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EVERY FREE BIRESIDUATED LATTICE IS SEMISIMPLE

A b s t r a c t. In this paper, we prove the semisimplicity of free biresiduated lattices, more precisely, integral residuated lattices. In [4], authors show that variety of residuated lattices, more precisely, commutative integral residuated lattices, is generated by its finite simple members. The result is obtained by showing that every *free* residuated lattice is semisimple and then showing that every variety generated by a simple residuated lattice is generated by a set of finite simple residuated lattices. The proof of the former is based on Grišin's idea in [2]. We show that their proof of the semisimplicity works well also for free biresiduated lattices.

1. Introduction

In [4], authors show that variety of residuated lattices, more precisely, commutative integral residuated lattices, is generated by its finite simple members. The result is obtained by first showing that every *free* residuated

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lattice is semisimple and then showing that every variety generated by a simple residuated lattice is generated by a set of finite simple residuated lattices. To show the former, based on Grišin's idea in [2] authors introduced a sequent system SFL_{ew}^{+} such that

1. algebras for SFL_{ew}^{+} are exactly equal to residuated lattices,
2. cut elimination theorem holds for SFL_{ew}^{+}.

Then, using proof-theoretic properties of SFL_{ew}^{+}, the semisimplicity of free residuated lattices is obtained.

In this paper, we show that their proof of the semisimplicity works well also for integral (not necessarily commutative) residuated lattices, which we call simply biresiduated lattices in the present paper. We assume a familiarity with the paper [4].

2. Biresiduated lattices

We give a precise definition of biresiduated lattices. An algebraic structure $\mathbf{A} = \langle A, \cdot, \backslash, /, \wedge, \vee, 0, 1\rangle$ is called a biresiduated lattice, if it satisfies the following conditions:

(1)$\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice,
(2)$\langle A, \cdot, 1\rangle$ is a monoid,
(3) for any $x, y, z \in A, x \cdot y \leq z \Leftrightarrow y \leq x\backslash z \Leftrightarrow x \leq z/y$.

Operations $\backslash$ and $/$ are called *left* and *right* residuation, respectively. If we assume the commutativity of the monoid operation $\cdot$, then these two residuations become identical and hence the algebra **A** becomes a (commutative, integral) residuated lattices. When we assume only the existence of the left (right) residuation, algebras thus obtained are called left (right, respectively) residuated lattice. In the rest of the paper, we write xy instead of $x \cdot y$. The following characterization of congruence filters is shown in [1].

Proposition 1 ([1]). *A nonempty set F of a biresiduated lattice* **A** *is a congruence filter* (*called simply, a filter*) *if and only if F satisfies the following:* (1) *F is closed under $\cdot$,* (2) *F is upward closed with respect to the lattice order of* **A**, *and* (3) *$a \in F$ implies $\lambda_c(a)$, $\rho_c(a) \in F$ for any $c \in A$, where $\lambda_c(a) = c\backslash ac$ and $\rho_c(a) = ca/c$.*

Using this proposition, we can have the following representation of the filter generated by a given nonempty set S of a biresiduated lattice **A**. Let

μ_u be either λ_u or ρ_u. Next, we define the subset $\Gamma(S), \Pi(S)$ of A as follows.

$$\Gamma(S) = \{\mu_{u_1} \cdots \mu_{u_n}(s) : n \in \omega, u_i \in A, s \in S\}$$

$$\Pi(S) = \{s_1 s_2 \cdots s_n : n \in \omega, s_i \in S\} \cup \{1\}$$

Let H_S be $\{x | z \leq x \ \ for \ \ some \ \ z \in \Pi\Gamma(S)\}$. Then we have the following.

Lemma 2 ([3]). *For each nonempty set S of a biresiduated lattice $\mathbf{A}$, the filter generated by S is equal to H_S.*

When S is a singlton set $\{a\}$, we will denote H_a instead of $H_{\{a\}}$ and also denote $\Gamma_a, \Pi\Gamma_a$ instead of $\Gamma_{\{a\}}, \Pi\Gamma_{\{a\}}$. Recall that a biresiduated lattice $\mathbf{A}$ is *semisimple* if it has a subdirect representation with simple factors. Let Φ be the set of all maximal filters of a biresiduated lattice $\mathbf{A}$. Define the radical $Rad_{\mathbf{A}}$ of $\mathbf{A}$ by $Rad_{\mathbf{A}} = \bigcap_{F \in \Phi} F$. Then, the following can be easily shown.

Lemma 3. *For any biresiduated lattices, $\mathbf{A}$ is semisimple if and only if $Rad_{\mathbf{A}} = \{1\}$.*

Corresponding to Theorem 2.3 of [4], we can show the following result, which, however gives only a necessary condition for $x \in \mathbf{A}$ to be a member of $Rad_{\mathbf{A}}$.

Proposition 4. *Let $\mathbf{A}$ be a biresiduated lattice. If an element $x \in \mathbf{A}$ is in $Rad_{\mathbf{A}}$ then for any $n \geq 1$ there exist $d_1, \cdots, d_t \in \Gamma(\neg x^n)$ such that $d_1 \cdots d_t = 0$, where $\neg x$ stands for either $x \backslash 0$ or $0/x$.*

Proof. It is easy to see that it is enough to show that if $x \in \mathbf{A}$ is in $Rad_{\mathbf{A}}$ then $0 \in \Pi\Gamma(\neg x^n)$ for any $n \geq 1$. Taking the contraposition, suppose that there exists $n \geq 1$ such that $0 \notin \Pi\Gamma(\neg x^n)$. This means that the filter H generated by $\{\neg x^n\}$ is proper. By Zorn's lemma, there exists a maximal filter G including H. If $x \in G$ then $x^n \in G$. On the other hand, $\neg x^n \in G$ since G includes H. This contradicts the fact that G is proper. Thus $x \notin G$. Hence $x \notin Rad_{\mathbf{A}}$.

3. Semisimplicity of free biresiduated lattices

In this section we show that every free biresiduated lattice is semisimple, using the sequent system FL_w^+ introduceed below. Our proof proceeds similarly to Grišin[2] and [4].

Similarly to the sequent system SFL_{ew} introduced in [4], we introduce a sequent system, which we call FL_w^+ as follows. A sequent is of the form $\Gamma \to \alpha$ where Γ is a finite sequence of formulas:

1. initial sequents

   (1) $\Gamma, p, \Delta \to p$ where p is a propositional variable,
   (2) $\Gamma, 0, \Delta \to \alpha$.

2. rules of inference

$$\frac{\Gamma \to \alpha \quad \Delta, \alpha, \Sigma \to \theta}{\Delta, \Gamma, \Sigma \to \theta} \; (cut)$$

$$\frac{\Gamma, \alpha, \Delta \to \theta \quad \Gamma, \beta, \Delta \to \theta}{\Gamma, \alpha \vee \beta, \Delta \to \theta} \; (\vee \to) \quad \frac{\Gamma \to \alpha}{\Gamma \to \alpha \vee \beta} \; (\to \vee 1) \quad \frac{\Gamma \to \beta}{\Gamma \to \alpha \vee \beta} \; (\to \vee 2)$$

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} \; (\to \wedge) \quad \frac{\Gamma, \alpha, \Delta \to \theta}{\Gamma, \alpha \wedge \beta, \Delta \to \theta} \; (\wedge 1 \to) \quad \frac{\Gamma, \beta, \Delta \to \theta}{\Gamma, \alpha \wedge \beta, \Delta \to \theta} \; (\wedge 2 \to)$$

$$\frac{\Gamma \to \alpha \quad \Delta, \beta, \Sigma \to \theta}{\Delta, \Gamma, \alpha \backslash \beta, \Sigma \to \theta} \; (\backslash \to) \qquad \frac{\alpha, \Gamma \to \beta}{\Gamma \to \alpha \backslash \beta} \; (\to \backslash)$$

$$\frac{\Gamma \to \alpha \quad \Delta, \beta, \Sigma \to \theta}{\Delta, \beta / \alpha, \Gamma, \Sigma \to \theta} \; (/ \to) \qquad \frac{\Gamma, \alpha \to \beta}{\Gamma \to \beta / \alpha} \; (\to /)$$

$$\frac{\Gamma_1 \to \alpha_1, \cdots, \Gamma_m \to \alpha_m}{\Gamma_1, \cdots, \Gamma_m \to \alpha_1 * \cdots * \alpha_m} \; (\to *) \qquad \frac{\Gamma, \alpha_1, \cdots, \alpha_m, \Delta \to \theta}{\Gamma, \alpha_1 * \cdots * \alpha_m, \Delta \to \theta} \; (* \to)$$

Here, we assume that in each application of rules $(\to *)$ and $(* \to)$, none of α_i must be fusion formulas, i,e, formulas whose outermost logical connective is the fusion $*$. The reason why we need the system FL_w^+ will be explained by the following Lemma, which can be shown in the same way as[4].

Lemma 5.
1. Cut elimination holds for FL_w^+.
2. Free biresiduated lattices are precisely Lindenbaum algebras of FL_w^+.

Let a formula α be given. In the following, $\neg\alpha$ denotes either $\alpha\backslash 0$ or $0/\alpha$. Also $\Gamma(\alpha)$ and $\Pi(\alpha)$ denote sets of formulas which are defined in the same way as those defined in Section 2, though in the present case, $*, \backslash$ and $/$ denote logical connectives. For each formula α, let $\ell(\alpha)$ denote the length of α as a sequence of symbols. For a sequence Γ of formulas $\alpha_1, \cdots, \alpha_m$, the lengh $\ell(\Gamma)$ is defined by $\ell(\Gamma) = \ell(\alpha_1) + \cdots + \ell(\alpha_m)$. Also we need to introduce some notations for our main lemma. The expression $\{\alpha^N\}^m$ stands for the sequence $\alpha^N, \cdots, \alpha^N$ with m times, where α^N is of the form $\alpha * \cdots * \alpha$ (N times). Let β be any member of $\Pi\Gamma(\alpha)$. Then β is of the form $\gamma_1 * \cdots * \gamma_n$ where each γ_i is of the form $\mu_{\delta_{1i}} \cdots \mu_{\delta_{m_i i}}(\alpha)$ for some formulas $\delta_{1i}, \cdots, \delta_{m_i i}$. Define the rank $r(\beta)$ by $r(\beta) = \sum_{i=1}^{n} m_i$. For each nonempty multiset X of $\Pi\Gamma(\alpha)$, where X is $\beta_1, \cdots, \beta_k$, define $r(X)$ and $|X|$ by $r(X) = \sum_{j=1}^{k} r(\beta_j)$ and $|X| = k$, respectively.

To show that any free biresiduated lattice A is semisimple, Lemma 3 says that it is enough to show that the radical $Rad_{\mathbf{A}}$ of any Lindenbaum algebra of FL_w^+ is equal to $\{1\}$, where 1 is the greatest element of a given Lindenbaum algebra. Since the element 1 consists of all provable formulas. By Lemma 5, this follow from the following lemma.

Lemma 6 (Main Lemma). *Suppose that a formula α is not provable in FL_w^+ and that $N \geq \ell(\alpha)$. For any sequent $\Gamma_1, \cdots, \Gamma_{K+1} \to \sigma$ such that $\ell(\Gamma, \sigma) \leq \ell(\alpha)$, where Γ is equal to $\Gamma_1, \cdots, \Gamma_{K+1}$ and any nonempty multisets $X_1, \cdots, X_K$ of $\Gamma(\neg\alpha^N)$, if $\Gamma_1, X_1, \cdots, X_k, \Gamma_{K+1} \to \sigma$ is provable in FL_w^+ then $\Gamma_1, \cdots .\Gamma_{K+1} \to \sigma$ is provable in FL_w^+.*

We will give a proof of Lemma 6 in the next section. Clearly, the sequent system FL_w^+ is consistent, i.e., the sequent $\to 0$ is not provable. Let $\delta_1, \cdots, \delta_n$ be arbitrary formulas in $\Gamma(\neg\alpha^N)$. Then the sequent $\delta_1, \cdots, \delta_n \to 0$ is not provable in FL_w^+. For, otherwise, $\to 0$ is provable in FL_w^+ by Lemma 6, which is a contradiction. Recall here that any formula β in $\Pi\Gamma(\neg\alpha^N)$ is of the form $\delta_1 * \cdots * \delta_n$ for some $\delta_1, \cdots, \delta_n \in \Gamma(\neg\alpha^N)$. Thus, we have the following.

Proposition 7. *Let α be any formula which is not provable in FL_w^+. Then there exists $N \geq 1$ such that for any $\delta_1, \cdots, \delta_n \in \Gamma(\neg\alpha^N)$, $\delta_1, \cdots, \delta_n \to 0$ is not provable in FL_w^+. Thus, for any β in $\Pi\Gamma(\neg\alpha^N)$, $\beta \to 0$ is neither provable in FL_w^+.*

In the term of Lindenbaum algebra $\mathbf{A}$ of FL_w^+ , the above proposition says that if $[\alpha] \neq [1]$ in $\mathbf{A}$ then $[0] \notin \Pi\Gamma([\neg\alpha^N])$ for some $N \geq 1$, where $[\gamma]$

denotes the equivalence class, to which a given formula γ belongs. Thus, using Proposition 4, we have the following main theorem.

Theorem 8 (Main Theorem). *Every free biresiduated lattice is semisimple.*

The proof of our main lemma work well also in the case of left (right) residuated lattices. Thus, we have the following corollary.

Corollary 9. *Every free left (right) residuated lattice is semisimple.*

4. A proof of main lemma

The proof will be given by using double induction on the total rank of multisets X_i's and $\ell(\Gamma, \sigma)$, where Γ is $\Gamma_1, \Gamma_2, \cdots \Gamma_{k+1}$. We note first that without loss of generality, it is enough to consider the case for $\neg\alpha$ is $\alpha\backslash 0$. Therefore our proof will be given by using double induction on $\langle \sum_{i=1}^{K} r(X_i), \ell(\Gamma, \sigma)\rangle$. When $\sum_{i=1}^{K} r(X_i) = 0$, i.e., every X_i consists only of the formula $\alpha^N\backslash 0$, the proof goes essentially the same as one in [4], as shown in the following and therefore our lemma can be regarded as an extension of the one for commutative case.

(1) Suppose that the given sequent $\Gamma_1, X_1, \cdots, X_K, \Gamma_{K+1} \to \sigma$ is an initial sequent. In this case, either σ is a propositional variable which occurs also in some Γ_i, or 0 occurs in some Γ_i. It is clear that $\Gamma_1, \cdots .\Gamma_{K+1} \to \sigma$ is provable in either case.

(2) Suppose next that the given sequent $\Gamma_1, X_1, \cdots, X_k, \Gamma_{K+1} \to \sigma$ is the lower sequent of an inference rule I. By Lemma 5, this sequent has a cut-free proof P. We need to consider two possibilities. The first case is that the principal formula of I is either in some Γ_i or in σ, and the second case is that the principal formula of I is one of an element in X_i.

Consider the first case. Since the proof P is a cut-free proof, it is easily seen that the length of each of the upper sequent of a given inference is smaller than that of the lower sequent by subformula property. Thus, we can use the hypotheses of induction and apply the same inference rule I, we conclude that $\Gamma_1, \cdots, \Gamma_{K+1} \to \sigma$ is provable.

Consider the second case. The principal formula of the inference rule I is one of the element γ in X_i. In this case, there two possibilities that (i) $r(\gamma) = 0$ or (ii) $r(\gamma) > 0$.

(i) In this case the given sequent of the form that

$$\Gamma_1, X_1, \cdots, X_i^l, \gamma, X_i^r, \cdots \Gamma_{K+1},$$

where X_i^l, γ, X_i^r is equal to X_i. So the inference rule I is of the form that;

$$\frac{\Pi_2, X_i^l \to \alpha^N \quad \Pi_1, 0, X_i^r, \Pi_3 \to \sigma}{\Pi_1, \Pi_2, X_i^l, \alpha^N \backslash 0, X_i^r, \Pi_3 \to \sigma}$$

or

$$\frac{X_i^{l_1} \to \alpha^N \quad \Pi_1, \Pi_2, X_i^{l_2}, 0, X_i^r, \Pi_3 \to \sigma}{\Pi_1, \Pi_2, X_i^l, \alpha^N \backslash 0, X_i^r, \Pi_3 \to \sigma}$$

where Π_1, Π_2 is equal to $\Gamma_1, X_i, \cdots, \Gamma_i$, and Π_3 is equal to $\Gamma_{i+1}, X_{i+1}, \cdots,$ Γ_{K+1}. and also X_i^l is equal to $X_i^{l_1}, X_i^{l_2}$.

Since the proof goes essentially in the same way, we consider only the first case. Consider the proof R of the left upper sequent $\Pi_2, X_i^l \to \alpha^N$. We will trace back branches of R, which consists of sequents having α^N in the conclusion, to the places where this α^N is introduced. Note that α^N is introduced at one place in each branch of R. It is easy to see that each α^N is introduced either as an initial sequent, or by $(\to *)$ rule. We will show that any α^N is introduced only as an initial sequent. Suppose that at least one place , α^N is introduced by $(\to *)$, whose lower sequent is of the form $\Delta \to \alpha^N$. We assume here that α is of the from $D_1 * \cdots * D_w$ and none of D_j are fusion-formulas. Then, I must have $N \cdot w$ upper sequents, each of which is of the form $\Delta_i \to D_{n_i}$, where $1 \leq n_i \leq w$ and the list $\Delta_1, \cdots, \Delta_{N \cdot w}$ is equal to Δ. For each j such that $1 \leq j \leq w$, there exists exactly N sequents with the conclusion D_j among those sequents. We enumerate them as $S_1^j, \cdots, S_N^j$. Next, for each h such that $1 \leq h \leq N$, take $S_h^1, \cdots, S_h^w$ for upper sequent and apply $(\to *)$ rule to them. Then we can have a sequent of the form $\Sigma_h \to \alpha$ for $1 \leq h \leq N$ and the list of $\Sigma_1, \cdots, \Sigma_N$ is equal to Δ. Now $\ell(\Delta) \leq \ell(\Pi_2) \leq \ell(\Gamma, \sigma) \leq \ell(\alpha) < N$. If we assume that $\Sigma_i > 0$ for any i such that $1 \leq i \leq N$ then $\ell(\Delta) \geq N$, which is a contradiction. Therefore, Σ_i must be empty for some i. But this means that $\to \alpha$ is provable. This contradicts the assumption that α is unprovable. Hence, we conclude that at any place α^N is introduced by initial sequent of the form $\Pi, 0, \Lambda \to \alpha^N$.

We will modify the proof R of $\Pi_2, \alpha^N \backslash 0 \to \alpha^N$ as follows. We replace every sequent $\Theta \to \alpha^N$ in a branch which we have traced in R, including initial sequent of the form $\Pi, 0, \Lambda \to \alpha^N$ mentioned above, by the sequent

$\Pi_1, \Theta, \Pi_3 \rightarrow \sigma$, which is equal to $\Gamma_1, X_1, \cdots, X_i^l, \Gamma_{i+1}, \cdots, \Gamma_{K+1} \rightarrow \sigma$. Then we will have the proof whose end sequent is $\Pi_1, \Pi_2, X_i^l, \Pi_3 \rightarrow \sigma$. This sequent has the smaller length than the original sequent $\Gamma_1, X_1, \cdots, \Gamma_{K+1} \rightarrow \sigma$. Hence, by hypothesis of induction, we conclude that $\Gamma_1, \cdots, \Gamma_{K+1} \rightarrow \sigma$ is provable. We note that the above proof (i) works well also for the case when $\sum_{i=1}^{K} r(X_i) = 0$, i.e., each X_i consists only of the formula $\alpha^N \backslash 0$. Hence, the above proof assures us that the base step of the induction of our proof holds.

(ii) The rank $r(\gamma)$ is greater than 0. In this case the principal formula γ is of the form $\mu_{\beta_1} \cdots \mu_{\beta_m}(\neg\alpha^N)$ with $m > 0$, where μ is either ρ or λ operator and $\beta_1, \cdots, \beta_m$ are formulas. There are two possible cases. The first case is that μ_{β_1} is $\rho_{\beta 1}$, and the second case is that μ_{β_1} is λ_{β_1}. We give here only a proof of the first case. In the following, we let χ denote $\mu_{\beta_2} \cdots \mu_{\beta_m}(\neg\alpha^N)$. Thus, γ is $\beta_1 \backslash \chi * \beta_1$. In this case, the inference rule I is either of the following form;

$$\frac{\Pi_2, X_i^l \rightarrow \beta_1 \quad \Pi_1, \chi * \beta_1, X_i^r, \Pi_3 \rightarrow \sigma}{\Pi_1, \Pi_2, X_i^l, \beta_1 \backslash \chi * \beta_1, X_i^r, \Pi_3 \rightarrow \sigma}$$

or

$$\frac{X_i^{l_1} \rightarrow \beta_1 \quad \Pi_1, \Pi_2, X_i^l, \chi * \beta_1, X_i^r, \Pi_3 \rightarrow \sigma}{\Pi_1, \Pi_2, X_i^l, \beta_1 \backslash \chi * \beta_1, X_i^r, \Pi_3 \rightarrow \sigma}$$

where (1) Π_1, Π_2 is equal to $\Gamma_1, X_1, \cdots, \Gamma_i$, (2) Π_3 is equal to $\Gamma_{i+1}, X_{i+1}, \cdots, \Gamma_{K+1}$, (3) X_i is equal to $X_i^l, \beta_1 \backslash \chi * \beta_1, X_i^r$ and (4) X_i^l is equal to $X_i^{l_1}, X_i^{l_2}$.

We consider the first case. In this case, the right upper sequent of the inference rule I implies the provability of $\Pi_1, \chi, \beta_1, X_i^r, \Pi_3 \rightarrow \sigma$. Taking the left upper sequent and $\Pi_1, \chi, \beta_1, X_i^r, \Pi_3 \rightarrow \sigma$, and applying the cut rule, as shown below,

$$\frac{\Pi_2, X_i^l \rightarrow \beta_1 \quad \Pi_1, \chi, \beta_1, X_i^r, \Pi_3 \rightarrow \sigma}{\Pi_1, \chi, \Pi_2, X_i^l, X_i^r, \Pi_3 \rightarrow \sigma}$$

We can see the sequent

$$\Pi_1, \chi, \Pi_2, X_i^l, X_i^r, \Pi_3 \rightarrow \sigma$$

is provable. In this sequent $\sum_{j \neq i} r(X_j) + r(\chi) < \sum_{i=1} r(X_i)$. Hence, by hypothesis of induction, we conclude that $\Gamma_1, \cdots, \Gamma_{K+1} \rightarrow \sigma$ is provable.

Similary, we can show this in the second case. This completes the proof of our lemma.

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