REPORTS ON MATHEMATICAL LOGIC
37 (2003), 41–57

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ON THE FIRST ORDER LOGIC TRUE IN EVERY UNIVERSE (INCLUDING THE EMPTY UNIVERSE)

A b s t r a c t. As it is well known, the set of theorems: Th_{LP} of the first order predicate calculus LP contains all formulae which are valid in every universe with non-empty set of individuals. In this paper I investigate a certain restriction LM of LP. Its set of theorems contains only the theorems valid also in the "empty-universe" (i.e. in the universe with empty set of individuals).

0.1. Historical note

Fifty years ago A.Mostowski published in the Journal of Symbolic Logic a paper in which he gives the definition of a new system F_p^* of first order predicate calculus F_p^1.

The idea of Mostowski (cf. Grzegorczyk [1], p.171) was to restrict the logic to the predicate calculus laws valid in all universes, including

Received 31 December 2001

the empty one. Grzegorczyk cites in his monograph [1] the arguments of Mostowski in favor of this restricted logic:

"Making a distinction between those formulae which are true in all domains, the empty domain included, and those which are only true in non-empty domains, at the first glance, seem to be unnecessary pedantry. This impression vanishes if we realize that in the applications of logic to mathematics we usually have to do with formulae in which quantifiers are restricted to certain sets (domains). In mathematics, we hardly ever say "for every x", but we almost always say "for every point x", "for every number x", etc. No less frequent are the cases in which the domain to which the quantifier is restricted depends on a parameter or parameters; for instance, in the case of the formulation "for every real root of the equation..." the coefficients of that equation are such parameters. In such a case the domain may become empty for new values of the parameter, and this is why only those logical theorems which are true in every domain, the empty domain included, are applicable to it without any reservation whatever. "

Mostowski modestly admits ([4], p.111) that his system, symbolized by F_p^*, "is by far not as elegant as the usual predicate calculus".

Ten years latter Grzegorczyk published the monograph [1] "An Outline of Mathematical Logic", where the axiomatic exposition of the classical first order predicate calculus is decomposed into two parts. The first one (pp. 172–181), based on 11 axioms schemata, contains the proofs of theorems: 11–79, which are true also in the empty domain. In the second part he adds a supplementary axiom, true only in *non-empty* domains. The entire axiom system defines the set of theorems called by Grzegorczyk: "classical logical calculus L". This set contains all *closed* theorems of the first order predicate calculus. Likewise, the first part (based on 11 axiom schemata) contains all *closed* theorems of the Mostowski's system F_p^*.

Grzegorczyk (p. 154) brings out two important characteristics of his approach to the classical logical calculus: the utilization of only one rule of inference and the form of particular axioms. He remarks justly that:

1^0 "The reduction of all steps in the proofs to successive operations of detachment (...), simplifies (...) the concept of proof and makes it possible to present more elegantly certain ideas on logic as a mathematical theory."

2^0 "In this approach the axioms are probably more convincing from the intuitive point of view then the rules which occur in other approaches based on rules and axioms."

Grzegorczyk, did not give "the logic restricted to laws true in empty universe" a name. Perhaps ten years of the existence of a theory were too few. Actually, when the usefulness of this logic is confirmed, it seems necessary to do this. In this paper the system created by Mostowski, as a restriction F_p^* of classical first order predicate calculus LP will be symbolized by LM and called "(minimized) Mostowski's logic".

0.2. Topics

I start here from the axiomatic system of the logic LM, presented by Grzegorczyk in his book, and based on research carried out by Mostowski, Hailperin and Quine ([4], [2], [5]). The Grzegorczyk's axiom system will be symbolized here by $AXLM1$.

In this paper

1. I present some modification $AXLM2$ of $AXLM1$, which differs only slightly from it, but permits to abridge proofs;
2. I prove the dependence of two Grzegorczyk's axioms;
3. I present the thirds equivalent axiom system $AXLM3$ composed of 3 axiom schemata of Grzegorczyk and two others. (One of new axioms was suggested by Grzegorczyk ([1], p. 180).)

This paper, which seems only a small contribution to the Grzegorczyk's monograph, has moreover two other scopes:

1^0 To do honor to the memory of A. Mostowski on the fiftieth anniversary of creation by him the logic LM (in 1951),

2^0 To prove the last metatheorem of this paper (THEOREM 3.1), which is indispensable for my next paper (in preparation) on the "logic of domains".

0.3. Definitions

In this paper two logical systems are examined:

1. The system LP of the (classical) first order predicate calculus,
2. Its subsystem LM, composed of only these theorems of LP, which hold even in the empty universe.

For example the formula $\exists x[P^1(x) \to P^1(x)]$ is a theorem of LP, but it does not belong to the set of theorems of LM.

We fix now some sets, which will be used in the metalanguage to describe the investigated logical systems. To distinguish metatheorems from logical theorems expressed in the object language, we write here the names of these metatheorems using capital letters (THEOREM instead of Theorem and similarly: LEMMA, DEFINITION).

DEFINITION 0.1.

a) To define the common language used in the both logical systems: LP and LM, we fix the set Sf of all well-formed sentential formulae (closed as well as open) and by $\overline{S}f$ we denote its subset composed of only closed sentences.

b) To construct sentential formulae, we use here
the (individual) variables: x, x_0, x_1, x_2, $x_3, \ldots$,
the n-ary predicative symbols P^n, P_0^n, P_1^n, P_2^n, $P_3^n, \ldots$,
the logical constants: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$,
parentheses and the comma.

c) The set of all *theorems* of LP will be denoted here by Th_{LP}, and its subset containing only closed formulae by $\overline{T}h_{LP}$.

d) The set of all *theorems* of LM will be denoted here by Th_{LM}. This set (based on the axiomatics due to Grzegorczyk) is a subset of the Mostowski's set F_p^*, and contains as theorems the closed formulae only, and hence one has $Th_{LM} \subseteq \overline{S}f$.

e) The Greek letters will be used to denote formulae belonging to Sf and their parts. In particular:

ε_k, δ_k stand for arbitrary concatenations of logical symbols,

ξ, η, ζ - stand for arbitrary individual variables: x_k,

α, β, γ, $\alpha', \beta', \gamma', \ldots$ stand for arbitrary (well-formed) sentential formulae i.e. the elements of the set Sf.

f) In the metalanguage we use the usual set-theoretic symbols: $\in$, $\subseteq$, $\notin$, $\cup$, $\cap$, $\{\varepsilon\}$, $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, $\{\varepsilon \in X : \ldots\}$.

g) The *identity* symbol: " $=$ " will be used for the identity of sets, as as well for the equality of formulae (e.g.($\overline{T}h_{LP} = \overline{S}f \cap Th_{LP}$, $\alpha = \neg P^1(x)$).

DEFINITION 0.2. In order to formalize the metalanguage applied in the description of schemata of logical axioms we propose to use in this paper the following supplementary meta-symbols:

a) The set of all *individual variables* will be denoted by Vr, and for every (well-formed) sentential formula α the symbol $Vr(\alpha)$ stands for the set of *free variables occurring in* the formula α.

b) $NVr(\alpha)$ stands for the number of *free variables occurring in* α i.e. the cardinality of the set $Vr(\alpha)$.

c) $Sb(\alpha : \eta)$ stands for the set of variables *substitutable in* α for η i.e. the set of such variables ξ that α does not contain any quantifier binding the variable ξ in the scope of which a variable η, free in α, occurs.

d) $sb(\alpha : \eta/\xi)$ is the sentential formula, which differs from α only in having all the variables η free in α replaced by the variables ξ.

e) Th_S stands for the set of all (well-formed) sentential formulae α belonging to Sf obtained from an arbitrary theorem of classical sentential calculus in which every sentential variable was substituted by some well-formed sentential formula from Sf.

To obtain theorems resulting from his axiom system, Grzegorczyk uses only one inference rule, namely the *rule of detachment*. This well known rule, can be defined as follows

DEFINITION 0.3. Let Z be a given set of formulae. We say that "*the rule RD holds in Z*" iff for every formulae α and β belonging to Sf, If $[\alpha \to \beta] \in Z$ and $\alpha \in Z$, then $\beta \in Z$.

We define the set of all theorems of Mostowski's logic LM, as the least set of sentential formulae containing $AXLM1$ in which the rule RD holds.

We can express this definition using Tarski's concept of *consequence by detachment* Cn defined as follows ([6], [7], [1] p.191):

DEFINITION 0.4. A formula α is a consequence by detachment of a set X of formulae (in symbols: $\alpha \in Cn(X)$) iff there exists a finite sequence of formulae: $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$, such that $\alpha = \alpha_n$, and for every $m \leq n$ either $\alpha_m \in X$ or there are i and j such that $1 \leq i, j < m$ and $\alpha_j = [\alpha_i \to \alpha_m]$.

The axiom system of Grzegorczyk is composed of 11 schemata of axioms : I, II.1-II.5, III.6-III.10 and for proving from them theorems he uses ([1], p. 161-3) only one *rule of detachment*.

To express 11 axioms of Grzegorczyk, we state here the following schemata, which serve to construct these axioms:

DEFINITION 0.5. For $n \in \{0, 1, ..., 10\}$ the symbol Ax_0n denotes the set of all formulae falling under the n-th axiom-scheme:

Let α, $\beta \in Sf$ and ξ, η, $\zeta \in Vr$.

0) If $\alpha \in Th_S$, then $\alpha \in Ax_0 0$;
1) $[\forall\xi\, [\alpha \to \beta] \to [\forall\xi\alpha \to \forall\xi\beta]] \in Ax_0 1$;
2) If $\xi \in Vr(\alpha)$, then $[\forall\xi\alpha \to \alpha] \in Ax_0 2$;
3) $[\forall\xi\forall\eta\alpha \to \forall\eta\forall\xi\alpha] \in Ax_0 3$;
4) If $\xi \in Sb(\alpha : \eta)$, then $[\forall\xi\forall\eta\alpha \to \forall\xi sb\,(\alpha : \eta/\xi)] \in Ax_0 4$;
5) If $\xi \notin Vr(\alpha)$, then $[\alpha \to \forall\xi\alpha] \in Ax_0 5$;

6) $[\forall\xi\,[\alpha \to \beta] \to [\exists\xi\alpha \to \exists\xi\beta]] \in Ax_0 6$;
7) If $\xi \in Vr(\alpha)$, then $[\alpha \to \exists\xi\alpha] \in Ax_0 7$;
8) $[\exists\xi\exists\eta\alpha \to \exists\eta\exists\xi\alpha] \in Ax_0 8$;
9) If $\eta \in Sb(\alpha : \xi)$, then $[\exists\eta sb\,(\alpha : \xi/\eta) \to \exists\eta\exists\xi\alpha] \in Ax_0 9$;
10) If $\xi \in Vr(\alpha)$, then $[\exists\xi\alpha \to \alpha] \in Ax_0 10$.

To formulate his schemata of axioms Grzegorczyk uses the notion of the *closure of a formula*. To formalize this notion, we propose to define here the concept of *prefixes*, which serve to obtain the *universal closure* $\delta\alpha$ *of a sentential formula* α.

DEFINITION 0.6. The set Pf of all *prefixes* is the smallest set of sequences: $\varepsilon_1\varepsilon_2\varepsilon_3 \ldots \varepsilon_k$ of logical symbols satisfying the following conditions:

1. The empty sequence (i.e. the sequence composed of 0 elements), is denoted by symbol ϑ and belongs to Pf.
2. For every $\xi \in Vr$, and $\delta \in Pf$, the sequence $\forall\xi\delta$ belongs to Pf.

The *number of quantifiers* $Nq(\delta)$ occurring in a prefix δ belonging to Pf is defined by the conditions:

1. $Nq(\vartheta) = 0$.
2. If $Nq(\delta) = n$, then $Nq(\forall\xi\delta) = n + 1$.

If δ has the form $\forall\xi_n \ldots \forall\xi_2\forall\xi_1$, then each $\forall\xi_k$ (where $k = 1, 2, ..., n$) is said *element of the prefix* δ.

The prefix ϑ will be called *empty prefix* and the other prefixes – *non-empty prefixes*.

For each prefix δ a prefix obtained by a change of order of its elements will be called a *permutation* of δ.

For an arbitrary sentential formula α:

a) $Pf(\alpha) = \{\delta \in Pf : \delta\alpha \in \overline{S}f\}$.

If $\delta \in Pf(\alpha)$, then one says: "δ is a *prefix closing* α".

b) $Pf^0(\alpha) = \{\delta \in Pf(\alpha) : Nq(\delta) = NVr(\alpha)\}$.

If $\delta \in Pf^0(\alpha)$, then δ is called "*minimal prefix closing* α".

Directly from DEFINITION 0.6 we deduce the following properties of prefixes, which can be used to prove some theorems and derived inference rules:

LEMMA 0.1. *For an arbitrary sentential formula* α*:*

a) $Pf^0(\alpha) \subset Pf(\alpha)$.

b) If δ, $\delta' \in Pf^0(\alpha)$, *then* δ' *is a permutation of* δ.

c) If $NVr(\alpha) = n$ *then the cardinality of* $Pf^0(\alpha) = n!$ *and the cardinality of* $Pf(\alpha) = \aleph_0$.

d) If $\alpha \in \overline{S}f$, *then* $Pf^0(\alpha) = \{\vartheta\}$.

e) If $\alpha \in Sf$, *then there exists* δ *belonging to* $Pf^0(\alpha)$.

f) If $\delta \in Pf$ *and* $\delta\alpha \in \overline{S}f$, *then* $\delta \in Pf(\alpha)$.

g) If $\delta\alpha = \forall\xi_1\forall\xi_2\ldots\forall\xi_{k-1}\forall\xi_k\forall\xi_{k+1\ldots}\forall\xi_m\alpha \in \overline{S}f$ *and if for each* k *belonging to* $\{1, m\}$ *one has* $\xi_k \in Vr\,(\forall\xi_{k+1\ldots}\forall\xi_m\alpha)$, *then* $\delta \in Pf^0(\alpha)$.

The syntactical definition of the first order logical system containing only theorems true in every universe, given in [1], can be expressed (taking into account DEFINITION 0.4, DEFINITION 0.5 and DEFINITION 0.6) as follows.

DEFINITION 1.

a) For $n \in \{0, 1, ..., 10\}$, a formula α will be called ”*an axiom falling under the schema* Ax_1n” iff there are $\gamma \in Ax_0n$ and $\delta \in Pf^0(\gamma)$, such that $\alpha = \delta\gamma$. The set of all these formulae α will be denoted by Ax_1n.

b) By the axiomatics $AXLM1$ we understand the following set of axioms

$$AXLM1 = \bigcup_{n\in\{0,1,...,10\}} Ax_1n.$$

c) By *Mostowski's Minimized Logic LM* we understand here the partial system of the First Order Predicate Calculus with the set of theorems $Th_{LM} = Cn(AXLM1)$.

Grzegorczyk in his monograph [1] gives as a supplementary axiom the following theorem of the First Order Predicate Calculus:

$Ax_P0.\quad \exists x\left[P^1(x) \rightarrow P^1(x)\right]$, such that[1] one has

THEOREM 0.1.

a) $Ax_P0 \notin TH_{LM}$,

b) If $\delta \in Pf^0(\alpha)$, *then* $\delta'\alpha \in Cn(AXLM1 \cup \{Ax_P0\}$ *iff* $\alpha \in Th_{LP}$.

In this paper I shall prove the dependence of some axioms belonging to $AXLM1$ and I shall present two other axiom systems: $AXLM2$ and $AXLM3$ of LM.

[1]See the “Theorem on the Equivalence of L and L^+” ([1], p. 181)

1. Consequences of $AXLM1$.

The first derived inference rule that one deduces from the axiomatics $AXLM1$, is the rule of "*transitivity*" RT, defined as follows:

DEFINITION 0.7. Let Z be a given set of formulae. We say that "*the rule RT holds in Z*" iff for every formulae α, β, γ belonging to Sf, and δ belonging to Pf, the hypotheses: $[\alpha \to \beta] \in Z$ and $[\beta \to \gamma] \in Z$ imply the thesis $[\alpha \to \gamma] \in Z$.

From the rule of detachment and from only one axiom belonging to the schema $Ax_1 0$ we deduce

LEMMA 1.1. *If $AX \subseteq \overline{S}f$, then the rule RT holds in $Cn(Ax_1 0 \cup AX$.*

The proof consists of applying twice RD to the axiom of the form: $[[\alpha \to \beta] \to [[\beta \to \gamma] \to [\alpha \to \gamma]]]$, belonging to $Ax_1 0$.

Next, using only the rule RT and axiom schemata $Ax_1 0$ and $Ax_1 1$, one proves, by induction with respect to the number of quantifiers of the prefix δ, the following theorem scheme:

T-S.1. If $\delta \in Pf^0([\alpha \to \beta])$, then $[\delta[\alpha \to \beta] \to [\delta\alpha \to \delta\beta]] \in Cn(Ax_1 0 \cup Ax_1 1) \subseteq Th_{LM}$.

From $Ax_1 7$ one obtains $\forall x[[P^1(x) \to P^1(x)] \to \exists x[P^1(x) \to P^1(x)]] \in Th_{LM}$ and hence from $Ax_1 1$ one deduces that $[\forall x[P^1(x) \to P^1(x)] \to \forall x \exists x[P^1(x) \to P^1(x)]] \in Th_{LM}$.

The antecedent of this formula is a theorem resulting $Ax_1 0$. Therefore, by RD, one obtains

T-S.2. $\forall x \exists x[P^1(x) \to P^1(x)] \in Th_{LM}$.

Directly from $Ax_1 3$, *T-S*.1 and RD one deduces that, if $\delta' \in Pf^0(\gamma)$, and δ'' is obtained from δ' by the permutation of two arbitrary successive elements, then $[\delta'\gamma \to \delta''\gamma] \in Th_{LM}$.

Repeating this operation, one arrives, by means of RT at the following conclusion:

LEMMA 1.2. *If $\delta' \in Pf^0(\gamma)$, then for each δ'' being a permutation of δ', one has $[\delta'\gamma \to \delta''\gamma] \in Th_{LM}$.*

Now suppose, that $\delta \in Pf(\gamma) \setminus Pf^0(\gamma)$, and hence $Nq(\delta) > NVr(\gamma)$. In this case some elements $\forall x_i$ of the prefix δ are superfluous i.e. do not bind any variables [2]. By means of $Ax_1 5$ one eliminates the first one from

[2]Mostowski in [4] uses the term "*non-vacuous quantifier*"

the prefix δ and using T-S.1 and RD one deduces that, there exists another prefix δ' such that $Nq(\delta') = Nq(\delta) - 1$ and $[\delta'\gamma \to \delta\gamma] \in Th_{LM}$.

Repeating this operation, one arrives, by means of RT, at the following conclusion:

LEMMA 1.3. *If $\delta \in Pf(\gamma)$, then there exists a prefix $\delta'' \in Pf^0(\gamma)$, such that $[\delta''\gamma \to \delta\gamma] \in Th_{LM}$.*

From LEMMA 1.2, LEMMA 1.3 and LEMMA 0.1b), one obtains, by RT, the following theorem scheme:

T-S.3 *If $\delta' \in Pf^0(\alpha)$ and $\delta \in Pf(\alpha)$, then $[\delta'\alpha \to \delta\alpha] \in Th_{LM}$.*

and (by RD) the derived inference rule

RP If $\delta' \in Pf^0(\alpha)$, $\delta \in Pf(\alpha)$ and $\delta'\alpha \in Th_{LM}$, then $\delta\alpha \in Th_{LM}$.

By substituting the weaker hypothesis: $\delta' \in Pf(\alpha)$, for $\delta' \in Pf^0(\alpha)$ in the schema: T-S.3 and in the rule RP, one obtains the schema of theorem of classical logic LP (valid only in non-empty universe):

T-S_{LP} If $\delta', \delta \in Pf(\alpha)$, then $[\delta'\alpha \to \delta\alpha] \in \overline{T}h_{LP}$

and the inference rule

RP_{LP} If $\delta', \delta \in Pf(\alpha)$ and $\delta'\alpha \in \overline{T}h_{LP}$, then $\delta\alpha \in \overline{T}h_{LP}$

that holds in the set [3] $\overline{T}h_{LP}$, but doesn't hold in the set Th_{LM}.

To show, that the rule RP_{LP} does not hold in the logic LM, it suffices to put $\alpha = \exists x[P^1(x) \to P^1(x)]$, $\delta' = \forall x$, $\delta = \vartheta$.

In this case the thesis: $\exists x[P^1(x) \to P^1(x)]$, according to THEOREM 0.1a), doesn't belong to Th_{LM}, whereas from T-S.2 results that the premise: $\forall x \exists x[P^1(x) \to P^1(x)]$ belongs to $Th_{LM} \subseteq \overline{T}h_{LP}$.

Likewise, one proves that $[\forall x \exists x[P^1(x) \to P^1(x)] \to \exists x[P^1(x) \to P^1(x)]]$ doesn't belong to Th_{LM} but belongs to $\overline{T}h_{LP}$ (cf. [4], p.107) .

Therefore we have

THEOREM 1.1. *Though the schema*

a) T-S_{LP}. If $\delta', \delta \in Pf(\alpha)$, then $[\delta'\alpha \to \delta\alpha] \in \overline{T}h_{LP}$

and the rule

b) RP_{LP} If $\delta', \delta \in Pf(\alpha)$ and $\delta'\alpha \in \overline{T}h_{LP}$, then $\delta\alpha \in \overline{T}h_{LP}$

[3] The set $\overline{T}h_L P$ is defined by DEFINITION 0.1c)

hold in the classical predicate calculus LP, nevertheless in the Mostowski's logic LM one can prove only the schema T-S.3 and the rule [4] *RP having the stronger premise:* $\delta' \in Pf^0(\alpha)$.

From DEFINITION 0.5, DEFINITION 1, LEMMA 0.1e) and T-$S.3$ one deduces

LEMMA 1.4. *If* $\gamma \in Ax_0n$ *and* $\delta \in Pf(\gamma)$, *then* $\delta\gamma \in Th_{LM}$ *(where* $n \in \{0, 1, ..., 10\}$*).*

2. Consequences of $AXLM2$

One sees that the substitution in DEFINITION 1 of $Pf(\gamma)$ for $Pf^0(\gamma)$ yields, in view of LEMMA 0.1a) and LEMMA 1.4, a new axiom system equivalent to $AXLM1$, which will be defined as follows:

DEFINITION 2.

a) For $n \in \{0, 1, ..., 10\}$, a formula α will be said "*an axiom falling under the schema* Ax_2n" iff there are: $\gamma \in Ax_0n$ and $\delta \in Pf(\gamma)$, such that $\alpha = \delta\gamma$. Ax_2n will denote the set of all these formulae α.

b) By the *axiomatics* $AXLM2$ we understand the set of axioms

$$AXLM2 = \bigcup_{n \in \{0,1,,...,10\}} Ax_2n.$$

Therefore one obtains

THEOREM 2.1. $Cn(AXLM2) = Th_{LM} = Cn(AXLM1)$,

which results directly from DEFINITION 1 by means of DEFINITION 0.4, DEFINITION 2, LEMMA 0.1a) and LEMMA 1.4.

In this section, one proves, on the basis of $AXLM2$, some theorems and rules of inference, which serve to create a third axiom system of the logic LM.

Comparing Ax_20 with Ax_10, it is easy to see that the reasoning analogous to the proof of LEMMA 1.1 gives us

LEMMA 2.1. *If* $AX \subseteq \overline{S}f$, *then the rule RT holds in* $Cn(Ax_20 \cup AX)$.

[4] To define LM in [4], Mostowski uses also some non-classical inference rules substituting them for the rules that hold only in non-empty universe. He observes that "A different set of rules for the functional calculus, such that theorems obtained by these rules are valid in each set I whether empty or not, has been given by Jaśkowski in [3]." (published in 1934)

Similarly, the substitution in the theorem T-$S.1$ (and in its proof) of the weaker premise: $\delta \in Pf(\alpha)$, for a stronger: $\delta \in Pf^0(\alpha)$, yields a new theorem (stronger than T-$S.1$) belonging to Th_{LM}:

T-$S.4$ If $\delta \in Pf([\alpha \to \beta])$, then $[\delta[\alpha \to \beta] \to [\delta\alpha \to \delta\beta]] \in Cn(Ax_2 0 \cup Ax_2 1) \subseteq Th_{LM}$.

Applying twice the rule RD, to the schema T-$S.4$, one can prove that in Th_{LM} the following derived rule of inference RDQ holds.

DEFINITION 0.8. Let Z be a given set of formulae. One says that "*the rule RDQ holds in Z*" iff for every formulae α and β belonging to Sf, and δ belonging to Pf, the hypotheses: $\delta[\alpha \to \beta] \in Z$ and $\delta a \in Z$, imply the thesis $\delta\beta \in Z$.

Likewise, applying twice the rule RDQ, to the schema $\delta[[\alpha \to \beta] \to [[\beta \to \gamma] \to [\alpha \to \gamma]]]$, belonging to $Ax_2 0$, one proves that in Th_{LM} the following derived rule of inference RTQ holds:

DEFINITION 0.9. Let Z be a given set of formulae. We say that "*the rule RTQ holds in Z*" iff for every formulae α, β, γ belonging to Sf, and δ belonging to Pf, the hypotheses: $\delta[\alpha \to \beta] \in Z$ and $\delta[\beta \to \gamma] \in Z$ imply the thesis $\delta]\alpha \to \gamma] \in Z$.

Putting in the rule RP (from): $\delta' = \vartheta$ one obtains the following derived rule of inference RP_0:

DEFINITION 0.10. Let a set Z of formulae be given. We say that "*the rule RP_0 holds in Z*" iff for every formula α belonging to Sf, and δ belonging to Pf, the hypothesis: $\alpha \in Z$ implies the thesis $\delta\alpha \in Z$.

The analysis of outlined above proofs of derived inference rules RDQ and RTQ, yields (taking into account T-$S.4$) the following

LEMMA 2.2. *If $AX \subseteq \overline{S}f$, then*
a) the rule RDQ holds in $Cn(Ax_2 0 \cup Ax_2 1 \cup AX)$,
b) the rule RTQ holds in $Cn(Ax_2 0 \cup Ax_2 1 \cup AX)$,

Putting in RP $\delta' = \vartheta$, one obtains

LEMMA 2.3. *The rule RP_0 holds in Th_{LM}.*

Now we shall prove that some schemata of axioms belonging to $AXLM2$ are dependent ones.

THEOREM 2.2. *In the axiomatics $AXLM2$ the axiom schema $Ax_2 3$ is a dependent one because $Ax_2 3 \subseteq Cn(Ax_2 0 \cup Ax_2 1 \cup Ax_2 2 \cup Ax_2 4 \cup Ax_2 5)$.*

To prove that

$Ax_2 3$. If $\delta \in PF(\alpha)$, then $\delta[\forall\xi\forall\eta\alpha \rightarrow \forall\eta\forall\xi\alpha] \in Th_{LM}$ holds, one examines two cases:

1) If $\eta \in Vr(\alpha)$, then, from $Ax_2 2$, $Ax_2 1$ and RDQ, one obtains

$$\delta[\forall\eta\forall\xi\forall\eta\alpha \rightarrow \forall\eta\forall\xi\alpha]. \tag{1.1}$$

On the other hand, from $Ax_2 5$, one has

$$\delta[\forall\xi\forall\eta\alpha \rightarrow \forall\eta\forall\xi\forall\eta\alpha]. \tag{1.2}$$

and hence the rule RTQ gives us the thesis.

2) If $\eta \notin Vr(\alpha)$, then, from $Ax_2 4$, one obtains

$$\delta[\forall\xi\forall\eta\alpha \rightarrow \forall\xi\alpha], \tag{2.1}$$

On the other hand, from $Ax_2 5$, one has

$$\delta[\forall\xi\alpha \rightarrow \forall\eta\forall\xi\forall\eta\alpha], \tag{2.2}$$

and hence the rule RTQ gives us the thesis too.

Likewise, one can prove

THEOREM 2.3. *In the axiom system $AXLM2$, the axiom schema $Ax_2 8$ is a dependent one, because it results from RD and only from the axiom schemata $Ax_2 0$, $Ax_2 1$, $Ax_2 6$, $Ax_2 7$, $Ax_2 9$, and $Ax_2 10$.*

Now we shall present some new schemata valid in Th_{LM}, which we will use to construct a new axiom system of the logic LM.

T-S.5. If $Vr(\alpha) = \{\xi\}$ and $\eta \in Sb(\alpha : \xi)$, then $[\exists\eta sb(\alpha : \xi/\eta) \rightarrow \exists\xi\alpha] \in Th_{LM}$.

Proof. From the schemata $Ax_2 9$ and $Ax_2 10$ one obtains $[\exists\eta sb(\alpha : \xi/\eta) \rightarrow \exists\eta\exists\xi\alpha] \in Th_{LM}$ and $[\exists\eta\exists\xi\alpha \rightarrow \exists\xi\alpha] \in Th_{LM}$. The thesis results from them by the rule RT.

T-S.6. If $\delta \in Pf(\forall\xi[\alpha \rightarrow \beta])$ and $\xi \notin Vr(\alpha)$, then $[\forall\xi[\alpha \rightarrow \beta] \rightarrow [\alpha \rightarrow \forall\xi\beta]] \in Th_{LM}$.

Proof. From [5] the theorem of classical sentential calculus:
$[[\alpha' \rightarrow [\alpha'' \rightarrow \alpha''']] \rightarrow [[\alpha \rightarrow \alpha''] \rightarrow [\alpha' \rightarrow [\alpha \rightarrow \alpha''']]]]$
one obtains

[5] The proof is outlined in [1] as schema (15*) on the page 173

$\delta[[\forall\xi[\alpha \to \beta] \to [\forall\xi\alpha \to \forall\xi\beta]] \to [[\alpha \to \forall\xi\alpha] \to [\forall\xi[\alpha \to \beta] \to [\alpha \to \forall\xi\beta]]]] \in Th_{LM}$.

Therefore the axioms $Ax_2 1$ and $Ax_2 5$ give us the thesis.

T-S.7. If $\delta \in Pf(\forall\xi\alpha)$, then $\delta\forall\xi[\forall\xi\alpha \to \alpha] \in Th_{LM}$.

Proof. To obtain *T-S*.7, one proves two implications concerning $\gamma = \forall\xi[\forall\xi\alpha \to \alpha]$.

The implication $[\forall\xi\alpha \to \gamma]$ results from $Ax_2 0$, $Ax_2 1$ and the rule RDQ, and the implication $[\neg\forall\xi\alpha \to \gamma]$ results from the schema $\forall\xi[\neg\forall\xi\alpha \to [\forall\xi\alpha \to \alpha]]$ by means of *T-S*.6.

The next theorem is a modification of $Ax_2 4$.

T-S.8. If $\delta \in Pf(\forall\eta[\forall\xi[\alpha \to sb(\alpha : \xi/\zeta)])$ and $\zeta \in Sb(\alpha : \xi)$, then $\delta\forall\eta[\forall\xi[\alpha \to sb(\alpha : \xi/\zeta)] \in Th_{LM}$.

Proof. Let β stands for $[\forall\xi\alpha \to \alpha]$. For an arbitrary prefix δ belonging to $Pf(\forall\eta sb(\beta : \xi/\zeta))$ one deduces from $Ax_2 4$, that

$$\delta\forall\eta[\forall\zeta\forall\xi\beta \to \forall\zeta sb(\beta : \xi/\zeta)] \in Th_{LM} \tag{1}$$

and from *T-S*.7,

$$\delta\forall\eta\forall\zeta\forall\xi\beta \in Th_{LM}. \tag{2}$$

Therefore, by RDQ, one obtains

$$\delta\forall\eta\forall\zeta sb(\beta : \xi/\zeta) \in Th_{LM}. \tag{3}$$

In the first case, when $\zeta \in Vr(sb(\beta : \xi/\zeta))$, the thesis $\delta\forall\eta sb(\beta : \xi/\zeta)$ results from (3) and $Ax_2 2$.

In the second case, when $\zeta \notin Vr(sb(\beta : \xi/\zeta))$, the thesis results from (3) and $Ax_2 4$.

We shall prove now, that from $AXLM2$ one deduces the following schema of De Morgan's law:

T-S.11. If $\delta \in Pf(\forall\xi\alpha)$, then $\delta[\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha] \in Th_{LM}$.

Grzegorczyk notes ([1], p.180, schemata 74*) that: "... this schema might serve as kind of definition of the existential quantifier in terms of the universal quantifier. It would suffice to adopt only schemata $Ax_2 1$-5 and schema 74*, and all the axioms $Ax_2 6$-10 could be deduced from them". Unfortunately, he gives the proof of *T-S*.11 only for the case when the condition (*) $\xi \in Vr(\alpha)$ holds.

Instead of completing the proof of Grzegorczyk (for the case $\xi \notin Vr(\alpha)$), we give here another proof of theorem T-S.11.

We begin with the following observation: It results, from THEOREM 2.1 and THEOREM 0.1b) proved by Grzegorczyk, that the set $\overline{T}h_{LP}$ of all closed theorems of the classical First Order Logic LP (true in the all *non-empty* universes) is defined by the following axiom system:

$AXLM2 \cup \{Ax_P0\}$, where $Ax_P0 := \exists x[P^1(x) \to P^1(x)]$.

Taking into account, that in LP, theorem T-S.11 holds always, one deduces from Tarski-Herbrand's Deduction Theorem (in the form presented in [1] p. 192), that one has

T-S.9. If $\delta \in Pf(\forall\xi\alpha)$, then $[Ax_P0 \to \delta[\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha]] \in Th_{LM}$.

Therefore, to prove that T-S.11 is always a theorem of LM, it suffices to prove it only with a supplementary hypothesis: $\neg Ax_P0$, which asserts that one examines the formula in the case of the empty universe.

From $AXLM2$ one deduces that

T-S.10. If $\delta \in Pf(\forall\xi\alpha)$, then $[\neg Ax_P0 \to \delta[\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha]] \in Th_{LM}$.

Proof. One puts $\beta = [P^1(\xi) \to P^1(\xi)]$. Therefore, from Ax_20 one obtains (2) $\delta\forall\xi[\alpha \to \beta] \in Th_{LM}$. From Ax_26 and (2) one obtains $\delta[\exists\xi\alpha \to \exists\xi\beta] \in Th_{LM}$, and hence (3) $\delta[\neg Ax_P0 \to \neg\exists\xi\alpha] \in Th_{LM}$.

On the other hand, from Ax_27, it results $\forall\xi[\beta \to \exists\xi\beta] \in Th_{LM}$, and hence $\forall\xi[\neg\exists\xi\beta \to \neg\beta] \in Th_{LM}$. Therefore, by (2) we obtain $\delta\forall\xi[\neg Ax_P0 \to \neg\alpha] \in Th_{LM}$, and from T-S.6: (4) $\delta[\neg Ax_P0 \to \forall\xi\neg\alpha] \in Th_{LM}$.

From the theorem of sentential calculus: $[[[\gamma \to \neg\gamma'] \wedge [\gamma \to \gamma'']] \to [\gamma \to [\gamma' \leftrightarrow \neg\gamma'']]]$, and from the implications (3), (4), one obtains $\delta[\neg Ax_P0 \to [\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha]] \in Th_{LM}$, and finally (5) $[\delta\neg Ax_P0 \to \delta[\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha]] \in Th_{LM}$.

The thesis results, by RT, from (5) and T-S.3.

Directly from T-S.9 and T-S.10 one obtains

T-S.11. If $\delta \in Pf(\forall\xi\alpha)$, then $\delta[\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha] \in Th_{LM}$.

3. Consequences of $AXLM3$.

As new axioms schemata one adopts the schemata of axioms: Ax_20, Ax_21, Ax_25, and the schemata of theorems: T-S.8 and T-S.11, i.e. we state

DEFINITION 3.

a) For $n \in \{0, 1, ..., 4\}$ the symbol Ax_3n denotes the sets of all axioms falling under the n-th new axiom-scheme:

Let $\alpha, \beta, \gamma \in Sf$, $\xi, \eta, \zeta \in Vr$ and $\delta \in Pf(\gamma)$.

0) If $\gamma \in Th_S$, then $\delta\gamma \in Ax_30$;
1) If $\gamma = [\forall\xi[\alpha \to \beta] \to [\forall\xi\alpha \to \forall\xi\beta]$, then $\delta\gamma \in Ax_31$;
2) If $\xi \notin Vr(\alpha)$ and $\gamma = [\alpha \to \forall\xi\alpha]$, then $\delta\gamma \in Ax_32$;
3) If $\zeta \in Sb(\alpha : \xi)$ and $\gamma = \forall\eta[\forall\xi\alpha \to sb(\alpha : \xi/\zeta)]$, then $\delta\gamma \in Ax_33$;
4) If $\gamma = [\exists\xi\alpha \leftrightarrow \neg\forall\xi\neg\alpha]$, then $\delta\gamma \in Ax_34$.

b) By the *axiomatics* $AXLM3$ we understand the set of axioms:

$$AXLM3 = \bigcup_{n \in \{0,1,2,3,4\}} Ax_3n.$$

Comparing DEFINITION 3 with DEFINITION 2, one sees that $Ax_30 = Ax_20$, $Ax_31 = Ax_21$ and $Ax_32 = Ax_25$, and hence one has

LEMMA 3.1.
a) $Ax_20, Ax_21, Ax_25 \in AXLM3$ *;*
b) $Ax_30, Ax_31, Ax_32 \in AXLM2 \subseteq Th_{LM}$.

We shall prove that this axiom system defines the same logic LM, i.e. that $Cn(AXLM3) = Th_{LM}$.

This results from DEFINITION 3, THEOREM 2.1, LEMMA 3.1b), *T-S*.8, *T-S*.11 that all axioms belonging to $AXLM3$ are theorems of LM, i.e. one has

LEMMA 3.2. $AXLM3 \subseteq Th_{LM}$.

LEMMA 3.3. *If* $AX \subseteq \overline{S}f$, *then the rules* RT, RDQ, RTQ, RP_0 *hold in* $Cn(Ax_30 \cup Ax_31 \cup Ax_32 \cup AX)$ *and hence hold in* $Cn(AXLM3)$.

Proof. To prove that RP_0 holds in $Cn(Ax_32 \cup AX) \subseteq Cn(Ax_30 \cup Ax_31 \cup Ax_32 \cup AX)$, it suffices to apply $Nq(\delta)$ times the axiom scheme Ax_32 and the rule RD, beginning from a theorem α.

For the other rules the lemma results from LEMMA 2.1, LEMMA 2.2 and DEFINITION 3.

To prove, that all theorems belonging to Th_{LM} result from $AXLM3$ by means of the rule: RD, it suffices to prove now

LEMMA 3.4. $AXLM2 \subseteq Cn(AXLM3)$.

Proof.

1) From LEMMA 3.1a) one obtains

$$Ax_20, Ax_21, Ax_25 \in Cn(AXLM3). \tag{1}$$

2) One proves Ax_22 as follows:

Suppose that $\delta \in Pf(\alpha)$ and $\xi \in Vr(\alpha)$, and hence $Nq(\delta) > 0$. Therefore there exist: a variable η and a prefix δ' such that $\delta = \delta'\forall\eta$. Taking into account, that $\delta' \in Pf(\forall\eta[\forall\xi\alpha \to \alpha])$ and that always $\xi \in Sb(\alpha : \xi)$ holds, one obtains from Ax_33 the thesis $\delta'\forall\eta[\forall\xi\alpha \to \alpha]$, and hence

$$Ax_22 \in Cn(AXLM3). \tag{2}$$

3) Suppose that $\delta \in Pf(\forall\eta\forall\xi\alpha)$ and $\delta \in Sb(\alpha : \xi)$.The substitution in Ax_33 of η for ζ yields the theorem $\delta\forall\eta[\forall\xi\alpha \to sb(\alpha : \xi/\eta)]$, from which one obtains (by means of Ax_31 and RDQ). the thesis $\delta[\forall\eta\forall\xi\alpha \to \forall\eta sb(\alpha : \xi/\eta)]$. Therefore one has

$$Ax_24 \in Cn(AXLM3). \tag{3}$$

4) From (1)-(3) and THEOREM 2.2 one obtains

$$Ax_23 \in Cn(AXLM3). \tag{4}$$

5) To prove every one of remaining 5 axioms: Ax_26, Ax_27, Ax_28, Ax_29, Ax_210, one uses respectively axioms: Ax_21, Ax_22, Ax_23, Ax_24, Ax_25, as follows. At first one eliminates from Ax_2n, the existential quantifier $\exists$ (by means of definition Ax_34, and next, one proves it using the respective axiom scheme $Ax_2(n-5)$.

For example, to prove Ax_210, if $\delta \in Pf(\alpha)$ and $\xi \notin Vr(\alpha)$, then $\delta[\exists\xi\alpha \to \alpha] \in Cn(AXLM3)$, one uses Ax_25 as follows.

One deduces from Ax_34, that (0.1) $\delta[\exists\xi\alpha \to \neg\forall\xi\neg\alpha]$ holds, and on the other hand, the axiom $Ax_25(= Ax_32)$ yields (0.2) $\delta[\neg\forall\xi\neg\alpha \to \alpha] \in Cn(AXLM3)$. The thesis results, by RTQ, from (0.1) and (0.2).

Likewise one proves Ax_26, Ax_27, Ax_28, Ax_29, and hence one obtains

$$Ax_26, Ax_27, Ax_28, Ax_29, Ax_210 \in Cn(AXLM3). \tag{5}$$

Finally from LEMMA 3.2, LEMMA 3.4 and from THEOREM 2.1, one obtains:

THEOREM 3.1. $Th_{LM} = Cn(AXLM3)$.

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