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MILDLY INEFFABLE CARDINALS AND HYPERUNIVERSES

A b s t r a c t. In this paper, we give an alternative to M. Forti and F. Honsell's construction of hyperuniverses. We use this construction to see precisely on what conditions there can exist a κ-hyperuniverse of given uniform weight. This condition is expressed in terms of *mildly ineffable* cardinals.

Introduction

The aim of this paper is to see how the existence of certain *hyperuniverses* and the notions of mildly ineffable cardinal and ramifiable directed set are connected. Hyperuniverses were introduced by M. Forti and R. Hinnion ([11]) mainly in order to show the consistency of the *generalized comprehension scheme* (GPK). Hyperuniverses were deeply investigated by M. Forti and F. Honsell (see for example [13, 14]). We

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prove a theorem giving a general way to construct hyperuniverses (theorem 2.8): in order to obtain a hyperuniverse, we take the Cauchy-completion of some suitable first-order structure. The notion of *ramifiable directed set*, in the sense of R. Hinnion, comes into play to show the κ-compactness of the completion. This construction presents similarities with a construction of M. Forti and F. Honsell where they take the quotient of the universe by suitable equivalences. They assumed the existence of a strongly compact cardinal in order to construct a non-κ-metrizable κ-hyperuniverse, which appears here to be a too strong hypothesis. Also, we prove a theorem stating that the set of entourages, directed by reverse inclusion, of the uniformity of a hyperuniverse is ramifiable (theorem 2.9). We use these two results to characterize the existence of a κ-hyperuniverse of given uniform weight in terms of *mild ineffability* in the sense of Di Prisco C. and Zwicker S. (theorem 2.10).

In the first part of section 1, we recall the prerequisites about *ramifiable directed sets* and *mildly ineffable cardinals*. The notion of *ramifiable directed set*, generalizing the tree-property to directed sets, has been introduced by R. Hinnion ([15, 16]) where he studies the notion of Cauchy-completion of first order-structures. He needs this notion of ramifiability to get his Cauchy-completion κ-compact. This notion of ramifiability has then been studied by R. Hinnion and the author ([9, 10]); it is connected to the notion of *mild ineffability* for cardinals (see [20, 4, 3]). In the second part of the first section we see more precisely how these two notions are connected (theorem 1.6).

1. Ramifiability and ineffability

1.1 Recallings

A *directed* set $(D, \leqslant)$ is a set D endowed with an order relation $\leqslant$ such that any pair of D is upperly bounded. From now, by D, we mean always a directed set; we omit to mention $\leqslant$ as usual. A κ-*directed* set is a directed set where any κ-finite (i.e. of size $< \kappa$) subset of D is bounded. For a directed set $(D, \leqslant)$ without maximum, we denote δ_D the largest cardinal δ such that D is δ-directed; when using this notation, we assume that D has no maximum. The directed set D is said *degree-regular* iff [1] $(\forall d \in D)\{e \in D \mid e \leqslant d\}$ is δ_D-finite.

[1] *iff* abbreviates *if and only if*.

A *tree* $(T, \leqslant_T, \mathrm{lev}_T)$ (or simply T or $(T, \leqslant_T)$) on D is a structure satisfying the following conditions:

(i) $(T, \leqslant_T)$ is an ordered set (not necessarily directed).

(ii) lev_T is a function: $D \to \mathcal{P}T$.

(iii) The *downwards uniqueness principle* holds i.e.

$$x \leqslant y \Rightarrow (\forall \xi \in \mathrm{lev}_T(y))(\exists! \eta \in \mathrm{lev}_T(x))(\eta \leqslant_T \xi)$$

If moreover, we have the *upwards access principle*:

$$x \leqslant y \Rightarrow (\forall \xi \in \mathrm{lev}_T(x))(\exists \eta \in \mathrm{lev}_T(y))(\xi \leqslant_T \eta)$$

then the tree is said to be *well-pruned*[2]. If $\xi \in \mathrm{lev}_T(d)$, we say that the level of ξ is d. We may write "lev" instead of "lev_T" when no confusion can occur. The unique η in (iii) above is denoted $\xi \downarrow x$. A tree T is said to be a θ-tree iff $(\forall d \in D)(|\mathrm{lev}_T(d)| < \theta)$. A branch h of T (notation $h : D \hookrightarrow T$) is an embedding (i.e. $(\forall d, d' \in D)(d \leqslant d') \Leftrightarrow h(d) \leqslant_T h(d'))$ such that $h(a) \in \mathrm{lev}_T(a)$ for all $a \in D$. We say that D is *θ-ramifiable* if for each tree T on D, there is a branch h of T (we say that T has a branch). It is said *ramifiable* iff it is δ_D-ramifiable or has a maximum[3].

Remark 1.1. [4]We can define a tree T on $(D, \leqslant)$ as being a projective system on $(D, \leqslant)$. A branch of the tree T is an element of the projective limit $\varprojlim A$.

One can show that ramifiability, i.e. δ_D-ramifiability is the only interesting case since D is always θ-ramifiable for $\theta < \delta_D$ and never θ-ramifiable for $\theta > \delta_D$ (see [9]). We check that, as for cardinals, D is ramifiable iff each *well-pruned* δ_D-tree on D has a branch. It is easy to see that a regular cardinal κ is ramifiable iff it has the tree-property.

If $D' \subset D$ are two directed sets and D' is cofinal in D, then D is ramifiable iff D' is. We denote by $\overset{\mathrm{cof}}{\cong}$ the least equivalence relation (on the class of directed sets) such that any D is equivalent with any cofinal subset of D, and isomorphic directed sets are equivalent.

[2]Such trees are called *arborescences* in [15, 16, 17, 9, 10].

[3]This is justified by the fact that if D has a maximum, it is θ-ramifiable for each cardinal θ.

[4]This remark is due to A. Rigo.

For a set A, $\mathcal{P}_\kappa A$ denotes the set of κ-finite subsets of A; this set is κ-directed by inclusion if κ is regular; if we consider the directed set $\mathcal{P}_\kappa A$, the order is always assumed to be $\subset$. We denote $\mathcal{P}_{\mathrm{bd}} D$, the set of bounded subsets of D. We see that $(D, \leqslant) \overset{\mathrm{cof}}{\cong} (\mathcal{P}_{\mathrm{bd}} D, \subset)$, so $(D, \leqslant)$ is ramifiable iff $(\mathcal{P}_{\mathrm{bd}} D, \subset)$ is. We denote $\mathrm{cf}(D)$ (the *cofinality* of D), the smallest cardinal κ such that there is $D' \subset D$, D' cofinal in D and $|D'| = \kappa$. Notice that $\mathrm{cf}(D)$ is not necessarily a regular cardinal. Let us recall the definition of mild ineffability ([20]).

Definition 1.2. A cardinal κ is said to be *mildly γ-ineffable* iff given a family $\{A_x \mid x \in \mathcal{P}_\kappa \gamma\}$ satisfying $A_x \subset x$ for each $x \in \mathcal{P}_\kappa \gamma$, there is $A \subset \gamma$ such that $(\forall x \in \mathcal{P}_\kappa \gamma)(\exists y \in \mathcal{P}_\kappa \gamma)(x \subset y \;\&\; A_y \cap x = A \cap x)$.

Let us recall the following proposition about mild ineffability ([20, 4]):

Proposition 1.3.

- *A cardinal κ is mildly κ-ineffable iff it is strongly inaccessible*[5] *and has the tree-property.*
- *If a cardinal κ is mildly γ-ineffable then it is mildly δ-ineffable for all cardinals δ with $\kappa \leqslant \delta \leqslant \gamma$.*
- *A cardinal κ is strongly compact iff it is mildly γ- ineffable for all cardinals $\gamma \geqslant \kappa$.*

1.2 Connection between ramifiability and mild ineffability

The notion of ramifiable directed set is closely related to the notion of mild ineffability. This connection however works well only for directed sets D with δ_D strongly inaccessible. The following lemma with its corollary will be useful.

Lemma 1.4. *Let $(D, \leqslant)$ be a degree-regular directed set with $\kappa = \delta_D$ strongly inaccessible. Then D is ramifiable iff $\mathcal{P}_\kappa |D|$ is. Moreover, we have $\mathrm{cf}(D) = |D|$ and $|D|^{<\kappa} = |D|$.*

Proof. The first part comes from the fact that $\mathcal{P}_{\mathrm{bd}}(D) = \mathcal{P}_\kappa D$.

For the second part let $\gamma = |D|$ and let us prove that $\mathrm{cf}(\mathcal{P}_\kappa \gamma) = \gamma^{<\kappa} = \gamma$. We first prove that $\mathrm{cf}(\mathcal{P}_\kappa \gamma) \geqslant \gamma$. Let $\theta = \mathrm{cf}(\mathcal{P}_\kappa \gamma)$ and suppose on the contrary that $\theta < \gamma$. Let $H \subset \mathcal{P}_\kappa \gamma$ be cofinal in $\mathcal{P}_\kappa \gamma$ with $|H| = \theta$, we

[5] We include ω among the strongly inaccessible cardinals.

have $(\forall x \in \gamma)(\exists A \in H)(x \in A)$. This implies that $\gamma = \theta \cdot \kappa$. We have $\mathrm{cf}(\mathcal{P}_\kappa \kappa) = \kappa$ using the fact that κ is strongly inaccessible. Now we have $|D| = \gamma \geqslant \mathrm{cf}(D) = \mathrm{cf}(\mathcal{P}_\kappa D) = \mathrm{cf}(\mathcal{P}_\kappa \mathcal{P}_\kappa D) = \mathrm{cf}(\mathcal{P}_\kappa \gamma^{<\kappa}) \geqslant \gamma^{<\kappa} \geqslant \gamma$ and the result is proved. □

Corollary 1.5. *Let κ be a strongly inaccessible cardinal, then $\mathcal{P}_\kappa \gamma$ is ramifiable iff $\mathcal{P}_\kappa \gamma^{<\kappa}$ is.*

Proof. This is an obvious consequence of lemma 1.4 since $|\mathcal{P}_\kappa \gamma| = \gamma^{<\kappa}$. □

We adopt the following notation. For an ordered set E and for $X \subset E$, $\mathrm{bd}(X) := \{e \in E \mid (\forall x \in X)(x \leqslant e)\}$.

For strongly inaccessible cardinals, we can entirely characterize the ramifiability of $\mathcal{P}_\kappa \gamma$ in terms of mild ineffability as is shown by the following theorem.

Theorem 1.6. *Let κ be a strongly inaccessible cardinal. Then $\mathcal{P}_\kappa \gamma$ is ramifiable iff κ is mildly $\gamma^{<\kappa}$-ineffable.*

Proof. Denote $D = \mathcal{P}_\kappa \gamma$. Let us prove

$$\mathcal{P}_\kappa \gamma \text{ is ramifiable } \Rightarrow \kappa \text{ is mildly } \gamma^{<\kappa}\text{-ineffable}$$

Using corollary 1.5, it suffices to prove

$$\mathcal{P}_\kappa \gamma \text{ is ramifiable } \Rightarrow \kappa \text{ is mildly } \gamma\text{-ineffable} \tag{1}$$

In order to prove (1), we suppose D is ramifiable and prove that κ is mildly γ-ineffable. Let us define the following tree T on D. For each $x \in D$, take A_x as in definition 1.2. Now define

$$\begin{cases} \mathrm{lev}_T(x) = \{A_y \cap x \mid y \geqslant x\} \times \{x\} \\ (\xi, x) \leqslant_T (\eta, z) \text{ iff } x \leqslant z \ \& \ \eta \cap x = \xi \end{cases}$$

Let h' be a branch of T and let h be the first component function corresponding to h'. We check that $A = \bigcup \mathrm{im}(h)$ answers the question.

Now let us prove

$$\kappa \text{ is mildly } \gamma^{<\kappa}\text{-ineffable } \Rightarrow \mathcal{P}_\kappa \gamma \text{ is ramifiable} \tag{2}$$

Let T be a well-pruned κ-tree on D. Clearly $|T| = \gamma^{<\kappa}$. To each $x \in \mathcal{P}_\kappa T$, denote $B(x) = \{d \in D \mid \mathrm{lev}_T(d) \cap x \neq \varnothing\}$. For each $x \in \mathcal{P}_\kappa B(x)$, choose $d_x \in \mathrm{bd}(B(x))$ and choose $\xi \in \mathrm{lev}_T(d_x)$. Define $A_x = \{\xi \downarrow d \mid d \in B(x)\} \cap x$. The A_x's satisfy the hypothesis of the A_x's in definition 1.2. A set A as in definition 1.2 gives the branch we were looking for. □

2. Construction of hyperuniverses

2.1 Recallings on κ-topologies and κ-uniformities

In this subsection, we recall some results about κ-topologies and κ-uniformities that will be needed in the paper.

For a cardinal κ, a κ-topological space is a topological space where the set of open sets is stable under κ-finite intersections. A κ-uniform space is a uniform space where the filter of entourages is κ-complete. From now, we consider only strongly inaccessible κ's. Many results on topologies and uniformities can be generalized to κ- topologies and κ-uniformities. We briefly recall what we need, for more details about uniformities and topologies see [2] or [18].

The topology induced by a κ-uniformity is a κ-topology. A T_2 κ-topological space is said *κ-compact* iff every open cover contains a κ-finite subcover. A κ-compact κ-topology is induced by a unique κ-uniformity; in this case all continuous functions are uniformly continuous. Every continuous bijection from a κ-compact κ-topological space to a T_2 κ-topological space is an homeomorphism. The Cauchy-completion of a κ-uniform space is a κ-uniform space. We will adopt the following definition.

Definition 2.1. An *admissible basis*[6] $\mathcal{F}$ on a set A is a basis of entourages for a non-discrete T_2 uniformity on A, made of equivalences.

An admissible basis induces a 0-dimensional topological space (a topology is said 0-*dimensional* iff it is T_2 and has a basis of open-closed sets). If we have a κ-compact 0-dimensional κ-topological space then it is induced by a κ-uniformity having an admissible basis.

If $\kappa > \omega$, every non-discrete T_2 κ-uniformity $\mathcal{U}$ has a basis of entourages which is an admissible basis. For the proof, notice that $\{V^\omega \mid V \text{ is an entourage of } \mathcal{U}\}$ (where $V^\omega = \bigcap_{n\in\omega} V^n$, V^n denoting $\underbrace{V \circ \cdots \circ V}_{n \text{ times}}$) is an admissible basis.

The interest in uniformities having an admissible basis is that the uniformity can be easily described in terms of *nets* generalizing the notion of sequences for metric spaces. Many results on such uniformities can be found in [15] and [16]. We briefly recall what we need.

Consider $\mathcal{F}$ an admissible basis on a set A and order $\mathcal{F}$ by reverse inclusion. An *$\mathcal{F}$-net in A* (or more simply an $\mathcal{F}$-net or a net) is a function

[6] An admissible basis is called a *Malitz family* in [15, 16].

$X : \mathcal{F} \to A$; we write $X_\sim$ instead of $X(\sim)$ for $\sim \in \mathcal{F}$. An *$\mathcal{F}$-subnet* Y of an $\mathcal{F}$-net X is an $\mathcal{F}$-net such that there is an expansive function $\sigma : \mathcal{F} \to \mathcal{F}$ (i.e. $\sigma(\sim) \geqslant \sim$ for all $\sim \in \mathcal{F}$) with $X_{\sigma(\sim)} = Y_\sim$. A strongly Cauchy $\mathcal{F}$-net $(X_\sim)_{\sim\in\mathcal{F}}$ is an $\mathcal{F}$-net such that: $(\forall \sim, \sim' \in \mathcal{F})(\sim \leqslant \sim' \Rightarrow x_\sim \sim x_{\sim'})$. We write *scnet* for strongly Cauchy net. An $\mathcal{F}$-net X strongly converges to x iff $X_\sim \sim x$ for all $\sim \in \mathcal{F}$. A *limit point* x of an $\mathcal{F}$-net X is an element of A such that there is an $\mathcal{F}$-subnet of X which strongly converges to x. The uniformity induced by $\mathcal{F}$ is κ-compact iff every $\mathcal{F}$-net has a limit point (see [15]).

Recall the following proposition ([15, theorem 5.7.8]); it is a generalization of [2, Chap. II, §4, theorem 4] to κ- topology.

Proposition 2.2. *Suppose to have $\mathcal{F}$ an admissible basis on a set A. If $(\mathcal{F}, \supset)$ is ramifiable as a directed set and $(\forall \sim \in \mathcal{F})(A/\sim$ is κ-finite) then the Cauchy-completion $\overline{A}$ of A is κ-compact.*

Recall now the notion of *Cauchy-completion of a first-order structure.* Assume to have a relational first order structure $(A, R_1, \ldots, R_n)$, where R_i is a j_i-ary relation on A $(i = 1, \ldots, n)$, with a uniformity on A. The Cauchy-completion $(\overline{A}, \overline{R_1}, \ldots, \overline{R_n})$ is the first order structure where $\overline{A} \supset A$ is the Cauchy-completion of A and $\overline{R_i}$ is a j_i-ary relation on $\overline{A}$ which is the closure of R_i in the product topology $\overline{A}^{j_i}$. This notion has been studied in details in [15, 16] in the case where we have a uniformity having an admissible basis. This is the case that mainly interests us for our purpose.

Consider thus a first order structure (A, R) where R is a binary relation on A (this in order to simplify the notations) and let $\mathcal{F}$ be an admissible basis on A. Consider S the set of strongly Cauchy $\mathcal{F}$-nets in A. On S, define the following equivalence: $X \approx Y$ iff $(\forall \sim \in \mathcal{F})(X_\sim \sim Y_\sim)$. Let us denote $[X]_\approx$ or simply $[X]$ for the equivalence class of $X \in S$ modulo $\approx$. The Cauchy-completion $(\overline{A}, \overline{R})$ can be seen as follows. The set $\overline{A}$ is $S/\approx$, the uniformity on $\overline{A}$ is given by the following equivalences $\{[\sim] \mid \sim \in \mathcal{F}\}$ on $\overline{A}$ where $[X]\,[\sim]\,[Y]$ iff $X_\sim \sim Y_\sim$. The relation $\overline{R}$ is defined as follows on $\overline{A}$: $[X]\,\overline{R}\,[Y]$ iff $(\forall \sim \in \mathcal{F})(\exists X', Y' \in A)(X' \sim X_\sim \;\&\; Y' \sim Y_\sim \;\&\; X' \,R\, Y')$.

At last, recall the notion of κ-Vietoris topology.

Definition 2.3. The κ-*Vietoris* topology on the set $\mathcal{P}_{\mathrm{cl}}X$ of the closed subsets of a topological space X is the κ-topology having the following sets as subbase of closed sets:

$$\Box(a) = \mathcal{P}_{\mathrm{cl}}X \cap \mathcal{P}(a) \text{ and } \Diamond(a) = \{c \in \mathcal{P}_{\mathrm{cl}}X \mid c \cap a \neq \varnothing\}$$

for every closed set a of X.

We will need the following proposition (see [5] for $\kappa = \omega$).

Proposition 2.4. *Let τ be a κ-topology on X and let v be the κ-Vietoris topology on $\mathcal{P}_{\mathrm{cl}}X$. Then*

(i) v is T_1 iff τ is T_1.

(ii) v is T_2 iff τ is T_3.

(iii) v is T_3 iff τ is T_4.

(iv) v is T_4 iff τ is κ-compact.

(v) v is κ-compact iff τ is κ-compact.

2.2 Hyperuniverses and mildly ineffable cardinals

Here, we relate the existence of hyperuniverses and notions of mild ineffability. We will work in $\mathrm{ZFC}_0 + U_l$ where ZFC_0 is the set theory of Zermelo-Fraenkel without foundation and U_l is the axiom of local universality of [1].

U_l: For any extensional relation $\in_A$ on a set A, there is a transitive set t such that $(A, \in_A)$ is isomorphic to $(t, \in)$.

The axiom U_l is only assumed for convenience, we could avoid it by replacing "unexisting transitive sets" by structures $(A, \in_A)$ and working only with these.

We begin with a few recallings on hyperuniverses, more details can be found in [11, 13, 14].

Definition 2.5. ([14])A structure $(H, \in_H)$ where H is a 0-dimensional topological space and $\in_H$ is an extensional binary relation on H is a κ-hyperuniverse iff

$$\psi_{\in_H} : H \to \mathcal{P}_{\mathrm{cl}}H : \psi_{\in_H}(x) = \{y \in H \mid y \in_H x\}$$

is an homeomorphism between H and its κ-Vietoris topology on $\mathcal{P}_{\mathrm{cl}}H$.

A κ-hyperuniverse $(H, \in_H)$ is said to be *standard* iff H is transitive and $\in_H$ is $\in$. By the axiom U_l, any κ-hyperuniverse is isomorphic to a standard κ-hyperuniverse. Such a κ-hyperuniverse is *equal* to the space of its closed subsets endowed with the κ-Vietoris topology (and both initial and κ-Vietoris topology coincide).

By proposition 2.4, the κ-topology on a κ-hyperuniverse is κ-compact and so it is induced by a unique κ-uniformity, which has an admissible basis since the κ-topology is 0-dimensional.

Notice that if $\kappa > \omega$, it suffices to ask that the κ-topology on a κ-hyperuniverse is T_2 instead of 0-dimensional since then it is T_4 by proposition 2.4 and thus 0-dimensional since $\kappa > \omega$.

We recall now a few facts about bounded positive comprehension. Consider the language of set-theory: $(\in, =)$. The class of Bounded Positive Formulas (BPF) is the smallest class containing atomic formulas ($x = y$ and $x \in y$) and such that if φ and ψ are BPF's then so are: $\varphi \vee \psi$, $\varphi \,\&\, \psi$, $\forall x \in y\ \varphi$, $(\exists x \in y\ \varphi)$, $(\forall x\ \varphi)$, $\exists x\ \varphi$ (the formulas between braces can be deduced BPF's from the others).

The *Bounded Positive Comprehension* is the following scheme:

BPC: the universal closure of $\exists u\ \forall t\ t \in u \Leftrightarrow \varphi$, where φ is BPF and u is not free in φ.

For more details about this theory see [6, 7, 8] (In these papers, it is in fact a stronger theory GPK^+ which is studied).

The following lemma will be useful:

Lemma 2.6. *Let $(H, \in_H)$ be a structure where H is a κ-compact 0-dimensional κ-topological space and $\in_H$ is a binary relation on H such that:*

$$(\forall X \subset H)(X \text{ is closed iff } \exists x \in H\ X = \{a \in H \mid a \in_H x\})$$

Then H is a κ-hyperuniverse iff it satisfies BPC.

Proof. Any hyperuniverse satisfies BPC by [11]. On the other hand take a standard structure $(H, \in)$ satisfying the hypotheses of the lemma (the standard aspect is not restrictive since we have U_l). In order to show that H is a κ-hyperuniverse, we have to show that the identity is an homeomorphism between H and $\mathcal{P}_{\mathrm{cl}}H$ endowed with the κ-Vietoris topology. We have that any closed set for the κ-Vietoris topology on $\mathcal{P}_{\mathrm{cl}}H$ is also a closed set for the topology on H: take a closed set in the natural subbase of the κ-Vietoris topology: $\Box(a) = \{x \in H \mid x \subset a\} = \{x \in H \mid (\forall t \in x)(t \in a)\}$ (where a is closed in H). We have that $\Box(a)$ is closed for the topology of H since it is described by a BPF. Proceed similarly for $\Diamond(a)$. This shows that the identity $H \to \mathcal{P}_{\mathrm{cl}}H$ is continuous and so is an homeomorphism since H is κ-compact. □

Definition 2.7. Consider an equivalence relation $\sim$ on a set A endowed with a binary relation R. We denote $\sim^+$, the following equivalence relation on A: $\sim^+ = \{(a,b) \in A^2 \mid \forall x$ with $x\,R\,a$ $\exists y$ with $y\,R\,b$ such that $x \sim y$ & $\forall y$ with $y\,R\,b$ $\exists x$ with $x\,R\,a$ such that $x \sim y\}$. In case of ambiguity we say that we take $\sim^+$ *relatively to* R.

Theorem 2.8. *Suppose to have an admissible basis for a first order structure* $(\mathcal{M}, \in_{\mathcal{M}})$ *where* $\in_{\mathcal{M}}$ *is a binary relation. Let us denote* $\kappa = \delta_{\mathcal{F}}$ *(where* $\mathcal{F}$ *is seen as the directed set* $(\mathcal{F}, \supset)$*). Suppose that the following conditions are satisfied:*

(i) $\in_{\mathcal{M}}$ *is extensional.*

(ii) For all $X \subset \mathcal{M}$ *such that* X *is* κ*-finite, there exists* $X_{\mathcal{M}} \in \mathcal{M}$, $X = \{y \mid y \in_{\mathcal{M}} X_{\mathcal{M}}\}$.

(iii) For all $\sim \in \mathcal{F}$*, the quotient* $A/\sim$ *is* κ*- finite.*

(iv) κ *is strongly inaccessible.*

(v) The directed set $(\mathcal{F}, \supset)$ *is ramifiable.*

(vi) $(\forall \sim \in \mathcal{F})(\sim^+ \subset \sim)$*, where* $\sim^+$ *is taken relatively to* $\in_{\mathcal{M}}$.

Under these conditions, the Cauchy-completion $(\overline{\mathcal{M}}, \overline{\in}_{\mathcal{M}})$ *is a* κ*-hyperuniverse.*

Proof. Use axiom U_l to assume without loss of generality that $\mathcal{M}$ is transitive and that $\in_{\mathcal{M}}$ is the real membership relation $\in$. The proof of the theorem is an adaptation of the proof of the theorem in [7, section 2]. The topology on $\overline{\mathcal{M}}$ is 0-dimensional since $\mathcal{F}$ is made of equivalences. By lemma 2.6, the proof will be complete if we prove that $(\overline{\mathcal{M}}, \overline{\in})$ is κ-compact and satisfies BPC. The structure $(\overline{\mathcal{M}}, \overline{\in})$ is κ-compact by proposition 2.2. Let us show that it satisfies BPC. In this proof, by *net* we mean $\mathcal{F}$-net in $\mathcal{M}$ (and similarly for scnet). We will need a few preliminary results. On $\mathcal{M}$, we define for each $\sim \in \mathcal{F}$, $a \in_\sim b$ iff $\exists a', b'$ $(a' \sim a$ & $b' \sim b$ & $a' \in b')$.

Claim 1 (Prolongation). *Let* A *be a* $\mathcal{F}$*-scnet in* $\mathcal{M}$ *and let* $x \in \mathcal{M}$ *with* $x \in_{\sim_0} A_{\sim_0}$*, then there exists a scnet* X *with* $X_{\sim_0} \sim_0 x$ *and* $[X] \mathrel{\overline{\in}} [A]$.

Proof. We define A' by $A'_\sim = A_{\sim^+}$. Obviously $A' \approx A$. Define the net $(x_\sim)_{\sim \in \mathcal{F}}$ by: for $\sim' \geqslant \sim_0$, pick $x_{\sim'} \in_{\sim'} A'_{\sim'}$, with $x_{\sim'} \sim x$ and for $\sim' \not\geqslant \sim_0$, pick any $x_{\sim'} \in \mathcal{M}$. Since $\overline{\mathcal{M}}$ is κ-compact, we can extract a strongly Cauchy subnet X of $(x_\sim)_{\sim \in \mathcal{F}}$. We see that $[X] \mathrel{\overline{\in}} [A]$ and $X_\sim \sim x$.

CLAIM 2 (Extensionality). *Let* $[X],[Y] \in \overline{\mathcal{M}}$ *be scnets such that* $\forall [Z]\ ([Z] \mathrel{\overline{\in}} [X] \Leftrightarrow [Z] \mathrel{\overline{\in}} [Y])$*, then* $[X] = [Y]$.

PROOF. The statement is equivalent to prove that for each $\sim \in \mathcal{F}$, we have:

$$\big((\forall x \in_\sim X_\sim)(\exists y \in_\sim Y_\sim)\ x \sim y\big)\ \ \&\ \ \big((\forall y \in_\sim Y_\sim)(\exists x \in_\sim X_\sim)\ x \sim y\big)$$

For $x \in_\sim Y_\sim$, use CLAIM 1 to find a scnet X with $X_\sim \sim x$ and $X \mathrel{\overline{\in}} Y$ and choose y to be $X_\sim$; similarly for $y \in_\sim Y_\sim$.

CLAIM 3. *Let* $\mathcal{A} \subset \overline{\mathcal{M}}$*, we have that* $\mathcal{A}$ *is closed iff there is a* $[A] \in \overline{\mathcal{M}}$ *with* $\{[X] \mid [X] \mathrel{\overline{\in}} [A]\} = \mathcal{A}$.

PROOF. The fact that $\{[X] \mid [X] \mathrel{\overline{\in}} [A]\}$ is closed comes from the fact that $\overline{\in}$ is closed in $\overline{\mathcal{M}} \times \overline{\mathcal{M}}$. On the other hand, suppose that $\mathcal{A} \subset \overline{\mathcal{M}}$ is closed. For each $\sim \in \mathcal{F}$, consider $C = \{Y_\sim \mid [Y] \mathrel{\overline{\in}} \mathcal{A}\}$, choose one element in each equivalence class modulo $\sim$ that intersects C and call the set of the chosen elements $B_\sim$. We have that $B_\sim \in \mathcal{M}$ by assumption (ii) and (iii). Take a limit point A of the net $(B_\sim)_{\sim \in \mathcal{F}}$. We check that A answers the question using the fact that $\mathcal{A}$ is closed.

In order to show that $(\overline{\mathcal{M}}, \overline{\in})$ satisfies BPC, we will show by induction on the BPF $\varphi \equiv \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ that for all $[X_1], \ldots, [X_n] \in \overline{\mathcal{M}}$, the set $\{([Y_1], \ldots, [Y_m]) \mid \varphi([X_1], \ldots, [X_n], [Y_1], \ldots, [Y_m])\}$ is closed in the product topology $\overline{\mathcal{M}}^{m+n}$. The case where φ is atomic and the stage where $\varphi \equiv \psi_1 \vee \psi_2$, $\varphi \equiv \psi_1\ \&\ \psi_2$ or $\varphi \equiv \exists x\ \psi$ are straightforward (see [15, fact 4.14]) (the case where $\varphi \equiv \exists x\ \psi$ corresponds to a projection, which remains closed since $\overline{M}$ is κ-compact).

Suppose at last that φ is $\forall x \in y\ \psi$. We have to show that for all $[Z_1], \ldots, [Z_n] \in \overline{\mathcal{M}}$:

$$\{([Y], [Y_1], \ldots, [Y_m]) \mid \forall [X] \mathrel{\overline{\in}} [Y]\ \psi([X], [Y], [Y_1], \ldots, [Y_m], [Z_1], \ldots, [Z_n])\}$$

is closed. In order to simplify the notations suppose that there are no variables $[Y_1], \ldots, [Y_m], [Z_1], \ldots [Z_n]$ and let us show that

$$\mathcal{A} = \{[Y] \mid \forall [X] \in [Y]\ \psi([X], [Y])\}$$

is closed. so take $[Y] \in \overline{\mathcal{A}}$ and take any $[X] \mathrel{\overline{\in}} [Y]$ and show that $\psi([X], [Y])$. The formula ψ defining a closed set by hypothesis, it suffices to show that for any open set $\mathcal{U} \in \overline{\mathcal{M}}^2$ with $([X], [Y]) \in \mathcal{U}$ there is $([X'], [Y']) \in \mathcal{U}$ with $\psi([X], [Y])$. Take a basis open set $\mathcal{U}$ with $([X], [Y]) \in \mathcal{U}$: $[X]_\sim \times [Y]_\sim$ (where $[X]_\sim$ denotes $\{[Y] \in \overline{\mathcal{M}} \mid Y_\sim \sim X_\sim\}$). Now take $[Y'] \in [Y]_\sim \cap \mathcal{A}$

and use claim 1 to find X' with $X_\sim \sim X'_\sim$ and $[X'] \mathrel{\overline{\in}} [Y']$. The couple $([X'],[Y'])$ answers the question. □ The previous construction is general, it allows to construct all known hyperuniverses. As an example, let us briefly see how to obtain the hyperuniverse N_κ of [11]. Consider the axiom X_1 of [12]. We have that any model $(A, \in_A)$ of ZFC can be extended to a model of $\mathrm{ZFC}_0 + X_1$ (see [12]; recall that ZFC_0 denotes the Zermelo-Fraenkel set theory without foundation). Now consider $(V_\kappa, \in)$ where V_κ is the set of sets of rank less than κ: an inaccessible cardinal having the tree-property. Extend $(V_\kappa, \in)$ to $(V'_\kappa, \in')$ a model of $\mathrm{ZFC}_0 + X_1$ as in [12]. On $(V'_\kappa, \in')$ consider the equivalences relations $\sim_\alpha$ $(\alpha \in \kappa)$ defined as follows

$$\begin{cases} \sim_0 &= V'_\kappa \times V'_\kappa \\ \sim_{\alpha+1} &= \sim^+_\alpha \ \text{(taken relatively to } \in') \\ \sim_\lambda &= \bigcap\limits_{\alpha<\lambda} \sim_\alpha \ \ (\lambda \text{ limit}) \end{cases}$$

We have that $\mathcal{F} = \{\sim_\alpha \mid \alpha \in \kappa\}$ is an admissible basis on $(V'_\kappa, \in')$. This structure satisfies all conditions of theorem 2.8. The Cauchy-completion $(\overline{V'_\kappa}, \overline{\in'})$ is the N_κ of [11].

For κ strongly inaccessible $> \omega$ and having the tree-property, we can also obtain the model M_κ of Malitz ([19]) by completing $(V_\kappa, \in)$, $\mathcal{F}$ being defined as before.

The following theorem gives a kind of reciprocal of theorem 2.8. It shows that any admissible basis for the uniformity of a hyperuniverse satisfies condition (iii) of the previous theorem (the other conditions are easily seen to be satisfied for a κ-hyperuniverse with $\kappa > \omega$). This generalizes a theorem of M. Forti and F. Honsell stating that if we have a κ-hyperuniverse then κ has the tree-property (adapt [13, lemma 2.3]).

Theorem 2.9. *Let H be a κ-hyperuniverse and let $\mathcal{U}$ be the set of entourages for the uniformity of H. Then $(\mathcal{U}, \supset)$ is a ramifiable directed set.*

Proof. We know that κ is strongly inaccessible by [13]. Suppose without loss of generality that H is a standard hyperuniverse. Let $\mathcal{F}$ be an admissible basis of entourages and denote $\gamma = |\mathcal{F}|$. We will show that $(\mathcal{F}, \supset)$ is ramifiable. For $X \subset H$, we denote by $\overline{X}$ the closure of X for the topology of H. For each $\sim \in \mathcal{F}$, $H/\sim$ is κ-finite as otherwise it would be an open covering of H with no κ-finite subcovering. We will need the following preliminary results.

Claim 1. *There exists a set $B \subset H$ with $|B| = \gamma$ such that B is discrete i.e.* $(\forall b \in B)(\overline{B \setminus \{b\}} = \overline{B} \setminus \{b\})$.

Proof. For any equivalence $e \in \mathcal{F}$, we have that $e = \{(a,b) \mid (a,b) \in e\}$ and $e' = \{(a,b) \mid (a,b) \notin e\}$ are both closed. Define $B = \{(a,a') \mid a \in \mathcal{F}\}$. We have $|B| = \gamma$. Moreover B is discrete since for any $e \in \mathcal{F}$

$$\begin{aligned} &\{(x,x') \in B \mid (x,x') \neq (e,e')\} \\ =& \{(x,x') \in B \mid (\exists y \in x)(y \in e') \vee (\exists y \in x')(y \in e)\} \end{aligned}$$

which achieves the proof since this last formula is BPF.

Claim 2. *For all $A \subset H$, $\bigcup \overline{A} = \overline{\bigcup A}$.*

Proof. The fact that $\overline{\bigcup A} \subset \bigcup \overline{A}$ follows from the fact that $\bigcup A \subset \bigcup \overline{A}$ and that $\bigcup \overline{A}$ is closed. Reciprocally suppose $x \in \bigcup \overline{A}$ and let U be an open set with $x \in U$. We will find $x' \in U$ with $x' \in \bigcup A$. We have $x \in y \in \overline{A}$ for some y. Define $W = \{z \mid z \cap U \neq \varnothing\}$. W is an open set with $y \in W$, so $W \cap A \neq \varnothing$. Choose $z \in W \cap A$ and $x' \in z \cap U$. We have $x' \in U$ and $x' \in z \in A$. This shows $x' \in \bigcup A$ and the result.

Consider now a κ-tree $(T, \leqslant_T)$ on $\mathcal{F}$. We will show that T has a branch. Consider a set B as in claim 1. We will suppose that $(T, \leqslant_T)$ satisfies $T \subset \mathcal{P}_\kappa B$ and that $\leqslant_T$ is $\subset$. This can be assumed without loss of generality: first replace the elements of T by elements of B (this can be done since $|T| = |B| = \gamma$) and then replace any $a \in T$ by $\{x \mid x \leqslant_T a\}$. Denote $L = \bigcup T$; L is discrete since $L \subset B$. Consider now a limit point s of the net $(\mathrm{lev}_T(\sim))_{(\sim \in \mathcal{F})}$. Such a limit point exists since H is κ-compact (see [18, theorem 2, chap. 5]). Pick any $a \in s$. In order to show that T has a branch, we will prove the following fact:

$$(\forall \sim \in \mathcal{F})(\exists! y \in \mathrm{lev}_T(\sim))(y \subset a) \tag{3}$$

Having proved 3, we find a branch in the following way: define for $\sim \in \mathcal{F}$, $h(\sim) =$ the unique $y \in \mathrm{lev}_T(\sim)$ with $y \subset a$. Now let us prove (3). Fix $\sim \in \mathcal{F}$ and let $t = \bigcup \mathrm{lev}_T(\sim)$; t is κ-finite by construction and so belongs to H. Let d be the following function of domain t: $d(x) = \overline{L \setminus \{x\}} = \overline{L} \setminus \{x\}$. The function d belongs to H since it is closed due to the fact that $\mathrm{dom}(d)$ is κ-finite (remember that we identify a function with its graph). Consider $G = \{\mathrm{lev}_T(\sim') \mid \sim' \geqslant \sim\}$ and denote $S = \bigcup G$. Clearly $s \in \overline{G}$ and so $a \in \overline{S}$ by claim 2. We have

$$(\forall x \in S)(\exists! y \in \mathrm{lev}_T(\sim))(y \subset x)$$

So (3) will be proved if we prove that

$$\{x \in \overline{S} \mid (\exists! y \in \mathrm{lev}_T(\sim))(y \subset x)\} \tag{4}$$

is closed. The formula $(\exists! y \in \mathrm{lev}_T(x))(y \subset x)$ is equivalent to the conjunction of the two following formulas:

$$(\exists y \in \mathrm{lev}_T(\sim))(y \subset x)$$

and

$$(\forall y, y' \in \mathrm{lev}_T(\sim))(y \not\subset x \vee y' \not\subset x \vee y = y')$$

For $x \in \overline{S}$ and for $y \in \mathrm{lev}_T(\sim)$, we have

$$y \not\subset x \Leftrightarrow (\exists v \in y)(\forall z \in x)(z \in d(v))$$

For the proof, use the fact that $z \in x \in \overline{S} \Rightarrow z \in \bigcup \overline{S} = \overline{\bigcup S} \subset \overline{L}$. Proceed similarly for $y' \not\subset x$. This proves that the set (4) is described by a BPF formula and achieves the proof. □ Consider a κ-hyperuniverse H, denote $\mathrm{uw}(H)$ the *uniform weight* of H, i.e. the smallest cardinal γ such that there is a basis of entourages $\mathcal{F}$ for the uniformity of H with $|\mathcal{F}| = \gamma$.

Theorem 2.10. *Let κ, γ be cardinals. The following conditions are equivalent:*

- *There exists a κ-hyperuniverse with* $\mathrm{uw}(H) = \gamma$.
- $\gamma^{<\kappa} = \gamma$ *and κ is mildly γ-ineffable.*

Proof. Suppose that there is a κ-hyperuniverse H with $\mathrm{uw}(H) = \gamma$. Take an admissible basis of entourages $\mathcal{F}$ of H, κ is strongly inaccessible by [13], $\mathcal{F}$ is degree-regular so $|\mathcal{F}| = \mathrm{cf}(\mathcal{F}) = \gamma$ and $\gamma^{<\kappa} = \gamma$ by lemma 1.4; $(\mathcal{F}, \supset)$ is ramifiable by theorem 2.9. Since $\mathcal{P}_{\mathrm{bd}}(\mathcal{F}) \overset{\mathrm{cof}}{\cong} \mathcal{P}_\kappa\gamma$, this implies that $\mathcal{P}_\kappa\gamma$ is ramifiable which is equivalent to say that κ is mildly γ-ineffable by theorem 1.6 since $\gamma^{<\kappa} = \gamma$.

On the other hand, suppose that $\gamma^{<\kappa} = \gamma$ and that κ is mildly γ-ineffable. By theorem 1.6, $\mathcal{P}_\kappa\gamma$ is ramifiable. Let us consider the κ-Cantor space $D = \{0,1\}^\gamma$ with the κ-Cantor uniformity: $\{\sim_X \mid X \in \mathcal{P}_\kappa\gamma\}$ where $a \sim_X b$ is satisfied iff a and b coincide on X. Take Z a set of self-singletons in bijection with D (this exists by U_l) and transfer the uniformity of D on Z. Let us consider $V_\kappa[Z]$ (for $\alpha \in On$, $V_\alpha[A]$ is defined by induction as follows: $V_0[A] = A$, $V_{\alpha+1}[A] = \mathcal{P}V_\alpha[A]$, $V_\lambda[A] = \bigcup_{\alpha<\lambda} V_\alpha[A]$ for λ limit). On $R = V_\kappa[Z]$, consider the equivalences $\sim_{X,\alpha}$ ($X \in \mathcal{P}_\kappa\gamma$, $\alpha \in \kappa$) defined as follows:

$$\begin{cases} \sim_{X,0} & = \sim_X \cup (R \setminus X)^2 \\ \sim_{X,\alpha+1} & = (\sim_{X,\alpha})^+ \text{ (relatively to } \in) \\ \sim_{X,\lambda} & = \bigcap_{\beta\in\lambda} \sim_{X,\beta} \text{ (for } \lambda \text{ limit)} \end{cases}$$

Obviously $\mathcal{F} = \{\sim_{X,\alpha} \mid X \in \mathcal{P}_\kappa\gamma,\ \alpha \in \kappa\}$ is degree-regular with $|\mathcal{F}| = \gamma$ so $\mathcal{F}$ is ramifiable by theorem 1.6 and lemma 1.4. It is easy to see that $(R, \in)$ with the equivalences $\sim_{X,\alpha}$ satisfies the conditions of theorem 2.8. The Cauchy-completion $(\overline{R}, \overline{\in})$ is the hyperuniverse we were looking for. □
This theorem permits to solve a question of M. Forti, asking when there can exist a non κ-metrizable κ-hyperuniverse. A κ-hyperuniverse is said to be *κ-metrizable* iff its uniformity is induced by a "κ-distance" $d : H \to \mathbb{R}^*$ where $\mathbb{R}^*$ is a non-standard model of the real numbers of cofinality κ. To say that a κ-hyperuniverse H if κ-metrizable is equivalent to say that $\mathrm{uw}(H) = \kappa$. Using the previous theorem and proposition 1.3, we have

Corollary 2.11. *Consider a cardinal κ.*

- *There exists a non κ-metrizable κ-hyperuniverse iff κ is mildly κ^+-ineffable.*
- *For all γ, there exists a κ-hyperuniverse H with* $\mathrm{uw}(H) = \gamma$ *iff κ is strongly compact.*

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