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SOME PROOF SYSTEMS FOR PREDICATE COMMON KNOWLEDGE LOGIC

A b s t r a c t. Common knowledge logic is a multi-modal epistemic logic with a modal operator which describes common knowledge condition. In this paper, we discuss some proof systems for the logic **CKL**, the predicate common knowledge logic characterized by the class of all Kripke frames with constant domains. Various systems for **CKL** and other related logics are surveyed by Kaneko-Nagashima-Suzuki-Tanaka, however, they did not give a proof of the completeness theorem of their main system for **CKL**. We first give a proof of their completeness theorem by an algebraic method. Next, we give a cut-free system for **CKL**, and show that the class of formulas in **CKL** which have some positive occurrences of the common knowledge operator and some occurrences of knowledge operators and quantifiers is not recursively axiomatizable and its complement is recursively axiomatizable. As a corollary, we obtain a cut-free system for the predicate modal logic **K** with the Barcan formula $\forall x \Box \phi \supset \Box \forall x \phi$.

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1. Introduction

Common knowledge logic (for n-agents) is a multi-modal logic with knowledge operators $\Box_i$ $(i \in n)$ and an operator $\Box_{\mathrm{c}}$ which describes common knowledge condition, where the formula $\Box_i\phi$ denotes that "the agent i knows ϕ" and $\Box_{\mathrm{c}}\,\phi$ denotes that "ϕ is common knowledge among the agents $0, \ldots, n-1$" (see, [3, 9, 2, 5, 4, 13]). From the meaning of $\Box_i\phi$ and $\Box_{\mathrm{c}}\,\phi$, the semantical relation in a possible world w is defined as follows:

$$w \models \Box_{\mathrm{c}}\,\phi \;\Leftrightarrow\; w \models \Box_{i_1} \cdots \Box_{i_k}\phi \text{ for all } k \in \omega,\; i_1, \ldots, i_k \in n.$$

Common knowledge logic is used in various areas, such as philosophy, artificial intelligence, and game theory, but to describe concrete problems, we need predicate common knowledge logic, instead of propositional logic. However, Wolter [13] proved that the logic **CKL**, that is, the predicate common knowledge logic characterized by the class of all Kripke frames with constant domains, cannot be recursively axiomatized [1], and therefore, any system for **CKL** has some non-recursive factors in it. On the other hand, the propositional fragment of **CKL**, that is, the propositional common knowledge logic characterized by the class of all Kripke frames, can be recursively axiomatized. Indeed, the following inference rules, which is called fixed point formalization, axiomatizes the propositional fragment of **CKL** (see [3]) [2]:

1. $\vdash \Box_{\mathrm{c}}\,\phi \supset \Box_{\mathrm{e}}(\phi \wedge \Box_{\mathrm{c}}\,\phi)$ (where $\Box_{\mathrm{e}}\,\phi = \bigwedge_{i\in n} \Box_i\phi$);
2. $\vdash \psi \supset \Box_{\mathrm{e}}(\phi \wedge \psi) \;\Rightarrow \vdash \psi \supset \Box_{\mathrm{c}}\,\phi$.

By Wolter's paper [13], fixed point formalization cannot be recursively extended for the predicate common knowledge logic **CKL**. But some systems for propositional common knowledge logic can be recursively extended for **CKL**, if they have some non-recursive factors. Indeed, Kaneko-Nagashima-Suzuki-Tanaka [6] surveyed the alternative ways of axiomatizing the predicate common knowledge logic **CKL** and other related logics, and presented a system for propositional common knowledge logic which can be recursively extended for **CKL**. However, they did not give a proof of the completeness theorem for their system. One aim of this paper is to give a proof of the

[1] Even the monadic predicates fragment without equality and function symbol is not recursively enumerable (see [13]).

[2] $\Box_{\mathrm{c}}\,\phi$ in [3] is defined by $w \models \Box_{\mathrm{c}}\,\phi \;\Leftrightarrow\; w \models \Box_{i_1} \cdots \Box_{i_k}\phi$ for all $k \geq 1,\; i_1, \ldots, i_k \in n$.

completeness theorem for their system [3] (Section 5). The proof is given by an algebraic method, which makes use of the representation theorem of modal algebras preserving infinite meets and joins.

The other aim is to give a cut-free system for **CKL** (Section 6 and Section 7). It is known that the Barcan formula $\forall x \Box \phi \supset \Box \forall x \phi$ is necessary to axiomatize modal logics characterized by classes of Kripke frames with constant domains. In [7], Pliuškevičienė presented cut-free systems for various modal logics such as **K**, **T**, **K4**, **S4**, **KB**, and **B** with the Barcan formula by indexing. We introduce a cut-free system for **CKL** by *tree sequent calculus* developed by Kashima. Then, we will show that the class of formulas in **CKL** which have some positive occurrences of the common knowledge operator and some occurrences of knowledge operators and quantifiers is not recursively axiomatizable and its complement is recursively axiomatizable. As a corollary, we obtain a cut-free system for the predicate (mono-)modal logic **K** with the Barcan formula.

2. Syntax and semantics for CKL

The language $\mathcal{L}$ consists of logical connectives $\wedge$, $\vee$, $\supset$, $\neg$, quantifiers $\forall$ and $\exists$, countable lists of variables and constant symbols, countable set of m-ary predicate symbols for each $m \in \omega$, n modal operators $\Box_i$ $(i \in n)$, and another modal operator $\Box_c$. Since we do not have any function symbols and equality sign in the language $\mathcal{L}$, a *term* in $\mathcal{L}$ is either a variable or a constant symbol, and a *closed term* in $\mathcal{L}$ is a constant symbol. We sometimes call $\Box_i$ $(i \in n)$ a *knowledge operator*, and $\Box_c$ the *common knowledge operator*. The set of formulas is defined in a standard way. We write $\Box_i \Box_j \phi$ for $\Box_i(\Box_j \phi)$, $\mathcal{K}$ for the set $\{\Box_i : i \in n\}$, and $\mathcal{K}^*$ for the set of all words of finite length of the alphabet $\mathcal{K}$. For example, $\{\kappa\phi : \kappa \in \mathcal{K}^*\}$ denotes the set $\{\Box_{i_1} \cdots \Box_{i_k} \phi : k \in \omega, i_1, \ldots, i_k \in n\}$ of formulas.

A *Kripke frame* with a constant domain for common knowledge predicate logic is a triple $\langle W, \{R_i\}_{i \in n}, D\rangle$, where W is a non-empty set, R_i is a binary relation on W for each $i \in n$, and D is a set called a *domain*. In this paper, we deal with only Kripke frames with constant domains, since the Barcan formula $\forall x \Box \phi \supset \Box \forall x \phi$ is always derivable in each of our systems.

[3] Although their system is given in Hilbert style, we discuss in Gentzen style, since we also discuss cut-free system. The proof of the completeness theorem works for both of the systems.

A *Kripke model* is a four tuple $\langle W, \{R_i\}_{i\in n}, D, I\rangle$, where $\langle W, \{R_i\}_{i\in n}, D\rangle$ is a Kripke frame with a constant domain and I is a function such that for each $w \in W$, $I(w)$ maps each predicate symbol P and each constant symbol c in the following way:

1. $P \mapsto P^{I(w)} \subset D^k$, for each predicate P of arity $k \in \omega$;
2. $c \mapsto c^{I(w)} \in D$, for each constant symbol c;
3. $c^{I(w)} = c^{I(w')}$, for any w and w' in W.

I is called an *interpretation*. An *assignment* $\mathcal{A}$ into a Kripke model $\langle W$, $\{R_i\}_{i\in n}$, D, $I\rangle$ is a function from the set of all variables to D. Define the mapping $v_{I(w),\mathcal{A}}$ from the set of all terms to D by

$$v_{I(w),\mathcal{A}}(t) = \begin{cases} \mathcal{A}(x) & \text{if } t \text{ is a variable } x \\ c^{I(w)} & \text{if } t \text{ is a constant symbol } c. \end{cases}$$

The relation $\models_{\mathcal{A}}$ between a formula ϕ and a world $w \in W$ is defined inductively as follows:

1. for each predicate P of arity $k \in \omega$ and each terms $t_1, \ldots, t_k$,
   $w \models_{\mathcal{A}} P(t_1, \ldots, t_k) \;\Leftrightarrow\; (v_{I(w),\mathcal{A}}(t_1), \ldots, v_{I(w),\mathcal{A}}(t_k)) \in P^{I(w)}$;
2. $w \models_{\mathcal{A}} \phi \wedge \psi \;\Leftrightarrow\; w \models_{\mathcal{A}} \phi$ and $w \models_{\mathcal{A}} \psi$;
3. $w \models_{\mathcal{A}} \phi \vee \psi \;\Leftrightarrow\; w \models_{\mathcal{A}} \phi$ or $w \models_{\mathcal{A}} \psi$;
4. $w \models_{\mathcal{A}} \phi \supset \psi \;\Leftrightarrow\; w \not\models_{\mathcal{A}} \phi$ or $w \models_{\mathcal{A}} \psi$;
5. $w \models_{\mathcal{A}} \neg\phi \;\Leftrightarrow\; w \not\models_{\mathcal{A}} \phi$;
6. $w \models_{\mathcal{A}} \forall x\phi \;\Leftrightarrow\; w \models_{\mathcal{A}'} \phi$ for all $\mathcal{A}'$ such that $\mathcal{A}'(y) = \mathcal{A}(y)$ for all $y \neq x$;
7. $w \models_{\mathcal{A}} \exists x\phi \;\Leftrightarrow\; w \models_{\mathcal{A}'} \phi$ for some $\mathcal{A}'$ such that $\mathcal{A}'(y) = \mathcal{A}(y)$ for all $y \neq x$;
8. $w \models_{\mathcal{A}} \Box_i\phi \;\Leftrightarrow\; w <_{R_i} w'$ implies $w' \models_{\mathcal{A}} \phi$ for all w' in W $(i \in n)$;
9. $w \models_{\mathcal{A}} \Box_{\mathrm{c}}\,\phi \;\Leftrightarrow\; w <_R w'$ implies $w' \models_{\mathcal{A}} \phi$ for all w' in W, where R is the reflexive and transitive closure of $\bigcup_{i\in n} R_i$.

It is easy to see that in any Kripke model,

$$w \models_{\mathcal{A}} \Box_{\mathrm{c}}\, \phi \;\Leftrightarrow\; w \models_{\mathcal{A}} \kappa\phi \text{ for all } \kappa \in \mathcal{K}^*,$$

that is, the formula $\Box_{\mathrm{c}}\, \phi$ is semantically equivalent to the infinite conjunction $\bigwedge\{\kappa\phi : \kappa \in \mathcal{K}^*\}$ of the set $\{\kappa\phi : \kappa \in \mathcal{K}^*\}$, although we do not have infinitary conjunction in our language.

Let ϕ be a formula and $x_1, \ldots, x_k$ the list of all free variables in ϕ. It is easy to see that if assignments $\mathcal{A}$ and $\mathcal{B}$ satisfy $\mathcal{A}(x_i) = \mathcal{B}(x_i)$ for every $i = 1, \ldots, k$, then $w \models_{\mathcal{A}} \phi \;\Leftrightarrow\; w \models_{\mathcal{B}} \phi$. So, for a closed formula ϕ, we write $w \models \phi$ for $w \models_{\mathcal{A}} \phi$. If a closed formula ϕ satisfies $w \models \phi$ in every $w \in W$ in a Kripke model $\mathcal{M}$, we write $\mathcal{M} \models \phi$. If $\mathcal{M} \models \phi$ for every $\mathcal{M}$ defined on a Kripke frame $\mathcal{F}$, we write $\mathcal{F} \models \phi$. The logic **CKL** is the set of all closed formulas ϕ that satisfy $\mathcal{F} \models \phi$ for every Kripke frame $\mathcal{F}$ with a constant domain.

3. The system CK

In this section, we introduce the system CK for **CKL**, which is based on Gentzen calculus LK (see, e.g., [1]). A sequent $\Gamma \to \Delta$ of CK is a pair of finite sets Γ and Δ of formulas. The axiom schemata are $p \to p$ and $\to \forall x \Box\phi \supset \Box\forall x\phi$ ($\Box \in \mathcal{K}$ or $\Box = \Box_{\mathrm{c}}$), which is known as the *axiom of constant domains* or the *Barcan formula*. The inference rules are the following:

set

$$\frac{\Gamma \to \Delta}{\Gamma' \to \Delta'}\ (\text{set}) \quad (\Gamma \subset \Gamma',\ \Delta \subset \Delta')$$

cut

$$\frac{\Gamma \to \Delta, \phi \quad \phi, \Lambda \to \Xi}{\Gamma, \Lambda \to \Delta, \Xi}\ (\text{cut})$$

conjunction

$$\frac{\Gamma \to \Delta, \phi \quad \Gamma \to \Delta, \psi}{\Gamma \to \Delta, \phi \wedge \psi}\ (\to \wedge) \quad \frac{\phi, \Gamma \to \Delta}{\phi \wedge \psi, \Gamma \to \Delta}\ (\wedge \to)_1 \quad \frac{\psi, \Gamma \to \Delta}{\phi \wedge \psi, \Gamma \to \Delta}\ (\wedge \to)_2$$

disjunction

$$\frac{\Gamma \to \Delta, \phi}{\Gamma \to \Delta, \phi \vee \psi}\ (\to \vee)_1 \quad \frac{\Gamma \to \Delta, \psi}{\Gamma \to \Delta, \phi \vee \psi}\ (\to \vee)_2 \quad \frac{\phi, \Gamma \to \Delta \quad \psi, \Gamma \to \Delta}{\phi \vee \psi, \Gamma \to \Delta}\ (\vee \to)$$

implication

$$\frac{\phi, \Gamma \to \Delta, \psi}{\Gamma \to \Delta, \phi \supset \psi}\ (\to\supset) \quad \frac{\Gamma \to \Delta, \phi \quad \psi, \Lambda \to \Xi}{\phi \supset \psi, \Gamma, \Lambda \to \Delta, \Xi}\ (\supset\to)$$

negation

$$\frac{\phi, \Gamma \to \Delta}{\Gamma \to \Delta, \neg\phi}\ (\to \neg) \quad \frac{\Gamma \to \Delta, \phi}{\neg\phi, \Gamma \to \Delta}\ (\neg \to)$$

forall

$$\frac{\Gamma \to \Delta, \phi[y/x]}{\Gamma \to \Delta, \forall x(\phi)}\ (\to \forall) \quad \frac{\phi[t/x], \Gamma \to \Delta}{\forall x(\phi), \Gamma \to \Delta}\ (\forall \to)$$

exists

$$\frac{\Gamma \to \Delta, \phi[t/x]}{\Gamma \to \Delta, \exists x(\phi)}\ (\to \exists) \quad \frac{\phi[y/x], \Gamma \to \Delta}{\exists x(\phi), \Gamma \to \Delta}\ (\exists \to)$$

The symbol t denotes any term which is free for x in ϕ and y denotes a variable which does not occur in any formulas in the lower sequent and free for x in ϕ.

necessitation

$$\frac{\Gamma \to \phi}{\Box_i\Gamma \to \Box_i\phi}\ (\text{nec}) \quad (\Box_i\Gamma = \{\Box_i\gamma : \gamma \in \Gamma\},\ i \in n)$$

common knowledge

$$\frac{\Gamma \to \Delta, \Box_{i_1}(\psi_1 \supset \Box_{i_2}(\psi_2 \supset \cdots \Box_{i_k}(\psi_k \supset \kappa\phi)\cdots)) \quad (\text{for all } \kappa \in \mathcal{K}^*)}{\Gamma \to \Delta, \Box_{i_1}(\psi_1 \supset \Box_{i_2}(\psi_2 \supset \cdots \Box_{i_k}(\psi_k \supset \Box_{\mathrm{c}}\,\phi)\cdots))}\ (\to \Box_{\mathrm{c}})$$

where $k \in \omega$ and $\Box_{i_1}, \ldots, \Box_{i_k} \in \mathcal{K}$. The set of upper sequents is countable.

$$\frac{\kappa\phi, \Gamma \to \Delta \quad (\text{for some } \kappa \in \mathcal{K}^*)}{\Box_{\mathrm{c}}\,\phi, \Gamma \to \Delta}\ (\Box_{\mathrm{c}} \to)$$

Note that the rule $(\to \Box_{\mathrm{c}})$ has infinitary many upper sequents, and is the only non-recursive inference rule in CK. It is easy to see that $(\to \Box_{\mathrm{c}})$ is an extension of

$$\frac{\Gamma \to \Delta, \kappa\phi \quad (\text{for all } \kappa \in \mathcal{K}^*)}{\Gamma \to \Delta, \Box_{\mathrm{c}}\,\phi}\ .$$

The rules $(\to \Box_{\mathrm{c}})$ and $(\Box_{\mathrm{c}} \to)$ mean that the formula $\Box_{\mathrm{c}}\,\phi$ is equivalent to the infinite conjunction $\bigwedge\{\kappa\phi : \kappa \in \mathcal{K}^*\}$. It should be remarked that the

Barcan formula $\forall x \Box \phi \supset \Box \forall x \phi$ is derivable without the axiom if we add the following rule to the system, instead of $(\to \forall)$ [4]:

$$\frac{\Gamma \to \Delta, \Box_{i_1}(\psi_1 \supset \Box_{i_2}(\psi_2 \supset \cdots \Box_{i_k}(\psi_k \supset \phi[y/x]) \cdots))}{\Gamma \to \Delta, \Box_{i_1}(\psi_1 \supset \Box_{i_2}(\psi_2 \supset \cdots \Box_{i_k}(\psi_k \supset \forall x \phi) \cdots))} (\to \forall)'.$$

In fact, $(\to \Box_{\mathrm{c}})$ and $(\to \forall)'$ correspond to the inference rules $(\to \Box_{\mathrm{c}})_+$ and $(\to \forall)_+$ in Section 6 (see, also, [11]).

Theorem 3.1. *(Soundness of* CK*). If a closed formula ϕ is derivable in* CK*, then $\mathcal{F} \models \phi$ for every Kripke frame $\mathcal{F}$ with a constant domain.*

Proof. Induction on the height of the derivation. $\Box$

Although a derivation $\mathcal{D}$ of CK generally includes countably many variables, for any derivation $\mathcal{D}$, we can obtain a derivation $\mathcal{D}'$ with the same conclusion as is derived in $\mathcal{D}$ and fresh variables for $\mathcal{D}'$. Let $(y_i)_{i\in\omega}$ be an enumeration of all variables, $(z_i)_{i\in\omega}$ a list of mutually distinct variables, and m a mapping such that

1. $m(t) = \begin{cases} z_i & \text{if } t = y_i \\ c & \text{if } t \text{ is a constant symbol } c; \end{cases}$
2. $m(P(t_1, \ldots, t_n)) = P(m(t_1), \ldots, m(t_n))$;
3. $m(\phi \circ \psi) = m(\phi) \circ m(\psi)$, where $\circ = \wedge, \vee, \supset$;
4. $m(\circ\phi) = \circ m(\phi)$, where $\circ = \neg, \Box_i$ $(i \in n)$, $\Box_{\mathrm{c}}$;
5. $m(\forall x \phi) = \forall m(x) m(\phi)$, $m(\exists x \phi) = \exists m(x) m(\phi)$.

For any set Γ of formulas, $m(\Gamma)$ denotes the set $\{m(\gamma) : \gamma \in \Gamma\}$, and for any derivation $\mathcal{D}$, $m(\mathcal{D})$ denotes the figure which is obtained by replacing all formulas ϕ in $\mathcal{D}$ by $m(\phi)$. It is clear that if $\mathcal{D}$ is a well-defined derivation, then so is $m(\mathcal{D})$. Now, the following theorem holds:

Theorem 3.2. *For any derivation $\mathcal{D}$ of a sequent $\Gamma \to \Delta$, there exists a derivation $\mathcal{D}'$ of $\Gamma \to \Delta$ such that there exist countably many variables which do not occur in $\mathcal{D}'$.*

Proof. Let $v_1, \ldots, v_k$ be the list of all variables in $\Gamma \to \Delta$ and $(y_i)_{i\in\omega}$ an enumeration of all variables. We may assume that $v_i = y_i$ for $i = 1, \ldots, k$. Let $(z_i)_{i\in\omega}$ be a list of variable such that $\{y_i\}_{i\in\omega} \setminus \{z_i\}_{i\in\omega}$ is countable and $v_i = y_i = z_i$ for $i = 1, \ldots, k$. Then, $m(\mathcal{D})$ is a well-defined derivation of $m(\Gamma) \to m(\Delta) = \Gamma \to \Delta$. $\Box$

[4] This is pointed out by Takashi Nagashima.

4. Some properties of modal algebras

In this section, we present some properties of modal algebras which we need to show the completeness theorem of CK.

Definition 4.1. An algebra

$$\mathcal{A} = \langle A, \wedge, \vee, -, \{\Box_i\}_{i\in\omega}, 0, 1\rangle$$

is called a *modal algebra*, if $\langle A, \wedge, \vee, -, 0, 1\rangle$ is a Boolean algebra and each $\Box_i$ $(i \in \omega)$ is a function from A to A such that:

1. $\Box_i 1 = 1$;
2. $\Box_i(x \wedge y) = \Box_i x \wedge \Box_i y$ for any $x,\ y \in A$.

We write $x \to y$ for $-x \vee y$. Now, we introduce the notion of Q-filters, which is an infinitary extension of prime filters. Note that the properties for meets and joins in the definition of prime filters are extended to infinite only for fixed infinite meets and joins:

Definition 4.2. (see [8]). Let A be a Boolean algebra and $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ a fixed pair of subsets of $\mathcal{P}(A)$ such that $\bigwedge X_n \in A$ and $\bigvee Y_n \in A$. A prime filter F is called a *Q-filter*, if the following conditions are satisfied:

1. $\forall n \in \omega(X_n \subset F \ \Rightarrow\ \bigwedge X_n \in F)$;
2. $\forall n \in \omega(\bigvee Y_n \in F \ \Rightarrow\ Y_n \cap F \neq \emptyset)$.

We write $\mathcal{F}_Q(A)$ for the set of all Q-filters of A.

We need the next lemma to obtain the dual algebra $\mathcal{F}^+$ from a given Kripke frame $\mathcal{F}$:

Lemma 4.3. *Let S be a set and $\{R_i\}_{i\in\omega}$ a set of binary relations on S. Then,*

$$\langle \mathcal{P}(S), \cap, \cup, -, \{\Box_i\}_{i\in\omega}, \emptyset, S\rangle$$

is a modal algebra, where for each $X \in \mathcal{P}(S)$,

$$-X = S \setminus X,\ \Box_i X = \{y \in S : \forall x \in S(y <_{R_i} x \ \Rightarrow\ x \in X)\}\ (i \in \omega).$$

Especially, the set $\{R_i\}_{i\in\omega}$ of binary relations on $\mathcal{F}_Q(A)$ given by

$$F <_{R_i} G \ \Leftrightarrow\ \Box_i^{-1}[F] \subset G$$

$(i \in \omega)$ defines the dual modal algebra $\mathcal{F}_Q(A)^+$ on $\mathcal{P}(\mathcal{F}_Q(A))$.

We need the next lemma to show the completeness theorem of CK:

Lemma 4.4. *([12, 10]). Let A be a modal algebra and $Q = (\{X_n\}_{n\in\omega}, \{Y_n\}_{n\in\omega})$ a pair of countable subsets of $\mathcal{P}(A)$ such that $\bigwedge X_n \in A$ and $\bigvee Y_n \in A$. Suppose the following conditions are satisfied:*

1. $\forall i \in \omega \forall n \in \omega (\bigwedge \Box_i X_n \in A,\ \bigwedge \Box_i X_n = \Box_i \bigwedge X_n)$;
2. $\forall i \in \omega \forall z \in A \forall n \in \omega \exists m \in \omega (\{\Box_i(z \to x) : x \in X_n\} = X_m)$;
3. $\forall i \in \omega \forall z \in A \forall n \in \omega \exists m \in \omega (\{\Box_i(y \to z) : y \in Y_n\} = X_m)$.

Then, for any $i \in \omega$ and $F \in \mathcal{F}_Q(A)$, if $\Box_i x \notin F$, there exists $G \in \mathcal{F}_Q(A)$ such that $\Box_i^{-1}[F] \subset G$ and $x \notin G$.

Note that the first conditions of Lemma 4.4 corresponds to the Barcan formula $\forall x \Box \phi \supset \Box \forall x \phi$. Now, we present the infinitary representation theorem of modal algebras, which is essentially equivalent to the completeness theorem of CK:

Theorem 4.5. *([12, 10]). Suppose Q satisfies the conditions in Lemma 4.4. Then the function $\eta : x \mapsto \{F \in \mathcal{F}_Q(A) : x \in F\}$ from A to $\mathcal{F}_Q(A)^+$ is a monomorphism of modal algebras which satisfies the following equalities for any $n \in \omega$:*

$$\eta(\bigwedge X_n) = \bigcap \eta[X_n],\ \ \eta(\bigvee Y_n) = \bigcup \eta[Y_n].$$

5. Kripke completeness of CK

Let Ψ be the set of all closed formulas and $\equiv$ the binary relation on Ψ such that $\phi \equiv \psi$ if and only if $\phi \supset \psi$ and $\psi \supset \phi$ are derivable in CK. We write A for the Lindenbaum algebra $\Psi/\equiv$ and $|\phi|$ for the equivalence class of ϕ. Suppose $\psi_1, \ldots, \psi_k$ is a list of formulas and $\Box_{i_1}, \ldots, \Box_{i_k} \in \mathcal{K}$. For any formula χ, we write $\Box_{i_i}\psi_1 \cdots \Box_{i_k}\psi_k \triangleright \chi$ for the formula $\Box_{i_1}(\psi_1 \supset \Box_{i_2}(\psi_2 \supset \cdots \Box_{i_k}(\psi_k \supset \chi)\cdots))$.

Lemma 5.1. *Let ϕ be a formula, $\Box \in \mathcal{K}$, and $k \in \omega$. Suppose $\psi_1, \ldots, \psi_k$ is a list of formulas and $\Box_{i_1}, \ldots, \Box_{i_k} \in \mathcal{K}$. In the Lindenbaum algebra A,*

$$\bigwedge\{|\Box_{i_i}\psi_1 \cdots \Box_{i_k}\psi_k \triangleright \kappa\phi| : \kappa \in \mathcal{K}^*\} = |\Box_{i_i}\psi_1 \cdots \Box_{i_k}\psi_k \triangleright \Box_{\mathrm{c}}\,\phi|,$$

$$\bigwedge\{\Box|\Box_{i_i}\psi_1 \cdots \Box_{i_k}\psi_k \triangleright \kappa\phi| : \kappa \in \mathcal{K}^*\} = \Box|\Box_{i_i}\psi_1 \cdots \Box_{i_k}\psi_k \triangleright \Box_{\mathrm{c}}\,\phi|.$$

Especially, $\bigwedge\{|\kappa\phi| : \kappa \in \mathcal{K}^*\} = |\Box_{\mathrm{c}}\,\phi|$ and $\bigwedge\{\Box|\kappa\phi| : \kappa \in \mathcal{K}^*\} = \Box|\Box_{\mathrm{c}}\,\phi|$.

Proof. We only show the first one. By ($\Box_{\rm c} \to$), ($\supset\to$), and (nec), the sequent

$$\Box_{i_1}(\psi_1 \supset \cdots \Box_{i_k}(\psi_k \supset \Box_{\rm c}\,\phi)\cdots) \to \Box_{i_1}(\psi_1 \supset \cdots \Box_{i_k}(\psi_k \supset \kappa\phi)\cdots)$$

is derivable for any $\kappa \in \mathcal{K}^*$. Hence, $|\Box_{i_1}(\psi_1 \supset \cdots \Box_{i_k}(\psi_k \supset \Box_{\rm c}\,\phi)\cdots)|$ is a lower bound of the set $\{|\Box_{i_i}\psi_1 \cdots \Box_{i_k}\psi_k \rhd \kappa\phi| : \kappa \in \mathcal{K}^*\}$. If $|\chi|$ is another lower bound of this set, the sequent

$$\chi \to \Box_{i_1}(\psi_1 \supset \cdots \Box_{i_k}(\psi_k \supset \kappa\phi)\cdots)$$

is derivable for any $\kappa \in \mathcal{K}^*$. Then, by ($\to \Box_{\rm c}$),

$$\chi \to \Box_{i_1}(\psi_1 \supset \cdots \Box_{i_k}(\psi_k \supset \Box_{\rm c}\,\phi)\cdots)$$

is derivable. Hence $|\chi| \leq |\Box_{i_1}(\psi_1 \supset \cdots \Box_{i_k}(\psi_k \supset \Box_{\rm c}\,\phi)\cdots)|$. $\Box$

Lemma 5.2. *In the Lindenbaum algebra A, each of the following holds:*

1. $|\forall x\phi(x)| = \bigwedge\{|\phi(t)| : t \textit{ is closed}\}$;
2. $|\exists x\phi(x)| = \bigvee\{|\phi(t)| : t \textit{ is closed}\}$;
3. $\Box_i|\forall x(\psi \supset \phi(x))| = \bigwedge\{\Box_i(|\psi| \to |\phi(t)|) : t \textit{ is closed}\}$ $(i \in n)$;
4. $\Box_i|\forall x(\phi(x) \supset \psi)| = \bigwedge\{\Box_i(|\phi(t)| \to |\psi|) : t \textit{ is closed}\}$ $(i \in n)$.

Proof. 1. It is obvious that $|\forall x\phi| \leq |\phi[t/x]|$ for any t. Suppose $|\psi|$ is another lower bound of the set $\{|\phi(t)| : t \text{ is closed}\}$. Then, $\psi \to \phi[c/x]$ is derivable for a constant symbol c which does not occur in ψ and ϕ. By Theorem 3.2, there exist a derivation $\mathcal{D}$ of $\psi \to \phi[c/x]$ and a variable y which does not occur in $\mathcal{D}$. Then, by replacing c in $\mathcal{D}$ by y, $\psi \to \phi[y/x]$ is derivable. By choice of y, $\psi \to \forall x\phi$ is derivable. Hence, $|\psi| \leq |\forall x\phi|$. 2 follows similarly. Since

$$\Box_i\forall x(\psi \supset \phi(x)) \equiv \forall x\Box_i(\psi \supset \phi(x)), \quad \Box_i\forall x(\phi(x) \supset \psi) \equiv \forall x\Box_i(\phi(x) \supset \psi)$$

by Barcan formula, 3 and 4 are special cases of 1. $\Box$

Theorem 5.3. *If a closed formula ϕ is not derivable in* CK*, there exists a Kripke model $\mathcal{M} = \langle W, \{R_i\}_{i\in n}, D, I\rangle$ such that $\mathcal{M} \not\models \phi$.*

Proof. Let $(S_k)_{k\in\omega}$ be the sequence of subsets of $\mathcal{P}(A)$ such that

1. $S_0 = \{\{|\kappa\phi| : \kappa \in \mathcal{K}^*\} : \phi \text{ is a closed formula}\}$;

2. $S_{k+1} = \{\{\Box_i(z \to x) : x \in X\} : z \in A, \Box_i \in \mathcal{K}, X \in S_k\}$.

For each closed formula ψ of the shape $\forall x\chi(x)$ or $\exists x\chi(x)$, let $\bar{\psi}$ be the subset $\{|\chi(t)| : t \text{ is closed}\}$ of A. Let

$$T = \{\bar{\psi} : \psi \text{ is a closed formula of the shape } \forall x\chi(x) \text{ or } \exists x\chi(x)\}.$$

Now, let $Q = (\bigcup_{k\in\omega} S_k \cup T, T)$. By Lemma 5.1 and Lemma 5.2, Q satisfies the conditions in Lemma 4.4. By definition of Q, each $F \in \mathcal{F}_Q(A)$ satisfies the following:

(A) $|\forall x\chi(x)| \in F \;\Leftrightarrow\; |\chi(t)| \in F$ for all closed term t;

(B) $|\exists x\chi(x)| \in F \;\Leftrightarrow\; |\chi(t)| \in F$ for some closed term t;

(C) $|\Box_{\mathrm{c}}\,\chi| \in F \;\Leftrightarrow\; \kappa|\chi| \in F$ for all $\kappa \in \mathcal{K}^*$.

Now, let $\mathcal{M} = \langle W, \{R_i\}_{i\in n}, D, I\rangle$ be the Kripke model such that

1. $W = \mathcal{F}_Q(A)$;

2. $G <_{R_i} F \;\Leftrightarrow\; \Box_i^{-1}[G] \subset F$, for each $i \in n$;

3. $D = \{t : t \text{ is a closed term}\}$;

4. $c^{I(F)} = c$, for each $F \in W$ and each constant symbol c;

5. $(c_1, \ldots, c_k) \in P^{I(F)} \;\Leftrightarrow\; |P(c_1, \ldots, c_k)| \in F$, for each $F \in W$ and each predicate P of k variables.

We claim that for any closed formula ψ and any $F \in W$, $F \models \psi$ if and only if $|\psi| \in F$. The cases where $\psi = \forall x\chi$ and $\psi = \exists x\chi$ follow from (A) and (B). The case where $\psi = \Box_i\chi$ follows from Lemma 4.4. Suppose $|\Box_{\mathrm{c}}\,\chi| \in F$. Then, by (C), $\kappa|\chi| \in F$ for any $\kappa \in \mathcal{K}^*$. Hence, $F <_R G$ implies $G \models \chi$. Suppose $|\Box_{\mathrm{c}}\,\chi| \notin F$. By (C), there exists $\Box_{i_1} \ldots \Box_{i_k}|\chi| \notin A$ which is not in F. By k applications of Lemma 4.4, there exists $G \in \mathcal{F}_Q(A)$ such that $|\chi| \notin G$ and $F <_{R_{i_1}} \ldots <_{R_{i_k}} G$. Hence, $F \not\models \Box_{\mathrm{c}}\,\chi$. The other cases are straightforward.

Now, suppose ϕ is not derivable in CK. Then $|\phi| \neq 1$ in A, hence, there exists a Q-filter F such that $|\phi| \notin F$ by Theorem 4.5. Thus $F \not\models \phi$, and hence $\mathcal{M} \not\models \phi$. $\Box$

6. The cut-free system TCK

In this section, we introduce the system TCK, which is a cut-free system for **CKL**. A *tree sequent* of TCK is a finite tree T such that each node is a sequent $\Gamma \to \Delta$ of CK and each edge is labeled by one of the symbols $\Box_i \in \mathcal{K}$. (see figure 1).

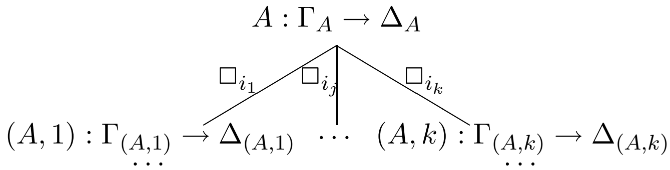

Figure 1: A tree sequent

We assume that each node N of a tree sequent has an address (A, k), where A denotes the address of the immediate predecessor N' of N and k denotes that N is the k-th immediate successor of N'. The address of the root is 0. We write $A : \Gamma_A \to \Delta_A$ for the node of a tree sequent whose address is A and attached sequent is $\Gamma_A \to \Delta_A$. We sometimes identify an address A in T with the node $A : \Gamma_A \to \Delta_A$ in T, and the set of all addresses in T with T itself. In this manner, for any k nodes $A_1, \ldots, A_k$ in T, $T \setminus \{A_1, \ldots, A_k\}$ denotes the graph obtained from T by removing k nodes at $A_1, \ldots, A_k$, and

$$T \setminus \{A_1, \ldots, A_k\} \cup \{A_1 : \Gamma_1 \to \Delta_1\} \cup \ldots \cup \{A_k : \Gamma_k \to \Delta_k\}$$

denotes the tree sequent which is obtained from T by replacing the nodes at $A_1, \ldots, A_k$ with new nodes $A_1 : \Gamma_1 \to \Delta_1, \ldots, A_k : \Gamma_k \to \Delta_k$, respectively.

If B is an immediate successor of A and the edge (A, B) is labeled by $\Box_i$, we say that B is an immediate successor of A with respect to $\Box_i$. A node A_k is called a successor of a node A_0 with respect to $\Box_{i_1} \cdots \Box_{i_k}$, if there exists a sequence $A_0, \ldots, A_k$ such that each A_j is an immediate successor of A_{j-1} with respect to $\Box_{i_j}$. For each tree sequent T and each node A of T, we write $\downarrow A$ for the maximum subtree of T with the root A.

An axiom of TCK is a tree sequent T which has a node of the shape $A : \phi \to \phi$ in it. The inference rules of TCK for Boolean connectives and quantifiers are same as those of CK applied to a node of a tree sequent. The list of all inference rules is the following:

set

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \Gamma' \to \Delta'\}} \ (\text{set})_+ \quad (\Gamma \subset \Gamma',\ \Delta \subset \Delta')$$

conjunction

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi\} \quad T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \psi\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi \wedge \psi\}} \ (\to \wedge)_+$$

$$\frac{T \setminus \{A\} \cup \{A : \phi, \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \phi \wedge \psi, \Gamma \to \Delta\}} \ (\wedge \to)_{+1}$$

$$\frac{T \setminus \{A\} \cup \{A : \psi, \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \phi \wedge \psi, \Gamma \to \Delta\}} \ (\wedge \to)_{+2}$$

disjunction

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi \vee \psi\}} \ (\to \vee)_{+1}$$

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \psi\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi \vee \psi\}} \ (\to \vee)_{+2}$$

$$\frac{T \setminus \{A\} \cup \{A : \phi, \Gamma \to \Delta\} \quad T \setminus \{A\} \cup \{A : \psi, \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \phi \vee \psi, \Gamma \to \Delta\}} \ (\vee \to)_+$$

implication

$$\frac{T \setminus \{A\} \cup \{A : \phi, \Gamma \to \Delta, \psi\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi \supset \psi\}} \ (\to\supset)_+$$

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi\} \quad T \setminus \{A\} \cup \{A : \psi, \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \phi \supset \psi, \Gamma \to \Delta\}} \ (\supset\to)_+$$

negation

$$\frac{T \setminus \{A\} \cup \{A : \phi, \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \neg\phi\}} \ (\to \neg)_+$$

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi\}}{T \setminus \{A\} \cup \{A : \neg\phi, \Gamma \to \Delta\}} \ (\neg \to)_+$$

forall

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi[y/x]\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \forall x\phi\}}\ (\to \forall)_+$$

$$\frac{T \setminus \{A\} \cup \{A : \phi[t/x], \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \forall x\phi, \Gamma \to \Delta\}}\ (\forall \to)_+$$

exists

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \phi[t/x]\}}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \exists x\phi\}}\ (\to \exists)_+$$

$$\frac{T \setminus \{A\} \cup \{A : \phi[y/x], \Gamma \to \Delta\}}{T \setminus \{A\} \cup \{A : \exists x\phi, \Gamma \to \Delta\}}\ (\exists \to)_+$$

The symbol t denotes any term which is free for x in ϕ and y denotes a variable which does not have any free occurrence in the lower sequent and free for x in ϕ.

necessitation 1

$$\frac{T \setminus \{A, (A,k)\} \cup \{A : \Gamma \to \Delta\} \cup \{(A,k) :\to \phi\}}{T \setminus \{A, (A,k)\} \cup \{A : \Gamma \to \Delta, \Box_i\phi\}}\ (\to \Box)_+$$

The node $(A,k) :\to \phi$ is an immediate successor of A with respect to $\Box_i$ and is a leaf of the upper tree sequent. The node $(A,k) :\to \phi$ will disappear after the application of $(\to \Box)_+$ (see figure 2);

necessitation 2

$$\frac{T\setminus\{A, (A,1), \dots, (A,k)\} \cup \{A : \Gamma \to \Delta\} \cup \bigcup_{j=1}^{k}\{(A,j) : \phi, \Gamma_j \to \Delta_j\}}{T\setminus\{A, (A,1), \dots, (A,k)\} \cup \{A : \Box_i\phi, \Gamma \to \Delta\} \cup \bigcup_{j=1}^{k}\{(A,j) : \Gamma_j \to \Delta_j\}}\ (\Box \to)_+$$

The set $\{(A,1), \dots, (A,k)\}$ is the collection of all immediate successors of A with respect to $\Box_i$. The formula ϕ is included in the left hand sides of $(A,1), \dots, (A,k)$ of the upper tree sequent (see figure 3).

common knowledge

$$\frac{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \kappa\phi\} \quad (\text{for all } \kappa \in \mathcal{K}^*)}{T \setminus \{A\} \cup \{A : \Gamma \to \Delta, \Box_{\mathrm{c}}\,\phi\}}\ (\to \Box_{\mathrm{c}})_+$$

$$\frac{T \setminus \{A\} \cup \{A : \kappa\phi, \Gamma \to \Delta\} \quad (\text{for some } \kappa \in \mathcal{K}^*)}{T \setminus \{A\} \cup \{A : \Box_{\mathrm{c}}\,\phi, \Gamma \to \Delta\}}\ (\Box_{\mathrm{c}} \to)_+$$

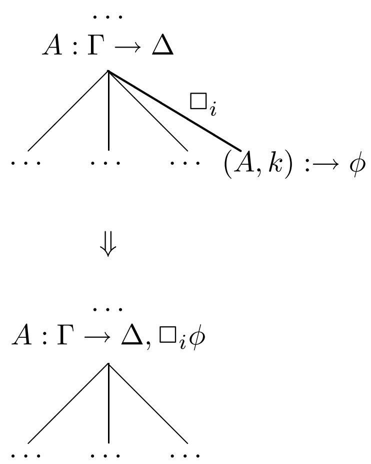

Figure 2: $(\to \Box)_+$

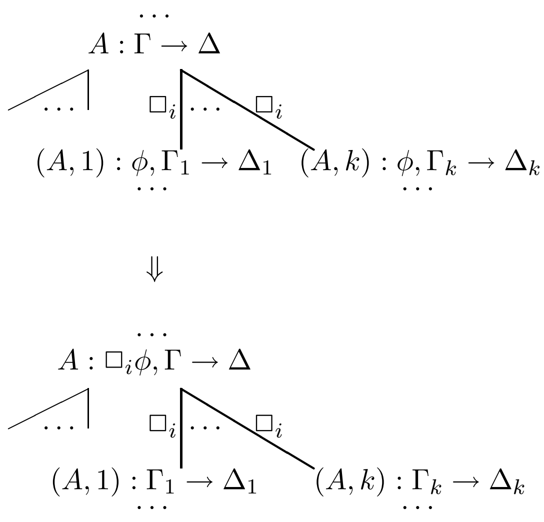

Figure 3: $(\Box \to)_+$

Note that TCK is cut-free and has $(\to \Box_{\mathrm{c}})$ as the only non-recursive inference rule. A formula ϕ is said to be derivable in TCK if the tree sequent $\to \phi$, which consists only of the root, is derivable in TCK.

Theorem 6.1. *The Barcan formula* $\forall x \Box \phi \supset \Box \forall x \phi$ *is derivable in* TCK.

Proof. If $\Box = \Box_i$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{0 :\to\} \cup \{(0,1) : \phi \to \phi\}}{\{0 : \Box_i \phi \to\} \cup \{(0,1) :\to \phi\}}\,(\Box \to)_+}{\{0 : \forall x \Box_i \phi \to\} \cup \{(0,1) :\to \phi\}}\,(\forall \to)_+}{\{0 : \forall x \Box_i \phi \to\} \cup \{(0,1) :\to \forall x \phi\}}\,(\to \forall)_+}{\{0 : \forall x \Box_i \phi \to \Box_i \forall x \phi\}}\,(\to \Box)_+}{\{0 :\to \forall x \Box_i \phi \supset \Box_i \forall x \phi\}}\,(\to\supset)_+ \quad .$$

If $\Box = \Box_{\mathrm{c}}$:

$$\{0 :\to\} \cup \{(0,1) :\to\} \cup \cdots \cup \{A :\to\} \cup \{(A,1) : \phi \to \phi\}$$
$$\vdots\ (\Box \to)_+$$
$$\dfrac{\dfrac{\dfrac{\{0 : \Box_{i_1} \cdots \Box_{i_k} \phi \to\} \cup \{(0,1) :\to\} \cup \cdots \cup \{A :\to\} \cup \{(A,1) :\to \phi\}}{\{0 : \Box_{\mathrm{c}}\, \phi \to\} \cup \{(0,1) :\to\} \cup \cdots \cup \{A :\to\} \cup \{(A,1) :\to \phi\}}\,(\Box_{\mathrm{c}} \to)_+}{\{0 : \forall x\, \Box_{\mathrm{c}}\, \phi \to\} \cup \{(0,1) :\to\} \cup \cdots \cup \{A :\to\} \cup \{(A,1) :\to \phi\}}\,(\forall \to)_+}{\{0 : \forall x\, \Box_{\mathrm{c}}\, \phi \to\} \cup \{(0,1) :\to\} \cup \cdots \cup \{A :\to\} \cup \{(A,1) :\to \forall x \phi\}}\,(\to \forall)_+$$
$$\vdots\ (\to \Box)_+$$
$$\dfrac{\dfrac{\{0 : \forall x\, \Box_{\mathrm{c}}\, \phi \to \Box_{i_1} \cdots \Box_{i_k} \forall x \phi\} \quad \text{for all } \Box_{i_1} \cdots \Box_{i_k} \in \mathcal{K}^*}{\{0 : \forall x\, \Box_{\mathrm{c}}\, \phi \to \Box_{\mathrm{c}}\, \forall x \phi\}}\,(\to \Box_{\mathrm{c}})_+}{\{0 :\to \forall x\, \Box_{\mathrm{c}}\, \phi \supset \Box_{\mathrm{c}}\, \forall x \phi\}}\,(\to\supset)_+ \quad .$$

Here, A is the address $((\ldots((0,1),1),\ldots),1)$ with n 1's and is the successor of 0 with respect to $\Box_{i_1} \cdots \Box_{i_{k-1}}$, and $(A,1)$ is the successor of A with respect to $\Box_{i_k}$. □

7. Completeness theorem for TCK

The embedding $*$ from the set of all tree sequents of TCK to the set of all formulas is defined inductively as follows:

1. if $T = \{0 : \Gamma \to \Delta\}$, then $T^* := \bigwedge \Gamma \supset \bigvee \Delta$;

2. if $0 : \Gamma \to \Delta$ is the root of T and S_i is the set of all immediate successors of the root with respect to $\Box_i$ for each $i \in n$, then:

$$T^* := \bigwedge \Gamma \supset \bigvee \Delta \vee \bigvee_{A \in S_0} \Box_0(\downarrow A)^* \vee \cdots \vee \bigvee_{A \in S_{n-1}} \Box_{n-1}(\downarrow A)^*.$$

Theorem 7.1. *If a closed formula ϕ is derivable in* TCK*, then $\mathcal{F} \models \phi$ for every Kripke frame $\mathcal{F}$ with a constant domain.*

Proof. It is enough to show that if a sequent T is derivable in TCK then $\mathcal{F} \models T^*$ for every Kripke frame $\mathcal{F}$ with a constant domain. But this is obtained by an easy induction on the height of the derivation. □

Now, we prove that TCK is complete with respect to the class of Kripke frames for predicate common knowledge logic. First, we show the following lemma:

Lemma 7.2. *Let T be a tree sequent of* TCK *which is not derivable. Suppose a node of T at an address A is denoted by $A : \Gamma_A \to \Delta_A$. Then, each of the following holds for any node $A : \Gamma_A \to \Delta_A$ in T.*

1. *If $\phi \wedge \psi \in \Gamma_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \phi, \psi, \Gamma_A \to \Delta_A\}$ is not derivable. If $\phi \wedge \psi \in \Delta_A$, then one of the tree sequents $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \phi\}$ and $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \psi\}$ is not derivable.*

2. *If $\phi \vee \psi \in \Gamma_A$, then one of the tree sequents $T \setminus \{A\} \cup \{A : \phi, \Gamma_A \to \Delta_A\}$ and $T \setminus \{A\} \cup \{A : \psi, \Gamma_A \to \Delta_A\}$ is not derivable. If $\phi \vee \psi \in \Delta_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \phi, \psi\}$ is not derivable.*

3. *If $\phi \supset \psi \in \Gamma_A$, then one of the tree sequents $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \phi\}$ and $T \setminus \{A\} \cup \{A : \psi, \Gamma_A \to \Delta_A\}$ is not derivable. If $\phi \supset \psi \in \Delta_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \phi, \Gamma_A \to \Delta_A, \psi\}$ is not derivable.*

4. *If $\neg\phi \in \Gamma_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \phi\}$ is not derivable. If $\neg\phi \in \Delta_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \phi, \Gamma_A \to \Delta_A\}$ is not derivable.*

5. *If $\forall x \phi(x) \in \Gamma_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \phi(c), \Gamma_A \to \Delta_A\}$ is not derivable for any constant symbol c. If $\forall x \phi(x) \in \Delta_A$ then the tree sequent $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \phi(c)\}$ is not derivable for some constant symbol c.*

6. *If $\exists x\phi(x) \in \Gamma_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \phi(c), \Gamma_A \to \Delta_A\}$ is not derivable for some constant symbol c. If $\exists x\phi(x) \in \Delta_A$ then the tree sequent $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \phi(c)\}$ is not derivable for any constant symbol c.*

7. *If $\Box_i\phi \in \Gamma_A$, then the tree sequent*
$$T \setminus \{(A,1),\ldots,(A,k)\} \cup \bigcup_{j=1,\ldots,k} \{(A,j) : \phi, \Gamma_{(A,j)} \to \Delta_{(A,j)}\}$$
is not derivable, where $(A,1),\ldots,(A,k)$ is the list of all immediate successors of A with respect to $\Box_i$. If $\Box_i\phi \in \Delta_A$, then the tree sequent
$$T \cup \{(A,k+1) :\to \phi\}$$
is not derivable, where $(A,k+1)$ is a new node and the edge $(A,(A,k+1))$ is labeled by $\Box_i$.

8. *If $\Box_c\,\phi \in \Gamma_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \kappa\phi, \Gamma_A \to \Delta_A\}$ is not derivable for any $\kappa \in \mathcal{K}^*$. If $\Box_c\,\phi \in \Delta_A$, then the tree sequent $T \setminus \{A\} \cup \{A : \Gamma_A \to \Delta_A, \kappa\phi\}$ is not derivable for some $\kappa \in \mathcal{K}^*$.*

Proof. Straightforward from the inference rules of TCK. □

Theorem 7.3. *Let ϕ be a closed formula. If $\mathcal{F} \models \phi$ for every Kripke frame $\mathcal{F}$ with a constant domain, then ϕ is derivable in* TCK.

Proof. Suppose ϕ is not derivable. We show that there exists a Kripke model $\mathcal{M}$ such that $\mathcal{M} \not\models \phi$. Let $\{\phi_i\}_{i\in\omega}$ be the set of all closed formulas and $(\psi_i)_{i\in\omega}$ the sequence
$$\phi_0, \phi_0, \phi_1, \phi_0, \phi_1, \phi_2, \phi_0, \phi_1, \phi_2, \phi_3, \ldots,$$
in which each ϕ_i occurs infinitary many times. We define a sequence $(T_i)_{i\in\omega}$ of tree sequents such that each T_i is not derivable. Let $T_0 := \{0 :\to \phi\}$. Suppose T_k is given. Let $\{A_1,\ldots,A_l\}$ be the list of all addresses of T_k and $A_j : \Gamma_{A_j} \to \Delta_{A_j}$ the node of T_k at A_j. We define T_{k+1} according as the construction of ψ_k as follows:

1. ψ_k is atomic: define $T_{k+1} := T_k$;
2. $\psi_k = \chi \wedge \psi$: for $j = 1$ to l, if $\chi \wedge \psi \in \Gamma_{A_j}$ then add χ and ψ to Γ_{A_j}, and if $\chi \wedge \psi \in \Delta_{A_j}$ then add χ or ψ given by Lemma 7.2 to Δ_{A_j};

3. $\psi_k = \chi \vee \psi$: dual to the previous case;

4. $\psi_k = \chi \supset \psi$: for $j = 1$ to l, if $\chi \supset \psi \in \Gamma_{A_j}$ then add χ to Δ_{A_j} or ψ to Γ_{A_j}, where each of χ or ψ is given by Lemma 7.2, and if $\chi \supset \psi \in \Delta_{A_j}$ then add χ to Γ_{A_j} and ψ to Δ_{A_j};

5. $\psi_k = \neg\psi$: for $j = 1$ to l, if $\neg\psi \in \Gamma_{A_j}$ then add ψ to Δ_{A_j}, and if $\neg\psi \in \Delta_{A_j}$ then add ψ to Γ_{A_j};

6. $\psi_k = \forall x\psi(x)$: for $j = 1$ to l, if $\forall x\psi(x) \in \Gamma_{A_j}$ then add $\psi(c_1),\dots,\psi(c_m)$ to Γ_{A_j}, where $c_1, \dots, c_m$ is the list of all constant symbols which occur in T_k, and if $\forall x\psi(x) \in \Delta_{A_j}$ then add $\psi(c)$ given by Lemma 7.2 to Δ_{A_j};

7. $\psi_k = \exists x\psi(x)$: dual to the previous case;

8. $\psi_k = \Box_i\psi$: for $j = 1$ to l, if $\Box_i\psi \in \Gamma_{A_j}$ then add ψ to the left hand sides of all immediate successors of A_j with respect to $\Box_i$, and if $\Box_i\psi \in \Delta_{A_j}$ add new node $B :\to \psi$ under A_j and label the edge (A_j, B) by $\Box_i$.

9. $\psi_k = \Box_{\mathrm{c}}\,\psi$: for $j = 1$ to l, if $\Box_{\mathrm{c}}\,\psi \in \Gamma_{A_j}$ then add the formulas of the shape $\Box_{i_1} \cdots \Box_{i_k}\phi$ to Γ_{A_j}, whenever there exists a successor of A_j with respect to $\Box_{i_1} \cdots \Box_{i_k}$. If $\Box_{\mathrm{c}}\,\psi \in \Delta_{A_j}$ then add $\kappa\psi$ given by Lemma 7.2 to Δ_{A_j}.

By Lemma 7.2, T_{k+1} is well-defined. Now, we take the *limit* of the sequence $(T_i)_{i\in\omega}$. For each $i \in \omega$, suppose the node of T_i at an address A is denoted by $A : \Gamma^i_A \to \Delta^i_A$. For each A, let

$$\lim \Gamma_A = \bigcup_{i\in\omega} \Gamma^i_A, \quad \lim \Delta_A = \bigcup_{i\in\omega} \Delta^i_A.$$

We write $\lim A$ for the pair $(\lim \Gamma_A, \lim \Delta_A)$.

Now, we define a model $\mathcal{M} = \langle W, \{R_i\}_{i\in\omega}, D, I\rangle$. Let W be the collection of all $\lim A$ and R_j ($j \in n$) the relation on W such that

$$\lim A <_{R_j} \lim B \;\Leftrightarrow\; \text{there exists an edge } (A, B) \text{ labeled by } \Box_j \text{ in some } T_i.$$

Let D be the set of all constant symbols which have some occurrences in some T_i ($i \in \omega$) and I be the interpretation such that

1. $c^{I(\lim A)} = a$, for each constant symbol $c \notin D$ (a is a fixed constant symbol in D);

2. $d^{I(\lim A)} = d$, for each constant symbol $d \in D$;

3. $(d_1, \ldots, d_k) \in P^{I(\lim A)} \Leftrightarrow P(d_1, \ldots, d_k) \in \lim \Gamma_A$, for each predicate symbol P.

Then, a simple induction shows that for any closed formula ψ, if $\psi \in \lim \Gamma_A$ then $\lim A \models \psi$ and if $\psi \in \lim \Delta_A$ then $\lim A \not\models \psi$. Since $\phi \in \lim \Delta_0$, $\lim 0 \not\models \phi$, hence $\mathcal{M} \not\models \phi$. □

An easy induction shows that if T is derivable in TCK then T^* is derivable in CK. Hence, Theorem 7.3 provides another proof of the completeness theorem of CK.

8. Miscellaneous results

Since TCK is cut-free, it follows that if a formula $\phi \in \mathbf{CKL}$ has no positive occurrence of $\Box_c$, it is derivable in TCK without $(\rightarrow \Box_c)_+$. Since $(\rightarrow \Box_c)_+$ is the only non-recursive inference rule in TCK, the set of formulas in **CKL** with no positive occurrence of $\Box_c$ is recursively axiomatizable. Let Φ be the set of formulas, in which any of the symbols $\Box_i \in \mathcal{K}$ does not occur. Since the accessibility relation for $\Box_c$ is reflexive and transitive, the set $\mathbf{CKL} \cap \Phi$ is characterized by the class of reflexive and transitive Kripke frames with constant domains, hence, it is equivalent to predicate **S4** with the Barcan formula. Therefore, it is recursively axiomatizable. Moreover, the propositional fragment of **CKL** is recursively axiomatizable by fixed point formalization (e.g. [3]). Since **CKL** is not recursively axiomatizable ([13]), we have proved the following:

Theorem 8.1. *The class of formulas in* **CKL** *which have some positive occurrences of the common knowledge operator* $\Box_c$ *and some occurrences of knowledge operators* $\Box_i$ *and quantifiers is not recursively axiomatizable, and its complement is recursively axiomatizable.*

Finally, we consider the predicate (mono-)modal logic which is characterized by the class of Kripke frames with constant domains. It is known that the Barcan formula is necessary to axiomatize modal logics characterized by classes of Kripke frames with constant domains. In [7], Pliuškevičienė presented cut-free systems for various modal logics with the Barcan formula by indexing. Now, it is easy to see that from TCK we obtain another cut-free system for the predicate modal logic **K** with the Barcan formula.

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