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FULLY FREGEAN LOGICS

A b s t r a c t. Frege's Principle asserts that the denotation of a propositional sentence coincides with its truth value. In the context of algebraizable logics the principle can be interpreted as the compositionality of interderivability relation $\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}$, defined formally by $\widetilde{\mathbf{\Lambda}}_{\mathcal{S}}\, T = \{\langle \phi, \psi \rangle \in \mathbf{Fm}_{\Lambda}{}^2 \mid T, \phi \dashv\vdash_{\mathcal{S}} T, \psi\}$, for given deductive system $\mathcal{S}$ and any $\mathcal{S}$-theory T. Of special interest are the deductive systems for which the property of being Fregean is inherited by all full 2nd-order models, so called, *fully Fregean* deductive systems. The main result of this paper is a characterization of fully Fregean deductive systems over countable languages using properties of the strong Frege operator on the formula algebra. The example of a Fregean, but not fully Fregean deductive system $\mathcal{S}$ is provided. $\mathcal{S}$ also turns out to be selfextensional, but not fully selfextensional, and, in addition, the three principal algebraic semantics for $\mathcal{S}$ are different, i.e., $\mathrm{Alg}^* \mathcal{S} \subsetneq \mathrm{Alg}\, \mathcal{S} \subsetneq \mathrm{Var}(\mathrm{Alg}\, \mathcal{S})$.

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Introduction

The Frege Principle (in various forms and, in particular, in the context of modal logics) was formulated by Suszko, who has also initiated the investigation of the sentential logic with identity [1, 2]. According to the Frege Principle the interderivability relation of classical logic is compositional. The notion of a Fregean logic in the broader context of Algebraic Logic is due to Czelakowski and Pigozzi (for references see [3, 4]). Consider a deductive system $\mathcal{S}$ in the sense of Tarski over a language of functional type Λ. $\mathcal{S}$ is completely determined by its substitution-invariant consequence relation $\vdash_{\mathcal{S}} \subseteq \mathcal{P}_\omega(\mathrm{Fm}_\Lambda) \times \mathrm{Fm}_\Lambda$, where Fm_Λ is the set of all formulas of type Λ and $\mathcal{P}_\omega(\mathrm{Fm}_\Lambda)$ is the set of all finite subsets of Fm_Λ. Then *Frege's principle* applied to $\mathcal{S}$ asserts the compositionality of the interderivability relation. In other words, a deductive system $\mathcal{S}$ is *Fregean* when for all $\Gamma \subseteq \mathrm{Fm}_\Lambda$ and all $\lambda \in \Lambda$ if $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_n \in \mathrm{Fm}_\Lambda$ are formulas such that $\Gamma, \phi_i \vdash_{\mathcal{S}} \psi_i$ and $\Gamma, \psi_i \vdash_{\mathcal{S}} \phi_i$ hold, then $\Gamma, \lambda(\phi_1, \ldots, \phi_n) \vdash_{\mathcal{S}} \lambda(\psi_1, \ldots, \psi_n)$ holds, where n is the arity of λ.

A weaker principle, the principle of *self-extensionality*, was introduced by Wójcicki. A deductive system $\mathcal{S}$ is *self-extensional* when for all $\lambda \in \Lambda$ if $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_n$ are such that $\phi_i \vdash_{\mathcal{S}} \psi_i$ and $\psi_i \vdash_{\mathcal{S}} \phi_i$ hold, then $\lambda(\phi_1, \ldots, \phi_n) \vdash_{\mathcal{S}} \lambda(\psi_1, \ldots, \psi_n)$ holds.

There are two kinds of matrix semantics naturally associated with a deductive system. Given a deductive system $\mathcal{S}$ over the language of type Λ and an algebra $\mathbf{A}$ of the same type Λ, an *$\mathcal{S}$-filter* over $\mathbf{A}$ is any subset F of A (A is the universe of $\mathbf{A}$) closed under rules of $\mathcal{S}$ in the following sense: F is closed under the rule $\gamma_1(\bar{x}), \ldots, \gamma_n(\bar{x}) \vdash \gamma(\bar{x})$ means that, for any assignment $x_i \mapsto a_i$ of elements of $\mathbf{A}$ to variables, if $\{\gamma_1^{\mathbf{A}}(\bar{a}), \ldots, \gamma_n^{\mathbf{A}}(\bar{a})\} \subseteq F$, then $\gamma^{\mathbf{A}}(\bar{a}) \in F$. A matrix $\langle \mathbf{A}, F \rangle$ is *a model of* $\mathcal{S}$ if F is an $\mathcal{S}$-filter.

The various Gentzen-style systems associated with $\mathcal{S}$ are intended to formalize the interrelation between the $\mathcal{S}$-filters on an algebra. The appropriate matrix semantics for Gentzen-style systems are *generalized* or *2nd-order matrices*. A pair $\langle \mathbf{A}, \mathcal{C} \rangle$ is a *generalized matrix* if $\mathcal{C} \subseteq \mathcal{P}(\mathrm{A})$ is an algebraic closed-set system. A generalized matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a *generalized model* for $\mathcal{S}$ if $\mathcal{C} \subseteq \mathcal{F}i_{\mathcal{S}}\mathbf{A}$, where $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$ is the set of all $\mathcal{S}$-filters on $\mathbf{A}$. Among all generalized models, of special interest are those of the form $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A} \rangle$ and also generalized matrices that are logically equivalent to them in a certain sense. Such models are called *full generalized models* (or *full models*) for $\mathcal{S}$.

The matrix semantics gives rise to the corresponding algebraic semantics. Thus, for any matrix $\langle \mathbf{A}, F\rangle$ there exists the largest congruence $\boldsymbol{\Omega}_{\mathbf{A}}F$ on $\mathbf{A}$ *compatible with* F, where a congruence θ on $\mathbf{A}$ is *compatible* with F when $F = \bigcup\{x/\theta \mid x \in F\}$. $\boldsymbol{\Omega}_{\mathbf{A}}F$ is called the *Leibniz congruence* for F. Then $\mathrm{Alg}^* \mathcal{S}$ is the class of all algebras such that each of them is isomorphic to some algebra of the form $\mathbf{A}/\boldsymbol{\Omega}_{\mathbf{A}}F$, where $\langle \mathbf{A}, F\rangle$ is a model for $\mathcal{S}$.

Similarly, for any generalized matrix $\langle \mathbf{A}, \mathcal{C}\rangle$, there exists the largest congruence $\widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\,\mathcal{C}$ on $\mathbf{A}$ compatible with all filters from $\mathcal{C}$. It is called *Tarski congruence* for $\langle \mathbf{A}, \mathcal{C}\rangle$. Then $\mathrm{Alg}\,\mathcal{S}$ is the class of all algebras isomorphic to an algebra of the form $\mathbf{A}/\widetilde{\boldsymbol{\Omega}}_{\mathbf{A}}\,\mathcal{C}$, where $\langle \mathbf{A}, \mathcal{C}\rangle$ is a generalized model for $\mathcal{S}$.

The problem of characterizing fully Fregean deductive systems is related to the problem of determining the conditions under that the validity of a Gentzen-style rule can be carried over all to full model of a given deductive system. A Gentzen-style rule

$$\frac{\Gamma_1 \vdash \gamma_1, \ldots, \Gamma_n \vdash \gamma_n}{\Gamma \vdash \gamma}$$

is *valid* in a generalized matrix $\langle \mathbf{A}, \mathcal{C}\rangle$ when for every substitution h of elements of $\mathbf{A}$ for variables, from $h(\gamma_i) \in \mathrm{Clo}_{\mathcal{C}}(h(\Gamma_i))$, $i = 1, \ldots, n$, it follows that $h(\gamma) \in \mathrm{Clo}_{\mathcal{C}}(h(\Gamma))$, where $\mathrm{Clo}_{\mathcal{C}}(\Delta) = \bigcap\{F \mid \Delta \subseteq F \in \mathcal{C}\}$ is the *closure of* Δ *in* $\mathcal{C}$. It is easy to see that a deductive system $\mathcal{S}$ is Fregean if each member of the following family of Gentzen-style rules

$$r_{\lambda,\bar{z}} := \frac{\bar{z}, x_1 \vdash y_1;\ \bar{z}, y_1 \vdash x_1;\ \ldots\ \bar{z}, x_n \vdash y_n;\ \bar{z}, y_n \vdash x_n}{\bar{z}, \lambda(x_1, \ldots, x_n) \vdash \lambda(y_1, \ldots, y_n)}, \quad \lambda \in \Lambda$$

(where $\bar{z}$ ranges over all finite sequences of variables), is valid in the special full generalized model $\langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S}\rangle$, where $\mathbf{Fm}_\Lambda$ is the formula algebra over the language of type Λ and $\mathrm{Th}\,\mathcal{S} = \mathcal{F}i_{\mathcal{S}}\,\mathbf{Fm}_\Lambda$. The notion of being Fregean can be naturally extended to any generalized model for $\mathcal{S}$. Then a deductive system $\mathcal{S}$ is *fully Fregean* if all $r_{\lambda,\bar{z}}$ are valid in all full models for $\mathcal{S}$. Similarly, a deductive system $\mathcal{S}$ is *fully self-extensional* if all Gentzen-style rules

$$r_\lambda := \frac{x_1 \vdash y_1;\ y_1 \vdash x_1;\ \ldots\ x_n \vdash y_n;\ y_n \vdash x_n}{\lambda(x_1, \ldots, x_n) \vdash \lambda(y_1, \ldots, y_n)}, \quad \lambda \in \Lambda,$$

are valid in all full models of $\mathcal{S}$.

For deductive systems there are intricate connections between the Fregean property or self-extensionality and Gentzen-style systems over $\mathcal{S}$ on the other hand. It was shown in [5] that in the deductive systems with the Property of Conjunction (PC) or with the Deduction-Detachment Theorem, self-extensionality leads to the existence of a fully adequate Gentzen system. A Gentzen-style system $\mathfrak{G}$ is *fully adequate for* $\mathcal{S}$, when finitary full models for $\mathcal{S}$ are exactly the models for $\mathfrak{G}$ without theorems.

A number of questions were intensively studied in [4, 5] but remained open in general case:

1. Is every self-extensional logic fully self-extensional? [5, p.45]

2. Is every Fregean logic fully self-extensional? [5, p.68]

3. Is every Fregean logic fully Fregean? (D. Pigozzi)

4. Does every Fregean logic $\mathcal{S}$ have the property that either $\mathrm{Alg}^* \mathcal{S} = \mathrm{Alg}\, \mathcal{S}$ or $\mathrm{Alg}\, \mathcal{S} = \mathrm{Var}(\mathrm{Alg}\, \mathcal{S})$ [5, p.38], where $\mathrm{Var}(\mathrm{Alg}\, \mathcal{S})$ is the variety generated by the class $\mathrm{Alg}\, \mathcal{S}$?

Obviously, a positive answer for (3) leads to a positive answer for (2).

Under the condition of protoalgebracity, questions (2)–(4) were successfully answered. So (2) and (3) are proven to be true in [4], and $\mathrm{Alg}^* \mathcal{S} = \mathrm{Alg}\, \mathcal{S}$ for any protoalgebraic deductive system [5, Proposition 3.2]. Also, (1) is true for systems with Deduction-Detachment Theorem (DDT) [5, Theorem 4.46], and $\mathrm{Alg}\, \mathcal{S} = \mathrm{Var}(\mathrm{Alg}\, \mathcal{S})$ for any self-extensional deductive system with DDT [5, Theorem 4.45].

Those questions were also studied in non protoalgebraic context. Thus, (1) is true for the systems with the Property of Conjunction (PC) [5, Theorem 4.28], and $\mathrm{Alg}\, \mathcal{S} = \mathrm{Var}(\mathrm{Alg}\, \mathcal{S})$ for any self-extensional deductive system with PC [5, Theorem 4.27]. Nevertheless, all questions (1)–(4) remained open in general case.

As we will see, in non protoalgebraic case there is one logic that gives a definite answer on all these questions.

Outline of the paper. The first section contains the basic definitions and description of concepts necessary for understanding the further results. For a historical and philosophical review of Frege's Principle the reader can refer to [4]. The second section contains the main result of the paper — the characterization of fully Fregean deductive systems over countable languages, and attendant lemmas and propositions. In the third section the counterexample of a Fregean, but not fully Fregean deductive system $\mathcal{S}$ is constructed. The last section discusses the interrelations between different algebraic semantics for deductive systems $\mathcal{S}$.

Deductive systems and their models

Let Λ be any nonempty set of functional symbols. With Λ associated an arity function $\rho : \Lambda \to \omega$, such that $\rho(\lambda)$ is the arity of the functional symbol $\lambda \in \Lambda$.

For a given algebra $\mathbf{A} = \langle A, \{\lambda^{\mathbf{A}}\}_{\lambda\in\Lambda}\rangle$ of type Λ, $\mathrm{Con}\,\mathbf{A}$ denotes the set of all congruences on the algebra $\mathbf{A}$, $\mathrm{Eq}\,\mathbf{A} = \mathrm{Eq}\,A$ is the set of all equivalence relations over the set A. Let $F \subseteq A$, then a congruence $\theta \in \mathrm{Con}\,\mathbf{A}$ is *compatible with* F if $F = \bigcup\{x/\theta \mid x \in F\}$, where x/θ is the θ-congruence class containing x. The *Leibniz congruence* for F, in symbols $\mathbf{\Omega_A} F$, is the largest congruence on $\mathbf{A}$ compatible with F. The Leibniz congruence always exists. Also, denote $\mathbf{\Lambda_A}\, F := \{\langle a, b\rangle \in \mathbf{A}^2 \mid a \in F \Leftrightarrow b \in F\}$, so $\mathbf{\Omega_A} F$ is the largest congruence θ such that $\theta \subseteq \mathbf{\Lambda_A}\, F$.

For a couple of algebras $\mathbf{A}$, $\mathbf{B}$ of the same type, $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ means, that h is a homomorphism from $\mathbf{A}$ onto $\mathbf{B}$.

A 2nd-order (or *generalized*) *matrix* is a pair $\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$, where $\mathbf{A} = \langle A, \{\lambda^{\mathbf{A}}\}_{\lambda\in\Lambda}\rangle$ is an algebra of type Λ, and $\mathcal{C}$ is a closed-set system, i.e., $\mathcal{C} \subseteq \mathcal{P}(A), A \in \mathcal{C}$, and $\mathcal{C}$ is closed under arbitrary intersection. For any $F \in \mathcal{C}$, denote $[F)_{\mathcal{C}} := \{G \in \mathcal{C} \mid F \subseteq G\}$.

The special kind of 2nd-order matrices are *1-deductive systems* or simply *deductive systems*, which are the matrices of the form $\mathcal{S} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{S}\rangle$, where $\mathbf{Fm}_\Lambda$ is the absolutely free algebra of countably many generators over the language of type Λ, and $\mathrm{Th}\,\mathcal{S}$ is a closed-set system which is, first, closed under inverse substitutions, i.e., for any substitution $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$, $\sigma^{-1}\,\mathrm{Th}\,\mathcal{S} \subseteq \mathrm{Th}\,\mathcal{S}$, where $\sigma^{-1}T = \{\alpha \in \mathbf{Fm}_\Lambda \mid \sigma\alpha \in T\}$; second, is algebraic, i.e., for any upward-directed subfamily $\mathcal{T} \subseteq \mathrm{Th}\,\mathcal{S}$ the direct limit of $\mathcal{T}$ is in $\mathrm{Th}\,\mathcal{S}$. The elements of $\mathrm{Th}\,\mathcal{S}$ are called *theories* of $\mathcal{S}$ or $\mathcal{S}$*-theories*.

For a given 2nd-order matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$,

$$\widetilde{\mathbf{\Lambda}}\,\mathcal{C} := \bigcap\{\mathbf{\Lambda}\,F \mid F \in \mathcal{C}\}, \quad \widetilde{\mathbf{\Omega}}\,\mathcal{C} := \bigcap\left\{\mathbf{\Omega_A} F \mid F \in \mathcal{C}\right\},$$

where $\widetilde{\mathbf{\Omega}}$ is called the Tarski operator, and is defined on a set of closed-set systems. More often we will use the relativized versions of these operators: the Frege operator $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}} : \mathcal{C} \mapsto \mathrm{Eq}\,A$ defined as follows

$$\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F := \widetilde{\mathbf{\Lambda}}[F)_{\mathcal{C}} = \bigcap\{\mathbf{\Lambda}\,G \mid F \subseteq G \in \mathcal{C}\},$$

and the Suszko operator $\widetilde{\mathbf{\Omega}}_{\mathfrak{A}} : \mathcal{C} \mapsto \mathrm{Con}\,\mathbf{A}$

$$\widetilde{\mathbf{\Omega}}_{\mathfrak{A}}\,F := \widetilde{\mathbf{\Omega}}[F)_{\mathcal{C}} = \bigcap\{\mathbf{\Omega_A} G \mid F \subseteq G \in \mathcal{C}\}.$$

The congruence $\widetilde{\mathbf{\Omega}}_{\mathfrak{A}}\,F$ is, in fact, the largest congruence compatible with all filters from $\mathcal{C}$ containing F. If the deductive system $\mathcal{S}$ is clear from context, then for brevity sake and better typesetting, we will write

$$\widetilde{\mathbf{\Lambda}} := \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}\ , \quad \widetilde{\mathbf{\Omega}} := \widetilde{\mathbf{\Omega}}_{\mathcal{S}}\ , \quad \mathbf{\Omega} := \mathbf{\Omega}_{\mathbf{Fm}_{\mathbf{\Lambda}}}.$$

Also, if the language type Λ is fixed we will write $\mathbf{Fm}$ instead of $\mathbf{Fm}_{\Lambda}$, especially in subscripts.

The set of all $\mathcal{S}$-filters on a given algebra $\mathbf{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}\mathbf{A}$. Any 2nd-order matrix of the form $\mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A}\rangle$ is called *a basic full model.* A 2nd-order matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$ is a *full model* if $\mathfrak{A}/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C} = \langle \mathbf{A}/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C}, \mathcal{C}/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C}\rangle$ is a basic full model, where $\mathcal{C}/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C} = \{F/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C} \mid F \in \mathcal{C}\}$ and $F/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C} = \{x/\,\widetilde{\mathbf{\Omega}}\,\mathcal{C} \mid x \in F\}$.

A deductive system is called *Fregean* if $\widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}} = \widetilde{\mathbf{\Omega}}_{\mathcal{S}}$. Similarly, a 2nd-order matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$ is *Fregean* if $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}} = \widetilde{\mathbf{\Omega}}_{\mathfrak{A}}$. A Fregean deductive system $\mathcal{S}$ is *fully Fregean* if all full models for $\mathcal{S}$ are Fregean.

Fully Fregean deductive systems

We will start from a simple, but useful characterization of Fregean deductive systems.

Proposition 0.1. *$\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$ is Fregean iff $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F \subseteq \mathbf{\Omega}_{\mathbf{A}}F$ for all $F \in \mathcal{C}$.*

Proof. $(\Rightarrow)$ $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F = \widetilde{\mathbf{\Omega}}_{\mathfrak{A}}\,F = \bigcap\{\mathbf{\Omega}_{\mathbf{A}}G \mid F \subseteq G \in \mathcal{C}\} \subseteq \mathbf{\Omega}_{\mathbf{A}}F$, for all $F \in \mathcal{C}$.

$(\Leftarrow)$ For any $F, G \in \mathcal{C}$, $F \subseteq G$ yields $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F \subseteq \widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,G \subseteq \mathbf{\Omega}_{\mathbf{A}}G$, by monotonicity of $\widetilde{\mathbf{\Lambda}}$ and by assumption. Thus $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F \subseteq \bigcap\left\{\mathbf{\Omega}_{\mathbf{A}}G \mid F \subseteq G\right\} = \widetilde{\mathbf{\Omega}}_{\mathfrak{A}}\,F$. The latter yields $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F = \widetilde{\mathbf{\Omega}}_{\mathfrak{A}}\,F$, since $\widetilde{\mathbf{\Omega}}_{\mathfrak{A}}\,F \subseteq \widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F$ always holds. $\square$

Denote $[F)^{L}_{\mathcal{C}} := \{G \in [F)_{\mathcal{C}} \mid \mathbf{\Omega}_{\mathbf{A}}F \subseteq \mathbf{\Omega}_{\mathbf{A}}G\}$.

Definition 0.2. The *strong Fregean operator* on a 2nd-order matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$ is defined as

$$\widetilde{\mathbf{\Lambda}}^{L}_{\mathcal{C}}F := \bigcap\left\{\mathbf{\Lambda}\,G \mid G \in [F)^{L}_{\mathcal{C}}\right\}.$$

Clearly, $\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\,F \subseteq \widetilde{\mathbf{\Lambda}}^{L}_{\mathcal{C}}F$, for every $F \in \mathcal{C}$. Also $\mathbf{\Omega}_{\mathbf{A}}F \subseteq \widetilde{\mathbf{\Lambda}}^{L}_{\mathcal{C}}F$, because

$$\mathbf{\Omega}_{\mathbf{A}}F = \bigcap\{\mathbf{\Omega}_{\mathbf{A}}G \mid G \in [F)^{L}_{\mathcal{C}}\} \subseteq \bigcap\{\mathbf{\Lambda}\,G \mid G \in [F)^{L}_{\mathcal{C}}\} = \widetilde{\mathbf{\Lambda}}^{L}_{\mathcal{C}}F$$

since $\mathbf{\Omega_A} G \subseteq \mathbf{\Lambda}\, G$, for all $G \in \mathcal{C}$.

The next proposition states that if the operator $\widetilde{\mathbf{\Lambda}}^L_{\mathcal{C}}$ on a matrix $\mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A}\rangle$ is strong enough (i.e., $\widetilde{\mathbf{\Lambda}}^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} F = \mathbf{\Omega_A} F$, for all $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$), then the property of being Fregean is inherited by all full models of $\mathcal{S}$ on homomorphic images of $\mathbf{A}$.

Proposition 0.3. *Let $\mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A}\rangle$ be a basic full model of a deductive system $\mathcal{S}$, such that $\widetilde{\mathbf{\Lambda}}^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} F = \mathbf{\Omega_A} F$, for all $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$. Then every full model $\mathfrak{B} = \langle \mathbf{B}, \mathcal{D}\rangle$, such that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, is Fregean.*

Proof. Let h be a homomorphism such that $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$, and let $F \in \mathcal{D}$. If $G \in [h^{-1}F)^L_{\mathcal{C}}$, then $\ker h = h^{-1}\,\mathrm{id}_{\mathbf{B}} \subseteq h^{-1}\,\mathbf{\Omega_A} F = \mathbf{\Omega_A} h^{-1}F \subseteq \mathbf{\Omega_A} G$, therefore $hG \in \mathcal{F}i_{\mathcal{S}}\mathbf{B}$, and also $F \subseteq hG$. Then $h^{-1}\,\mathbf{\Omega_B} F = \mathbf{\Omega_A} h^{-1}F \subseteq \mathbf{\Omega_A} G = \mathbf{\Omega_A} h^{-1}hG = h^{-1}\,\mathbf{\Omega_B} hG$, thus $\mathbf{\Omega_B} F \subseteq \mathbf{\Omega_B} hG$. [5, Theorem 2.14] tells us that $\mathcal{D} = \{G \in \mathcal{F}i_{\mathcal{S}}\mathbf{B} \mid \widetilde{\mathbf{\Omega}}\,\mathcal{D} \subseteq \mathbf{\Omega_B} G\}$. Thus $\widetilde{\mathbf{\Omega}}\,\mathcal{D} \subseteq \mathbf{\Omega_B} F \subseteq \mathbf{\Omega_B} hG$, so $hG \in \mathcal{D}$.

Thus, we have shown that $h[h^{-1}F)^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} \subseteq [F)_{\mathcal{D}}$. Now it is easy to see that

$$\begin{aligned}\widetilde{\mathbf{\Lambda}}_{\mathcal{D}}\, F &= \bigcap \{\mathbf{\Lambda}\, H \mid H \in [F)_{\mathcal{D}}\} \subseteq \bigcap \left\{\mathbf{\Lambda}\, H \mid H \in h\left[h^{-1}F\right)^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}\right\} \\ &= \bigcap h \left\{\mathbf{\Lambda}\, G \mid G \in \left[h^{-1}F\right)^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}\right\} = h \bigcap \left\{\mathbf{\Lambda}\, G \mid G \in \left[h^{-1}F\right)^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}\right\} \\ &= h\, \widetilde{\mathbf{\Lambda}}^L_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} h^{-1}F = h\,\mathbf{\Omega_A} h^{-1}F = hh^{-1}\,\mathbf{\Omega_B} F = \mathbf{\Omega_B} F\end{aligned}$$

Since the latter is true for every $F \in \mathcal{D}$, then, by Proposition 0.1, $\mathfrak{B}$ is Fregean. □

We will say that a 2nd-order matrix $\mathfrak{B} = \langle \mathbf{B}, \mathcal{B}\rangle$ is a *submatrix* of $\mathfrak{A} = \langle \mathbf{A}, \mathcal{A}\rangle$, in symbols $\mathfrak{B} \leqslant \mathfrak{A}$, if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\mathcal{B} = B \cap \mathcal{A}$.

We will need the following lemma

Lemma 0.4 ([4, Lemma 2.4]). *A Gentzen-style rule is valid in a 2nd-order matrix $\mathfrak{A}$ if, for every finitely generated $\mathfrak{B} \leq \mathfrak{A}$, there exists a countably generated $\mathfrak{B}'$ such that $\mathfrak{B} \leq \mathfrak{B}' \leq \mathfrak{A}$ and the rule is valid in $\mathfrak{B}'$.*

Next we will prove a lemma necessary for the proof of the main theorem of this section. The lemma was first proven by J.M. Font, R. Jansana, and D. Pigozzi [6] in, so called, the "closure-relation" form. We will need it formulated in the "closed-set" form and will split it into two sublemmas.

If X is a set and $\mathbf{C}$ is an algebra, we will write $X|_{\mathbf{C}}$ for $X \cap C$.

Lemma 0.5. *For every countably generated submatrix* $\mathfrak{C} = \langle \mathbf{C}, \mathcal{C} \rangle \leqslant \mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A} \rangle$*, there exists a countably generated submatrix* $\mathfrak{C}'$ *such that* $\mathfrak{C} \leqslant \mathfrak{C}' \leqslant \mathfrak{A}$*, and, for all* $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{C}$*,*

$$(\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{C}'} F)|_{\mathbf{C}} = (\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} F)|_{\mathbf{C}}.$$

Proof. According to [5, Lemma 1.18], for any given $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{C}$,

$$\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} F = \bigcup_{n\in\omega} X_n^F, \tag{$*$}$$

where $X_0^F := F$, and X_{n+1}^F consists of all $a \in A$ such that there are $\varphi_a^F(\bar{x}) \in \mathbf{Fm}_\Lambda$ and a finite $\Gamma_a^F(\bar{x}) \subseteq \mathbf{Fm}_\Lambda$ such that $\Gamma_a^F \vdash_{\mathcal{S}} \varphi_a^F$, and there is a homomorphism $h \in \mathrm{Hom}(\mathbf{Fm}_\Lambda, \mathbf{A})$ with $\Gamma_a^F(h\bar{x}) \subseteq X_n^F$, $\varphi_a^F(h\bar{x}) = a$. Denote $\widehat{a}^F$ the set of coordinates of the vector $h\bar{x}$.

Let $G := \bigcup\{\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}(F) - F \mid F \in \mathcal{F}i_{\mathcal{S}}\mathbf{C}\}$. For a given $a \in G$ and $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{C}$ such that $a \in \mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}(F) - F$, denote n the smallest natural number such that $a \in X_n^F$. Define, for every $a \in G|_{\mathbf{C}}$, the set $\widetilde{a}^F \subseteq \mathbf{A}$ inductively as follows:

$$\widetilde{a}_n^F := \{a\}, \quad \widetilde{a}_{k-1}^F := \bigcup\{\widehat{b}^F \mid b \in \widetilde{a}_k^F\},\ k = 1, \dots, n, \quad \widetilde{a}^F := \bigcup_{i\in n} \widetilde{a}_i^F.$$

Since we can establish 1-1 correspondence $\widetilde{a}_0^F \times \{a\} \leftrightarrow \langle a, F\rangle$, for any $a \in \mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}(F) - F$, then $\widetilde{G} := \bigcup\{\widetilde{a}^F \mid a \in G|_{\mathbf{C}}\}$ is countable. Let

$$\mathbf{C}' := \mathrm{Sub}_{\mathbf{A}}(\mathbf{C} \cup \widetilde{G}), \quad \mathcal{C}' := \mathcal{F}i_{\mathcal{S}}\mathbf{A} \cap \mathbf{C}',$$

where $\mathrm{Sub}_{\mathbf{A}}(X)$ is the subalgebra of $\mathbf{A}$ generated by X. Then $\mathfrak{C}' = \langle \mathbf{C}', \mathcal{C}' \rangle$ is countably generated and $\mathfrak{C} \leqslant \mathfrak{C}' \leqslant \mathfrak{A}$. Also, if $a \in (\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} F)|_{\mathbf{C}} - F$, then $\widetilde{a}^F \subseteq_\omega \widetilde{G} \subseteq \mathbf{C}'$. Thus, again by [5, Lemma 1.18], $a \in \mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{C}'} F$. Therefore $(\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{C}'} F)|_{\mathbf{C}} = (\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}} F)|_{\mathbf{C}}$, for all $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{C}$. □

The following result belongs to J.M. Font, R. Jansana and D. Pigozzi.

Lemma 0.6. *Let* $\mathfrak{B} = \langle \mathbf{B}, \mathcal{B} \rangle \leqslant \mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A} \rangle$ *be 2nd-order models for a deductive system* $\mathcal{S}$ *over a countable language. Suppose* $\mathbf{B}$ *is countably generated. Then there is a countably generated basic full model* $\mathfrak{B}^*$*, such that*

$$\mathfrak{B} \leqslant \mathfrak{B}^* \leqslant \mathfrak{A}.$$

Proof. Define $\mathfrak{B}_0 := \mathfrak{B}$, $\mathfrak{B}_{i+1} := (\mathfrak{B}_i)'$, where $(\,.\,)'$ as in Lemma 0.5, and let

$$\mathbf{B}^* := \bigcup_{i\in\omega} \mathbf{B}_i, \quad \mathcal{B}^* := \mathcal{F}i_{\mathcal{S}}\mathbf{A} \cap \mathbf{B}^*.$$

Then

• $\mathfrak{B}^* \leqslant \mathfrak{A}$, by definition.

• $\mathfrak{B} \leqslant \mathfrak{B}^*$, because $\mathbf{B} \leqslant \mathbf{B}^*$, and $\mathcal{B} = \mathcal{F}i_{\mathcal{S}}\mathbf{A} \cap \mathbf{B} = \mathcal{F}i_{\mathcal{S}}\mathbf{A} \cap \mathbf{B}^* \cap \mathbf{B} = \mathcal{B}^* \cap \mathbf{B}$.

• $\mathcal{B}^* = \mathcal{F}i_{\mathcal{S}}\mathbf{B}^*$. Suppose not. For any $X \subseteq A$, denote $X_i := X \cap B_i$. Then, for every $F \in \mathcal{F}i_{\mathcal{S}}\mathbf{B}^*$,

$$\begin{aligned}\mathrm{Clo}_{\mathcal{B}^*} F = \textstyle\bigcup_{i\in\omega} \mathrm{Clo}_{\mathcal{B}_i} F_i = \bigcup_{i\in\omega} \left(\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{B_{i+1}}} F_i\right)\big|_{\mathbf{B}_i} \\ \subseteq \bigcup_{i\in\omega} \mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{B_{i+1}}} F_{i+1} = \mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{B}^*} F = F.\end{aligned}$$

Thus $\mathfrak{B}^*$ is a basic full countably generated matrix and $\mathfrak{B} \leqslant \mathfrak{B}^* \leqslant \mathfrak{A}$. □

Theorem 0.7. *A deductive system $\mathcal{S}$ over a countable language of type Λ is fully Fregean iff for all $F \in \mathrm{Th}\,\mathcal{S}$*

$$\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} F = \mathbf{\Omega}_{\mathbf{Fm}_{\mathbf{\Lambda}}} F.$$

Proof. $(\Rightarrow)$ Let $\mathcal{S}$ be a fully Fregean deductive system, such that for some $F \in \mathrm{Th}\,\mathcal{S}$, $\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} F \neq \mathbf{\Omega}_{\mathbf{Fm}} F$. Then consider the factor–algebra $\mathbf{A} = \mathbf{Fm}_{\Lambda} \,/\, \mathbf{\Omega}_{\mathbf{Fm}} F$ and let h be the canonical homomorphism. Then

$$h[F)^{L}_{\mathrm{Th}\,\mathcal{S}} = [hF)_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}.$$

$(\subseteq)$ Indeed, if $G \in [F)^{L}_{\mathrm{Th}\,\mathcal{S}}$, then $\ker h = \mathbf{\Omega}_{\mathbf{Fm}} F \subseteq \mathbf{\Omega}_{\mathbf{Fm}} G$, therefore $hG \in \mathcal{F}i_{\mathcal{S}}\mathbf{A}$ and $F \subseteq G$. Thus $hG \in [hF)_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}$.

$(\supseteq)$ If $G \in [hF)_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}$, then $\mathrm{id}_{\mathbf{A}} \subseteq \mathbf{\Omega}_{\mathbf{A}} G$, that implies $\ker h = h^{-1}\,\mathrm{id}_{\mathbf{A}} \subseteq h^{-1}\,\mathbf{\Omega}_{\mathbf{A}} G$, so $\mathbf{\Omega}_{\mathbf{Fm}} F = \ker h \subseteq h^{-1}\,\mathbf{\Omega}_{\mathbf{A}} G = \mathbf{\Omega}_{\mathbf{Fm}} h^{-1} G$. Also $hF \subseteq G$ yields $h^{-1}hF = F \subseteq h^{-1}G$, so $h^{-1}G \in [F)^{L}_{\mathrm{Th}\,\mathcal{S}}$.

Therefore

$$\begin{aligned}\widetilde{\mathbf{\Lambda}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}\, hF &= \textstyle\bigcap \{\mathbf{\Lambda}\, G \mid G \in [hF)_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}\} \\ &= \textstyle\bigcap \left\{\mathbf{\Lambda}\, hG \mid G \in [F)^{L}_{\mathrm{Th}\,\mathcal{S}}\right\} = h \bigcap \left\{\mathbf{\Lambda}\, G \mid G \in [F)^{L}_{\mathrm{Th}\,\mathcal{S}}\right\} = h\, \widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} F,\end{aligned}$$

but $\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} F \supsetneq \mathbf{\Omega}_{\mathbf{Fm}} F$, therefore

$$\begin{aligned}\widetilde{\mathbf{\Lambda}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{A}}\, hF = h\, \widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} F \supsetneq h\, \mathbf{\Omega}_{\mathbf{Fm}} F = h\, \mathbf{\Omega}_{\mathbf{Fm}} h^{-1} hF \\ = hh^{-1}\, \mathbf{\Omega}_{\mathbf{A}} hF = \mathbf{\Omega}_{\mathbf{A}} hF.\end{aligned}$$

So, by Proposition 0.1, $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A} \rangle$ is not Fregean while full, a contradiction.

($\Leftarrow$) Let $\mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{S}}\mathbf{A}\rangle$ be any basic full model of the fully Fregean deductive system $\mathcal{S}$. By [4, Lemma 2.14], any 2nd-order matrix $\mathfrak{A}$ is Fregean iff all accumulative rules $R_\lambda := \{r_{\lambda,\bar{z}} \mid \bar{z} \in X^*_\omega\}$ are valid in $\mathfrak{A}$, where

$$r_{\lambda,\bar{z}} := \frac{\{\bar{z}, x_i \vdash y_i;\ \ \bar{z}, y_i \vdash x_i\}_{i\in\rho(\lambda)}}{\bar{z}, \lambda\langle x_i\rangle_{i\in\rho(\lambda)} \vdash \lambda\langle y_i\rangle_{i\in\rho(\lambda)}}, \quad \lambda \in \Lambda,$$

and $\bar{z} \in X^*_\omega$ means that $\bar{z}$ ranges over all finite sequences of free generators of $\mathbf{Fm}_\Lambda$.

By Lemma 0.6, for any finitely generated $\mathfrak{B} \leqslant \mathfrak{A}$, there exists countably generated basic full model $\mathfrak{B}'$, such that $\mathfrak{B} \leqslant \mathfrak{B}' \leqslant \mathfrak{A}$ and $r_{\lambda,\bar{z}}$ is valid in $\mathfrak{B}'$, since $\mathfrak{B}'$ is countably generated and hence Fregean, by Proposition 0.3. Thus, by Lemma 0.4, every $r_{\lambda,\bar{z}}$, $\lambda \in \Lambda, \bar{z} \in X^*_\omega$ is valid in $\mathfrak{A}$, i.e. $\mathfrak{A}$ is Fregean.

Now, suppose $\mathfrak{A} = \langle \mathbf{A}, \mathcal{C}\rangle$ is an arbitrary full model for $\mathcal{S}$. Then there exist a basic full model $\mathfrak{B} = \langle \mathbf{B}, \mathcal{F}i_{\mathcal{S}}\mathbf{B}\rangle$ for $\mathcal{S}$ and a homomorphism $h : \mathbf{A} \twoheadrightarrow \mathbf{B}$ such that $\mathcal{C} = h^{-1}\,\mathcal{F}i_{\mathcal{S}}\mathbf{B}$. For every $F \in \mathcal{C}$ holds $\ker h \subseteq \Lambda F$, hence $hF \in \mathcal{F}i_{\mathcal{S}}\mathbf{B}$. Then

$$\begin{aligned}\widetilde{\mathbf{\Lambda}}_{\mathcal{C}}\, F &= \bigcap\{G \mid F \subseteq G \in \mathcal{C}\} = \bigcap\{h^{-1}G \mid hF \subseteq G \in \mathcal{F}i_{\mathcal{S}}\mathbf{B}\} \\ &= h^{-1}\bigcap\{G \mid hF \subseteq G \in \mathcal{F}i_{\mathcal{S}}\mathbf{B}\} = h^{-1}\,\widetilde{\mathbf{\Lambda}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{B}}\, hF \in \operatorname{Con}\mathbf{A},\end{aligned}$$

since $\widetilde{\mathbf{\Lambda}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{B}}\, hF = \mathbf{\Omega}_{\mathbf{A}}F \in \operatorname{Con}\mathbf{B}$ for the Fregean basic full model $\mathfrak{B}$.

Thus any full model for $\mathcal{S}$ is Fregean. □

Theorem 0.7 provides a characterization for the fully Fregean deductive system in terms of properties of strong Fregean operator, but only for deductive systems over countable languages. The reason is that the given proof of the adequacy part is essentially based on Lemma 0.6, which doesn't hold in general, as shows the following counterexample.

Example 0.8. *There exists a deductive system $\mathcal{T}$ over a non countable language and 2nd-order models for $\mathcal{T}$*

$$\mathfrak{B} = \langle \mathbf{B}, \mathcal{B}\rangle \leqslant \mathfrak{A} = \langle \mathbf{A}, \mathcal{F}i_{\mathcal{T}}\mathbf{A}\rangle,$$

where $\mathbf{B}$ is countably generated, such that there is no a countably generated basic full model $\mathfrak{B}^*$ with the properties

$$\mathfrak{B} \leqslant \mathfrak{B}^* \leqslant \mathfrak{A}.$$

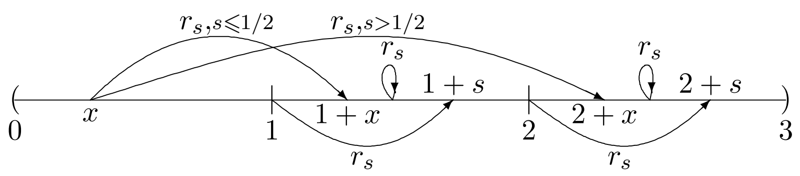

Figure 1: Algebra $\mathbf{A}$.

Proof. We will denote $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$, $[a,b) = \{x \in \mathbb{R} \mid a \leqslant x < b\}$, where $\mathbb{R}$ is the set of real numbers.

Let $\mathbf{A} := \left\langle (0,3); \{r_s^{\mathbf{A}}\}_{s\in(0,1)} \right\rangle$ (fig. 1), where

$$r_s^{\mathbf{A}}(x) = \begin{cases} x+1, & x \in (0,1), s \leqslant 1/2, \\ x+2, & x \in (0,1), s > 1/2, \\ x+s, & x = 1 \text{ or } 2, \\ x, & x \in (1,3) \setminus \{2\}. \end{cases}$$

Denote $\mathbf{B}$ the subalgebra of $\mathbf{A}$ generated by the set $\{1,2\}$. Then clearly

$$\mathbf{B} = \left\langle [1,3); \left\{ r_s^{\mathbf{A}} \restriction_{[1,3)} \mid s \in (0,1) \right\} \right\rangle.$$

Let $\vdash_{\mathcal{T}}$ be a consequence relation over the language of type $\Lambda = \{r_s^1\}_{s\in(0,1)}$

$$\vdash_{\mathcal{T}} := \{\Gamma \vdash x \mid x \in \Gamma\} \cup \{\Gamma, r_s(\phi) \vdash r_t(\phi) \mid s,t \in (0,1)\},$$

for all $\Gamma \subseteq \mathbf{Fm}_\Lambda$, $\phi \in \mathbf{Fm}_\Lambda$, and any variable x.

It is easy to check that $\vdash_{\mathcal{T}}$ is consequence relation. Therefore $\vdash_{\mathcal{T}}$ defines the deductive system $\mathcal{T} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\mathcal{T} \rangle$ as described in [5, p. 23].

Suppose $\mathbf{B}^*$ be a subalgebra of $\mathbf{A}$, such that

$$\langle \mathbf{B}, \mathcal{F}i_{\mathcal{T}}\mathbf{A} \cap \mathbf{B} \rangle \leqslant \langle \mathbf{B}^*, \mathcal{F}i_{\mathcal{T}}\mathbf{B}^* \rangle \leqslant \langle \mathbf{A}, \mathcal{F}i_{\mathcal{T}}\mathbf{A} \rangle.$$

We will establish several useful facts.

• $\mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{B}^*}([2,3)) = (1,3)$.

Let $a \in (1,2)$ and let $b = a-1$. Since $r_{3/4}(x) \vdash_{\mathcal{T}} r_{1/4}(x)$, and $r_{3/4}(b) = b+2 = a+1 \in [2,3) \subseteq \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{A}}([2,3))$, hence $r_{1/4}(b) = b+1 = a \in \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{A}}([2,3))$, so

$$(1,2) \subseteq \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{A}}([2,3)) = \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{A}\cap\mathbf{B}^*}([2,3)) = \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{B}^*}([2,3)),$$

since $[2,3)$ is a subset of $\mathbf{B}^*$, and $\langle \mathbf{B}^*, \mathcal{F}i_{\mathcal{T}}\mathbf{B}^*\rangle$ is a submatrix of $\langle \mathbf{A}, \mathcal{F}i_{\mathcal{T}}\mathbf{A}\rangle$.

Thus $(1,3) \subseteq \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{B}^*}([2,3))$, and it is easy to check that $(1,3)$ is a $\mathcal{T}$-filter itself.

• $\mathbf{B}^* = \mathbf{A}$.

Suppose $b \in (1,2)$. According to $(*)$

$$\mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{B}^*}([2,3)) = \bigcup_{n\in\omega}^{\infty} X_n^{[2,3)} \supseteq (1,2).$$

We will prove our statement by induction on n.

If $b \in X_{n+1}^{[2,3)} \setminus X_n^{[2,3)}$. Then there exists $\Gamma \vdash_{\mathcal{T}} \phi$ and $h : \mathbf{Fm}_\Lambda \to \mathbf{B}^*$, such that $h\Gamma \subseteq X_n^{[2,3]}$, $h\phi = b$. We have $\phi = r_t(\psi)$, since otherwise $\phi(x) = x$ and $b = hx \in h\Gamma \subseteq X_n^{[2,3)}$, a contradiction. Thus $b = r_t(h\psi)$ and there exists $s \in (0,1)$, such that $r_s(\psi) \in \Gamma$, so $r_s(h\psi) \in h\Gamma \subseteq X_n^{[2,3)}$. If

$h\psi \in (2,3)$, then $b = r_t(h\psi) = h\psi \in (2,3) \subseteq X_n^{[2,3)}$, a contradiction,

$h\psi \in \{1,2\}$, then $b = r_t(h\psi) = r_s(h\psi) \in h\Gamma \subseteq X_n^{[2,3)}$, a contradiction,

$h\psi \in (0,1)$, then $b = r_t(h\psi) = h\psi + 1$, so $h\psi = b - 1 \in \mathbf{B}^*$.

Thus, since $(1,2) \subseteq \mathrm{Clo}_{\mathcal{F}i_{\mathcal{T}}\mathbf{B}^*}([2,3))$, then $(0,1) \subseteq \mathbf{B}^*$, so $\mathbf{B}^* = \mathbf{A}$, which is not countably generated. □

Fregean, but not fully Fregean deductive system

In this section we will show the existence of a Fregean, but not fully Fregean deductive system. By Theorem 0.7, it suffices to construct a deductive system $\mathcal{S}$, for which $\widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}} = \widetilde{\mathbf{\Omega}}_{\mathcal{S}}$, but $\mathbf{\Omega}_{\mathbf{Fm}}F \subsetneq \widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}}F$, for some $F \in \mathrm{Th}\,\mathcal{S}$. Since $\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} = \mathbf{\Omega}_{\mathbf{Fm}}$ yields $\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} = \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}$, then for such deductive system $\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} \neq \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}$. Also, the existence of fully Fregean non protoalgebraic deductive system would mean that there exists $\mathcal{S}$, for which $\widetilde{\mathbf{\Lambda}}^{L}_{\mathrm{Th}\,\mathcal{S}} = \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}$, but $[F)^{L}_{\mathrm{Th}\,\mathcal{S}} \subsetneq [F)_{\mathrm{Th}\,\mathcal{S}}$ for some $F \in \mathrm{Th}\,\mathcal{S}$.

Let us consider the deductive systems $\mathcal{S}$ over the language of type $\Lambda = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, f^1\}$, such that

$$\mathrm{Ax}\,\mathcal{S} = \{\vdash \mathbf{1}, \vdash f\mathbf{2}\}, \quad \mathrm{Ru}\,\mathcal{S} = \left\{\frac{\mathbf{2}}{\mathbf{3}}, \frac{\mathbf{4}}{\mathbf{2}}, \frac{f\mathbf{4}}{\mathbf{3}}, \frac{\mathbf{3}}{f\mathbf{4}}, \frac{f\mathbf{3}}{\mathbf{3}}, \frac{x}{fx}, \frac{f^2x}{fx}\right\}$$

where, for brevity sake, we write fx instead of $f(x)$, and $f^{n+1}x := f(f^n x)$. Let $X_\omega = \{x_i\}_{i\in\omega}$ be the countable set of free generators for $\mathbf{Fm}_\Lambda$ (fig. 2).

We define, for every $a \in \mathbf{Fm}_\Lambda$, $a^\nabla := \{f^n a\}_{n\in\omega}$. Similarly, for any $S \subseteq \mathbf{Fm}_\Lambda$, $S^\nabla := \bigcup\{x^\nabla \mid x \in S\}$.

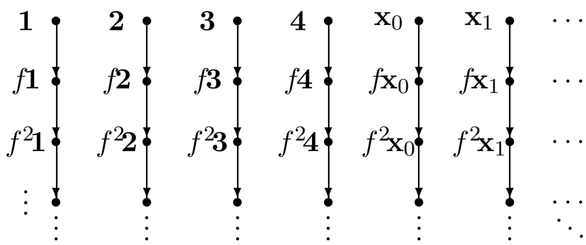

Figure 2: Algebra $\mathbf{Fm}_\Lambda$.

Lemma 0.9. *Let A, B, C, D be the following distinguished subsets of $\mathbf{Fm}_\Lambda$:*

$$A := \mathbf{1}^\nabla \cup (f\mathbf{2})^\nabla, \quad B := A \cup \mathbf{3}^\nabla \cup (f\mathbf{4})^\nabla, \quad C := B \cup \{\mathbf{2}\}, \quad D := C \cup \{\mathbf{4}\},$$

and let $\mathfrak{X}$ be the set of all subsets of $\mathbf{Fm}_\Lambda$ of the form $X \cup I^\nabla \cup f(J)^\nabla \in \mathrm{Th}\,\mathcal{S}$, where $X \in \{A, B, C, D\}$, $I, J \subseteq X_\omega$, $I \cap J = \emptyset$. Then

$$\mathfrak{X} = \mathrm{Th}\,\mathcal{S}.$$

Proof. In the closed form,

$$\mathfrak{X} = \{A, B, C, D\} \oplus \mathcal{P}^\nabla(X_\omega) \oplus \mathcal{P}^\nabla(fX_\omega),$$

where $U \oplus V := \{X \cup Y \mid X \in U, Y \in V\}$, $\mathcal{P}^\nabla(U) := \{Y^\nabla \mid Y \subseteq U\}$, $fU := \{fu\}_{u \in U}$. It easy to see (see fig. 3) that for every $T \in \mathrm{Th}\,\mathcal{S}$ and any $t \in \mathbf{Fm}_\Lambda$

(a) $t \in T \overset{x \vdash fx}{\Longrightarrow} t^\nabla \subseteq T$.

(b) $T \cap (ft)^\nabla \neq \emptyset \implies (ft)^\nabla \subseteq T$. Indeed

$$f^n t \in T \overset{f^2x \vdash fx}{\Longrightarrow} \{f^n t, f^{n-1} t, \ldots, ft\} \subseteq T \overset{(a)}{\Longrightarrow} (ft)^\nabla \subseteq T.$$

(c) $A \subseteq T \cap \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}^\nabla$. It follows from the fact that $A = \mathrm{Thm}\, S$.

(d) $((f\mathbf{3})^\nabla \cup (f\mathbf{4})^\nabla) \cap T \neq \emptyset \implies B \subseteq T$. Indeed, in either case:

$$(f\mathbf{3})^\nabla \cap T \neq \emptyset \overset{(b)}{\Rightarrow} (f\mathbf{3})^\nabla \subseteq T \overset{f\mathbf{3} \vdash \mathbf{3}}{\Longrightarrow} \mathbf{3} \in T,$$
$$(f\mathbf{4})^\nabla \cap T \neq \emptyset \overset{(b)}{\Rightarrow} (f\mathbf{4})^\nabla \subseteq T \overset{f\mathbf{4} \vdash \mathbf{3}}{\Longrightarrow} \mathbf{3} \in T.$$

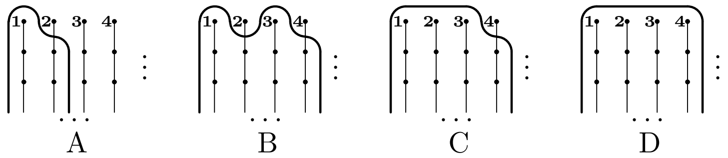

Figure 3: Theories A, B, C, D.

Furthermore

$$\mathbf{3}\in T\quad \left.\begin{array}{l} \overset{\mathbf{3}\vdash f\mathbf{4}}{\Longrightarrow}\ f\mathbf{4}\in T \overset{(a)}{\Rightarrow} (f\mathbf{4})^{\nabla}\in T \\ \overset{(a)}{\Longrightarrow}\ \mathbf{3}^{\nabla}\subseteq T \end{array}\right\} \Rightarrow A\cup\mathbf{3}^{\nabla}\cup(f\mathbf{4})^{\nabla} = B\subseteq T$$

$\boxed{\mathrm{Th}\,\mathcal{S}\subseteq\mathfrak{X}}$ Let $T\in\mathrm{Th}\,\mathcal{S}$ and denote $T_C := T\cap\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}^{\nabla}$.

Claim. *For any theory $T\in\mathrm{Th}\,\mathcal{S}$, $T_C\in\{A,B,C,D\}$.* **Proof.** Suppose

• $3\notin T_C$. Then $f\mathbf{4}\notin T_C$ holds, because of the rule $f\mathbf{4}\vdash\mathbf{3}$. Further, from $f^2x\vdash fx$, follows that $(f^2\mathbf{4})^{\nabla}\cap T_C=\emptyset$. Also, by $\mathbf{4}\vdash\mathbf{2}\vdash\mathbf{3}$, $\mathbf{4}\notin T_C$. Similarly $\mathbf{2}\vdash\mathbf{3}$ implies $\mathbf{2}\notin T_C$. Now, by $f\mathbf{3}\vdash\mathbf{3}$, $f\mathbf{3}\notin T_C$. Further, according to (b), $(f\mathbf{3})^{\nabla}\cap T_C=\emptyset$. Altogether, $T_C\subseteq\{\mathbf{1},f\mathbf{2}\}^{\nabla}=A$. Then, by (c), $T_C=A$.

• $\mathbf{3}\in T_C$. There are two possibilities

• $\mathbf{2}\notin T_C$. In this case $T_C=B$. Indeed, from $\mathbf{4}\vdash\mathbf{2}$ follows that $\mathbf{4}\notin T_C$. Since $\mathbf{3}\in T_C$, by (a), $\mathbf{3}^{\nabla}\subseteq T_C$. Also, by $\mathbf{3}\vdash f\mathbf{4}$, $f\mathbf{4}\in T_C$, and again, according to (a), $(f\mathbf{4})^{\nabla}\subseteq T_C$. Altogether, $T_C\subseteq A\cup(f\mathbf{4})^{\nabla}\cup\mathbf{3}^{\nabla}=B$, but $(f\mathbf{4})^{\nabla}\cup\mathbf{3}^{\nabla}\subseteq B$, therefore, by (d), $T_C=B$.

• $\mathbf{2}\in T_C$. Again, there are two cases

• $\mathbf{4}\notin T_C$. $\mathbf{2}\in T_C$ implies, by (a), that $\mathbf{2}^{\nabla}\subseteq T_C$. Since $\mathbf{3}\in T_C$, then again by (a), $\mathbf{3}^{\nabla}\subseteq T_C$. Also, according to the rule $\mathbf{3}\vdash f\mathbf{4}$, $\mathbf{3}\in T_C$, implies $f\mathbf{4}\in T_C$, and further, by (a), $(f\mathbf{4})^{\nabla}\subseteq T_C$. So $T_C=C$.

• $\mathbf{4}\in T_C$. Finally, if $\mathbf{4}\in T_C$, then $T_C=D$. □

Now, let denote for any $T\in\mathrm{Th}\,\mathcal{S}$

$$T_1 := \{x\in X_\omega \mid x^{\nabla}\cap T\neq\emptyset \ \&\ x\in T\}\subseteq X_\omega,$$
$$T_2 := \{x\in X_\omega \mid x^{\nabla}\cap T\neq\emptyset \ \&\ x\notin T\}\subseteq X_\omega.$$

By (a), $T_1^{\nabla}\subseteq T$, $(fT_2)^{\nabla}\subseteq T$. Also, if $f^n x\in T$ for some $n\in\omega$, then $x\in T_1$ or $x\in T_2$. Thus $T\cap X_\omega^{\nabla}=T_1^{\nabla}\cup(fT_2)^{\nabla}\in\mathcal{P}^{\nabla}(X_\omega)\oplus\mathcal{P}^{\nabla}(fX_\omega)$, therefore $T\in\mathfrak{X}$.

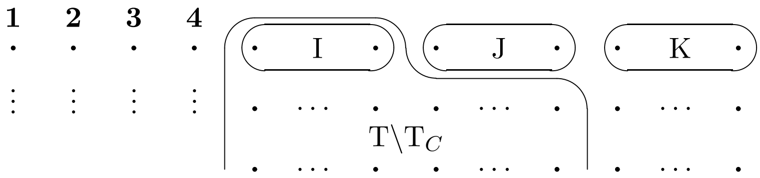

Figure 4: The sets I, J, K for a given theory T on the algebra $\mathbf{Fm}_\Lambda$.

$\boxed{\mathfrak{X} \subseteq \mathrm{Th}\,\mathcal{S}}$ Let $T \in \mathfrak{X}$. Then we can write $T = T_C \cup T_1 \cup T_2$, where $T_C \in \{A, B, C, D\}$, $T_1 \in \mathcal{P}^\nabla(X_\omega)$, $(fT_2) \in \mathcal{P}^\nabla(fX_\omega)$, and T_1, T_2 can be chosen disjoint. Obviously, A, B, C, D are closed under the rules: $\vdash \mathbf{1}$, $\vdash f\mathbf{2}$, $\mathbf{2} \vdash \mathbf{3}$, $\mathbf{4} \vdash \mathbf{2}$, $f\mathbf{4} \vdash \mathbf{3}$, $\mathbf{3} \vdash f\mathbf{4}$, $f\mathbf{3} \vdash \mathbf{3}$. Now, since $x \in T$ implies $fx \in T_1^\nabla$ and $T_1^\nabla \subseteq T$, therefore T is closed under the rule $x \vdash fx$. Finally, by definition of $\mathfrak{X}$, for every $y \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\} \cup X_\omega$, $y^\nabla \cap T$ equals to $\emptyset$, y^∇ or $(fy)^\nabla$. So if $f^2t \in T$ for some $t \in \mathbf{Fm}_\Lambda$, then $ft \in T$, therefore T is closed under the rule $f^2x \vdash fx$. Thus T is a $\mathcal{S}$-theory. □

Proposition 0.10. *$\mathcal{S}$ is Fregean.*

Proof. According to Lemma 0.1, it suffices to show that $\widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}} T \subseteq \mathbf{\Omega}_{\mathbf{Fm}} T$ for every $T \in \mathrm{Th}\,\mathcal{S}$. For a given $T \in \mathrm{Th}\,\mathcal{S}$, we split up $\mathbf{Fm}_\Lambda$ into three pieces

$$T,\ T^\emptyset := \{t \in \mathbf{Fm}_\Lambda \mid t^\nabla \cap T = \emptyset\},\ T^f := \{t \in \mathbf{Fm}_\Lambda \mid t \notin T \& ft \in T\}.$$

Then

- $\{T, T^\emptyset, T^f\}$ is a partition of $\mathbf{Fm}_\Lambda$. Clearly T, $T^\emptyset$, T^f are disjoint. Now, let $t \in \mathbf{Fm}_\Lambda$ and $t \notin T$, $t^\nabla \cap T \neq \emptyset$, then $f^n t \in T$ for some $n \in \omega - \{0\}$. By (b), $ft \in T$, so $t \in T^f$, therefore $T \cup T^\emptyset \cup T^f = \mathbf{Fm}_\Lambda$.
- $\{T, T^\emptyset, T^f\}$ is a partition of some congruence θ_T. Indeed, whenever $\langle a, b\rangle \in T \times T$, $T^\emptyset \times T^\emptyset$, $T^f \times T^f$, then $\langle fa, fb\rangle \in T \times T$, $T^\emptyset \times T^\emptyset$, $T \times T$, correspondingly.
- $\theta_T = \mathbf{\Omega}_{\mathbf{Fm}} T$. Clearly, θ_T is compatible with T. Now, suppose $\langle a, b\rangle \in T^\emptyset \times T^f$, then $\langle fa, fb\rangle \in T^\emptyset \times T$. Thus θ_T is the largest congruence compatible with T.

Therefore, for each $T = X \cup I^\nabla \cup (fJ)^\nabla \in \mathrm{Th}\,\mathcal{S}$, where $X \in \{A, B, C, D\}$, $I, J \subseteq X_\omega$, $I \cap J = \emptyset$, and $K := X_\omega \setminus (I \cup J)$ (see fig. 4)

if $X = A$, then	$T^f = \{\mathbf{2}\} \cup J$,	$T^\emptyset = \{\mathbf{3}, \mathbf{4}\}^\nabla \cup K^\nabla$,
if $X = B$, then	$T^f = \{\mathbf{2}, \mathbf{4}\} \cup J$,	$T^\emptyset = K^\nabla$,
if $X = C$, then	$T^f = \{\mathbf{4}\} \cup J$,	$T^\emptyset = K^\nabla$,
if $X = D$, then	$T^f = J$,	$T^\emptyset = K^\nabla$.

We will show now that, if $\langle a,b\rangle \notin \mathbf{\Omega_{Fm}}T$ (this means $\langle a,b\rangle \in T\times T^{\emptyset}$, $T\times T^{f}$ or $T^{\emptyset}\times T^{f}$), then $\langle a,b\rangle \notin \widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}T$ (i.e., there is a theory $S\supseteq T$, such that $a\in S$, $b\notin S$, or the other way around).

Obviously, if $\langle a,b\rangle \in T\times T^{\emptyset}$ or $T\times T^{f}$, then T itself separates a from b. Further, if $\langle a,b\rangle \in T^{\emptyset}\times T^{f}$, there are four different cases:

- If $a,b\in\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}^{\nabla}$, then $X=A$ and $a\in\{\mathbf{3},\mathbf{4}\}^{\nabla}$, $b=\mathbf{2}$. If $a\in\{\mathbf{3},f\mathbf{4}\}^{\nabla}$, then $a\in S:=B\cup I\cup(fJ)^{\nabla}\in[T)_{\mathrm{Th}\,\mathcal{S}}$, $b\notin S$. If $a=\mathbf{4}$, then $a\notin S:=C\cup I\cup(fJ)^{\nabla}\in[T)_{\mathrm{Th}\,\mathcal{S}}$, $b\in S$.
- If $a\in\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}^{\nabla}$, $b\in X_{\omega}^{\nabla}$, then $a\in S:=D\cup I\cup(fJ)^{\nabla}\in[T)_{\mathrm{Th}\,\mathcal{S}}$, $b\notin S$.
- If $a\in X_{\omega}^{\nabla}$, $b\in\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}^{\nabla}$, then $\mathbf{b}\in\{\mathbf{2},\mathbf{4}\}$, and $a\notin S:=D\cup I\cup(fJ)^{\nabla}\in[T)_{\mathrm{Th}\,\mathcal{S}}$, $b\in S$.
- If $a,b\in X_{\omega}^{\nabla}$, then $a\notin S:=X\cup(I\cup J)^{\nabla}\in[T)_{\mathrm{Th}\,\mathcal{S}}$, $b\in J\subseteq S$.

Thus, for every $T\in\mathrm{Th}\,\mathcal{S}$, $\widetilde{\mathbf{\Lambda}}_{\mathrm{Th}\,\mathcal{S}}T\subseteq\mathbf{\Omega_{Fm}}T$, therefore, by Lemma 0.1, $\mathcal{S}$ is Fregean. □

Proposition 0.11. *$\mathcal{S}$ is not fully self-extensional, hence not fully Fregean.*

Proof. Let $\mathbf{K}$ be an algebra over the language of type $\Lambda=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},f^{1}\}$

$$\mathbf{K}:=\left\langle\{a,b,c\};\mathbf{1^K},\mathbf{2^K},\mathbf{3^K},\mathbf{4^K},f^{\mathbf{K}}\right\rangle,$$

where $\mathbf{1^K}=a$, $\mathbf{2^K}=b$, $\mathbf{3^K}=\mathbf{4^K}=c$, $f^{\mathbf{K}}a=f^{\mathbf{K}}b=a$, $f^{\mathbf{K}}c=c$. Then

$$\langle\mathbf{K},\{\{a\},\{a,b,c\}\}\rangle$$

is a reduced full model for $\mathcal{S}$. Indeed $\{a\}$ is the smallest $\mathcal{S}$-filter on $\mathbf{K}$, since $\mathbf{1}=f\mathbf{2}=a\in\{a\}$, and if $f^{2}x\in\{a\}$, then $fx=a\in\{a\}$. Meanwhile, $\{a,b\},\{a,c\}$ are not $\mathcal{S}$-filters, because from $\mathbf{2}=b\in\{a,b\}$, by $\mathbf{2}\vdash\mathbf{3}$, follows $\mathbf{3}=c\in\{a,b\}$, a contradiction. Similarly, $\mathbf{4}=c\in\{a,c\}$ implies, by $\mathbf{4}\vdash\mathbf{2}$, that $\mathbf{2}=b\in\{a,c\}$, a contradiction.

Thus $\widetilde{\mathbf{\Lambda}}\,\mathcal{F}i_{\mathcal{S}}\mathbf{K}=\widetilde{\mathbf{\Lambda}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{K}}\{a\}=\{a\}^{2}\cup\{b,c\}^{2}$, where X^{2} is the direct product $X\times X$ of a set X. But $\widetilde{\mathbf{\Omega}}\,\mathcal{F}i_{\mathcal{S}}\mathbf{K}=\mathbf{\Omega_K}\{a\}=\mathrm{id}_{\mathbf{K}}$. The latter is true, since from $\langle b,c\rangle\in\mathbf{\Omega_K}\{a\}$ follows that $\langle fb,fc\rangle=\langle a,c\rangle\in\mathbf{\Omega_K}\{a\}$, that contradicts the fact, that $\mathbf{\Omega_K}\{a\}$ is compatible with $\{a\}$.

Thus $\widetilde{\mathbf{\Lambda}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{K}}\,\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{K}}(\emptyset)\neq\widetilde{\mathbf{\Omega}}_{\mathcal{F}i_{\mathcal{S}}\mathbf{K}}\,\mathrm{Clo}_{\mathcal{F}i_{\mathcal{S}}\mathbf{K}}(\emptyset)$, so $\langle\mathbf{K},\mathcal{F}i_{\mathcal{S}}\mathbf{K}\rangle$ is not self-extensional, and therefore also not Fregean. At the same time, by Lemma 0.10, $\mathcal{S}$ is Fregean, and therefore self-extensional. □

Relations between different semantics for $\mathcal{S}$

We will use definitions and notation from [5]: a 1st–order matrix $\langle \mathbf{A}, F\rangle$ is *reduced* if $\mathbf{\Omega_A} F = \mathrm{id}_\mathbf{A}$, $\mathrm{Matr}^*\,\mathcal{S}$ is the class of all reduced matrices of $\mathcal{S}$, $\mathrm{Alg}^*\,\mathcal{S}$ is the class of algebra reducts of all matrices in $\mathrm{Matr}^*\,\mathcal{S}$, $\mathrm{Alg}\,\mathcal{S}$ is the class of algebras of generalized models for $\mathcal{S}$ such that $\widetilde{\mathbf{\Omega}}_\mathbf{A}\,\mathcal{C} = \mathrm{id}_\mathbf{A}$. $\mathbf{K}_\mathcal{S}$ is the variety generated by $\mathbf{Fm}_\Lambda/\widetilde{\mathbf{\Omega}}\,\mathrm{Thm}\,\mathcal{S}$, $\mathbf{H}K$ is the class of homomorphic images of algebras from K.

Example 0.12. *For the deductive system $\mathcal{S}$:* $\mathrm{Alg}^*\,\mathcal{S} \subsetneq \mathrm{Alg}\,\mathcal{S} \subsetneq K_\mathcal{S}$.

Proof.
($\mathrm{Alg}^*\,\mathcal{S} \subsetneq \mathrm{Alg}\,\mathcal{S}$) Let $\mathbf{M}$ be an algebra over the language of type $\Lambda = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, f\}$ such that

$$\mathbf{M} := \left\langle \{1,2,3,4\}; \mathbf{1}^\mathbf{M}, \mathbf{2}^\mathbf{M}, \mathbf{3}^\mathbf{M}, \mathbf{4}^\mathbf{M}, f^\mathbf{M}\right\rangle,$$

where $\mathbf{1}^\mathbf{M} = 1, \mathbf{2}^\mathbf{M} = 2, \mathbf{3}^\mathbf{M} = 3, \mathbf{4}^\mathbf{M} = 4$, $f^\mathbf{M}1 = f^\mathbf{M}2 = 1, f^\mathbf{M}3 = f^\mathbf{M}4 = 3$.

It is easy to see, that $\mathcal{F}i_\mathcal{S}\mathbf{M} = \{\{1\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\}$, and

$$\begin{aligned}
&\mathbf{\Omega_M}\{1\} = \{1\}^2 \cup \{2\}^2 \cup \{3,4\}^2,\\
&\mathbf{\Omega_M}\{1,3\} = \{1,3\}^2 \cup \{2,4\}^2,\\
&\mathbf{\Omega_M}\{1,2,3\} = \{1,2,3\}^2 \cup \{4\}^2,\\
&\mathbf{\Omega_M}\{1,2,3,4\} = \{1,2,3,4\}^2.
\end{aligned}$$

Thus $\widetilde{\mathbf{\Omega}}\,\mathcal{F}i_\mathcal{S}\mathbf{M} = \bigcap\{\mathbf{\Omega_M}F \mid F \in \mathcal{F}i_\mathcal{S}\mathbf{M}\} = \mathrm{id}_\mathbf{M}$, therefore $\mathbf{M} \in \mathrm{Alg}\,\mathcal{S}$.

On the other hand, since for every $F \in \mathcal{F}i_\mathcal{S}\mathbf{M}$, $\mathbf{\Omega_M}F \neq \mathrm{id}_\mathbf{M}$, then $\mathbf{M} \notin \mathrm{Alg}^*\,\mathcal{S}$.
($\mathrm{Alg}\,\mathcal{S} \subsetneq K_\mathcal{S}$) Let

$$\mathbf{N} = \left\langle \{a,b\}, \mathbf{1}^\mathbf{N}, \mathbf{2}^\mathbf{N}, \mathbf{3}^\mathbf{N}, \mathbf{4}^\mathbf{N}, f^\mathbf{N}\right\rangle,$$

be an algebra over the language of type $\Lambda = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, f^1\}$, where $\mathbf{1}^\mathbf{N} = \mathbf{2}^\mathbf{N} = a, \mathbf{3}^\mathbf{N} = \mathbf{4}^\mathbf{N} = b, f^\mathbf{N}a = a, f^\mathbf{N}b = b$. Then $\mathcal{F}i_\mathcal{S}\mathbf{N} = \{\{a,b\}\}$ and $\widetilde{\mathbf{\Omega}}\,\mathcal{F}i_\mathcal{S}\mathbf{N} \neq \mathrm{id}_\mathbf{N}$, so $\mathbf{N} \notin \mathrm{Alg}\,\mathcal{S}$. But clearly $\mathbf{N} \in K_\mathcal{S} = \mathbf{H}\,\mathrm{Alg}^*\,\mathcal{S}$, as a homomorphic image of $\mathbf{M} \in \mathrm{Alg}\,\mathcal{S}$. □

Although the second strict inclusion $\mathrm{Alg}\,\mathcal{S} \subsetneq K_\mathcal{S}$ seems to require some additional assumptions, it is possible to show that the property $\mathrm{Alg}^*\,\mathcal{S} \subsetneq \mathrm{Alg}\,\mathcal{S}$ holds for any Fregean, but not fully Fregean deductive system with theorems.

Theorem 0.13. *Let $\mathcal{S}$ be a deductive system with* $\mathrm{Thm}\, S \neq \emptyset$ *and let* $t \in \mathrm{Thm}\, S$ *be any theorem. If* $F = t/\,\widetilde{\mathbf{\Omega}}\, F$ *for every theory* $F \in \mathrm{Th}\, \mathcal{S}$ *then* $\mathcal{S}$ *protoalgebraic if and only if* $\mathrm{Alg}^* \mathcal{S} = \mathrm{Alg}\, \mathcal{S}$.

Proof. If $\mathcal{S}$ is protoalgebraic then $\mathrm{Alg}^* \mathcal{S} = \mathrm{Alg}\, \mathcal{S}$, by [5, Proposition 3.2].

Let $\mathcal{S}$ be non-protoalgebraic. Suppose $\widetilde{\mathbf{\Omega}}\, \mathrm{Th}\, \mathcal{S} = \{\widetilde{\mathbf{\Omega}}\, T \mid T \in \mathrm{Th}\, \mathcal{S}\} \subseteq \mathbf{\Omega}\mathrm{Th}\, \mathcal{S} = \{\mathbf{\Omega}T \mid T \in \mathrm{Th}\, \mathcal{S}\}$. Then, for every $G \in \mathrm{Th}\, \mathcal{S}$, from $\widetilde{\mathbf{\Omega}}\, G = \mathbf{\Omega} H$, it follows that $G = t/\,\widetilde{\mathbf{\Omega}}\, G = t/\,\mathbf{\Omega} H = H$ for any $t \in \mathrm{Thm}\, S$. Thus $\widetilde{\mathbf{\Omega}} = \mathbf{\Omega}$. Therefore, by [4, Lemma 2.12], $\mathcal{S}$ is protoalgebraic, a contradiction.

Thus $\widetilde{\mathbf{\Omega}}\, \mathrm{Th}\, \mathcal{S} \not\subseteq \mathbf{\Omega}\mathrm{Th}\, \mathcal{S}$, i.e., there is a $F \in \mathrm{Th}\, \mathcal{S}$, such that $\widetilde{\mathbf{\Omega}}\, F \notin \mathbf{\Omega}\mathrm{Th}\, \mathcal{S}$. Let $\mathbf{A} = \mathbf{Fm}_\Lambda / \widetilde{\mathbf{\Omega}}\, F$ and let $h : \mathbf{Fm}_\Lambda \twoheadrightarrow \mathbf{A}$ be the canonical homomorphism. Thus $\mathbf{A} \in \mathrm{Alg}\, \mathcal{S}$. Suppose also $\mathbf{A} \in \mathrm{Alg}^* \mathcal{S}$ and G be a $\mathcal{S}$-filter on $\mathbf{A}$ such that $\mathbf{\Omega}_{\mathbf{A}} G = \mathrm{id}_{\mathbf{A}}$. Then $\mathbf{\Omega} h^{-1} G = h^{-1}\, \mathbf{\Omega}_{\mathbf{A}} G = h^{-1}\, \mathrm{id}_{\mathbf{A}} = \widetilde{\mathbf{\Omega}}\, F$, so $\widetilde{\mathbf{\Omega}}\, F \in \mathbf{\Omega}\mathrm{Th}\, \mathcal{S}$, a contradiction. Thus $\mathbf{A} \notin \mathrm{Alg}^* \mathcal{S}$, and $\mathrm{Alg}^* \mathcal{S} \subsetneq \mathrm{Alg}\, \mathcal{S}$. □

Note that the class of deductive systems in the Theorem 0.13 is significantly larger than just Fregean deductive systems and corresponds to the "algebraizable logics" (in the old sense), the logics that have an equivalent algebraic semantics (as opposed to the matrix semantics) for their theorems. In particular, the theorems of those logics are represented by a single element in the Lindenbaum-Tarski algebra. This class includes normal modal logics, which often are not Fregean.

Corollary 0.14. *For every Fregean deductive system $\mathcal{S}$ over a countable language Λ with* $\mathrm{Thm}\, \mathcal{S} \neq \emptyset$*, if* $\mathrm{Alg}^* \mathcal{S} = \mathrm{Alg}\, \mathcal{S}$ *then $\mathcal{S}$ is protoalgebraic, hence fully Fregean.*

Proof. Fix a term $t \in \mathrm{Thm}\, \mathcal{S}$. For any equivalence $\theta \in \mathrm{Eq}\, \mathbf{Fm}_\Lambda$, t/θ will denote the θ-equivalence class containing t.

Trivially, $t/\widetilde{\mathbf{\Lambda}} F = F$ for every $F \in \mathrm{Th}\, \mathcal{S}$. Since $\mathcal{S}$ is Fregean $\widetilde{\mathbf{\Lambda}} = \widetilde{\mathbf{\Omega}}$, therefore $F = t/\widetilde{\mathbf{\Lambda}} F = t/\widetilde{\mathbf{\Omega}} F$.

Thus the Theorem 0.13 is applicable and therefore $\mathcal{S}$ is protoalgebraic. So, by [4, Corollary 3.5], $\mathcal{S}$ is a fully Fregean deductive system. □

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