedge\}$.

Now, let us prove – a result by M. Szatkowski [11] – that Medvedev's logic ML is a conservative extension of the inuitionistic logic in the fragment $\{\rightarrow, \neg, \wedge\}$. By the same, ML is also a conservative extension, in $\{\rightarrow, \neg, \wedge\}$, of any structurally complete intermediate logic with the disjunction property.

Theorem 10. $\alpha \in ML \Leftrightarrow \alpha \in INT$, *for every formula* α *in* $\{\rightarrow, \neg, \wedge\}$.

Proof. One implication is clear as $INT \subseteq ML$. Our proof of $(\Rightarrow)$ is inductive with respect to the number of variables in α. If α is monadic, the implication holds. We assume the implication holds for formulas with less $(<)$ variables than α, assume that $\alpha \notin INT$, and show $\alpha \notin ML$.

Note that each formula in $\{\rightarrow, \neg, \wedge\}$ is equivalent to a conjunction of formulas in $\{\rightarrow, \neg\}$. So, we may assume that α is $\alpha_1 \wedge \ldots \wedge \alpha_n \rightarrow \beta$ for some

$\alpha_1, \ldots, \alpha_n, \beta$ in $\{\to, \neg\}$ where β is either a variable, or a negation of some formula. If β is a negation, then α is not classically valid (as $\alpha \notin \mathit{INT}$) and hence $\alpha \notin \mathit{ML}$. Thus, we can assume that β is a variable.

Let us extend $\{\alpha_1, \ldots, \alpha_n\}$ to a maximal set X of $\{\to, \neg\}$-formulas built up from the variables in α (see Theorem 9) such that

$$\bigwedge X \to \beta \notin \mathit{INT}.$$

Let Y be the set of all $\{\to, \neg\}$-formulas, built up from the variables in α, which do not belong to X. We have $\bigwedge X \to \gamma \notin X$ and $\gamma \to \beta \in X$ for every $\gamma \in Y$. Let X_0 and Y_0 be the subsets of X and Y, correspondingly, containing those formulas which do not contain the variable β.

a) If $\neg\neg\beta \notin X$, then $\neg\neg\beta \to \beta \in X$ and hence $\bigwedge X \to \neg\neg\beta \notin \mathit{INT}$. It also means that $\bigwedge X \to \neg\neg\beta$ is not classically valid and this clearly suffices for $\alpha \notin \mathit{ML}$. Thus, we may assume that $\neg\neg\beta \in X$.

b) Let $\beta \to \gamma \in X$ for some $\gamma \in Y_0$. Then $\beta \equiv \gamma \in X$ and hence the set X is equivalent (in INT) to $X_0 \cup \{\beta \equiv \gamma\}$. We also get $\bigwedge X_0 \to \gamma \notin \mathit{INT}$ and hence, by inductive hypothesis, we obtain $\bigwedge X_0 \to \gamma \notin \mathit{ML}$. This, in turn, gives us $\bigwedge X \to \beta \notin \mathit{ML}$ which completes our argument. Thus, we may also assume that $\beta \to \gamma \in Y$ and hence $(\beta \to \gamma) \equiv \gamma \in X$ for every $\gamma \in Y_0$.

c) Let p_i be a variable in α different from β. If $p_i \in X$, we can easily reduce X to an equivalent set $Z \cup \{p_i\}$ where Z does not contain p. Then using our inductive hypothesis we obtain $\bigwedge Z \to \beta \notin \mathit{ML}$ and hence $\bigwedge X \to \beta \notin \mathit{ML}$ which suffices for $\alpha \notin \mathit{ML}$. We can argue in the same way if $\neg p_i \in X$.

We conclude that the set X is (equivalent to)

$$X_0 \cup \{\neg\neg\beta\} \cup \{\beta \to \gamma) \to \beta : \gamma \in Y_0\}$$

where X_0 contains neither p_i nor $\neg p_i$ for any variable p.

If $\bigwedge X_0 \to \bigvee Y_0 \in \mathit{ML}$, then $\bigwedge X_0 \to \gamma \in \mathit{ML}$ for some $\gamma \in Y_0$, by Theorem 5 and the disjunction property in ML. This, however, contradicts our inductive hypothesis. Thus, $\bigwedge X_0 \to \bigvee Y_0 \notin ML$ and hence, by structural completeness of ML, we get $e(X_0) \subseteq ML$ and $e(Y_0) \cap ML = \emptyset$ for some substitution e. We may clearly assume that the substitution is not defined on the variable β. Next, let us consider the following possibilities:

(i) There is a formula $\gamma_0 \in Y_0$ such that $\bigvee Y_0 \equiv \gamma_0 \in X_0$. Then the set X is equivalent (in intuitionistic logic) to $X_0 \cup \{\neg\neg\beta, (\beta \to \gamma_0) \to \beta\}$. Let p_i be any variable which does not occur in $e(\gamma_0)$. If

$$((p_i \to e(\gamma_0)) \to p_i) \to (\neg\neg p_i \to p_i) \notin ML$$

we can extend the substitution e with the condition $e(\beta) = p_i$ to (a substitution such that) get $\bigwedge e(X) \to e(\beta) \notin ML$. Hence $\alpha \notin ML$. So, suppose that

$$(*) \qquad ((p_i \to e(\gamma_0)) \to p_i) \to (\neg\neg p_i \to p_i) \in ML$$

and take $\phi = (p_i \to e(\gamma_0)) \vee p_i$. Since ϕ is classically vaid, we get $\neg\neg\phi \in ML$. Let us note that $e(\gamma_0) \to \phi \in INT$ and $(\phi \to e(\gamma_0))) \to e(\gamma_0) \in INT$. Hence $(\phi \to e(\gamma_0))) \to \phi \in INT$ which gives $\phi \in ML$ by $(*)$. Then, by the disjunction property in ML, we get either $e(\gamma_0) \in ML$, or $p_i \to e(\gamma_0) \in ML$. Since p_i does not occur in $e(\gamma_0)$, in both cases $e(\gamma_0) \in ML$. This is, however, impossible as $\gamma_0 \in Y_0$ and $e(Y_0)$ is disjoint with ML.

(ii) Let $\{\gamma_1, \dots, \gamma_n\}$, with $n > 1$, be a minimal subset of Y_0 such that for every $\gamma \in Y_0$ there is an integer $i \le n$ with $\gamma \to \gamma_i \in X_0$. Clearly,

$$\bigvee Y_0 \equiv (\gamma_1 \vee \cdots \vee \gamma_n) \in X_0 \quad \text{and} \quad (\gamma_i \to \gamma_j) \equiv \gamma_j \in X_0 \quad \text{if } i \neq j.$$

Let us take $e(\beta) = e(\gamma_1) \vee \ldots \vee e(\gamma_n)$. Note that $e(\gamma_1 \vee \ldots \vee \gamma_n)$ is classically valid as Y_0 contains all variables (except of β) occurring in α and their negations. Hence $e(\neg\neg\beta) \in INT$.

Let us show that $(e(\beta) \to e(\gamma)) \to e(\gamma) \in ML$ for every $\gamma \in Y_0$. Since $e(X_0) \subseteq ML$, it suffices to show that

$$((\gamma_1 \vee \ldots \vee \gamma_n) \to \gamma) \to (\gamma_1 \vee \cdots \vee \gamma_n) \in X_0$$

for every $\gamma \in Y_0$. Let $\gamma \to \gamma_i \in X_0$ for an $i \leq n$. Then

$$((\gamma_1 \vee \ldots \vee \gamma_n) \to \gamma) \to ((\gamma_1 \vee \cdots \vee \gamma_n) \to \gamma_i) \in X_0$$

Since $(\gamma_j \to \gamma_i) \to \gamma_i \in X_0$ for any $j \neq i$ (and $n > 1$,) we get

$$((\gamma_1 \vee \cdots \vee \gamma_n) \to \gamma) \to \gamma_i \in X_0$$

Thus, we have shown that $e(X) \subseteq ML$. Since $e(\beta) \notin ML$ by the disjunction property in ML, we get $e(\alpha) \notin ML$ which completes our argument. ∎

Let P_n be the set of all (up to equivalence) consistent in intuitionistic logic (and hence in any intermediate logic) $\{\to, \neg, \wedge\}$-formulas built up from the variables $p_1, \ldots, p_n$. We define a Kripke model $\mathcal{P}_n = \langle P_n, \leq, \Vdash \rangle$ taking $\beta \leq \alpha$ iff $\alpha \to \beta \in INT$, and assuming that $\alpha \Vdash p_i$ iff $p_i \leq \alpha$. It is clear that $\mathcal{P}_n$ is an intuitionistic model. Next, let $M_n(e)$, for any $e \in E_n$, be the greatest in P_n formula α such that $e(\alpha) \in H$. Note that the definition of $\mathcal{P}_n$ does not depend on the logic H but that of $M_n(e)$ does. As it has been shown above $M_1 : E_1 \to P_1$ is a $p-$morphism. Unfortunately, $M_n : E_n \to P_n$ is not a $p-$morphism if $n \geq 2$. Thus, our approach fails.

The failure of this can be shown even for $n = 2$. Namely, let us take a substitution e such that $e(p_1) = p_1$ and $e(p_2) = \neg\neg p_1 \vee (\neg\neg p_1 \to p_1)$. Then $M_2(e)$ is a conjunction of $\neg\neg p_2$ and all formulas (up to equivalence) of the form $(\alpha \to p_2) \to \alpha$ for each invalid α in $\{\to, \neg\}$ built up from the variable p_1 only. We clearly get $M_2(e) \leq p_2$. But if $f(e(p_2)) \in H$, then $\neg\neg f(e(p_1)) \in H$ or $\neg\neg f(e(p_1)) \to f(e(p_1)) \in H$. Thus, $M_2(f \circ e) = p_2$ holds for none f.

Clearly, it is not possible that the models $\mathcal{P}_n$ and $\mathcal{E}_n(H)$ were equivalent for $n > 1$. One of them is finite and the other is a model for **ML**. Since the positive (without negation) fragment of **ML** coincides with that of intuitionistic logic, its model cannot be finite (if at least two variables are allowed).

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Institute of Mathematics
Silesian University
Bankowa 14
Katowice 40-007

e-mail: `wojtylak@ux2.math.us.edu.pl`