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GRABOWSKI LATTICES ARE GENERATED BY GRAPHS

A b s t r a c t. We show that Grabowski lattices of substitutions (see [1], [2]) are isomorphic to lattices generated by certain graphs, [3]. Since the latter are Heyting algebras, this will imply that so are the Grabowski lattices.

1. Preliminaries

The notions of a signature, an algebra of the signature τ, the algebra of terms, $\mathfrak{Trm}(\tau)$, of a signature τ as well as the notions of a congruence, a quotient algebra and of a free algebra with the set A as the set of free generators, can be found for instance in [4].

Let $E \subseteq \mathfrak{Trm}(\tau) \times \mathfrak{Trm}(\tau)$. We define $\approx_E$ as the least congruence in $\mathfrak{Trm}(\tau)$ containing E.

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The notion of a lattice, an ideal in a lattice, etc., can be found for instance in [5].

Let $\mathcal{K}$ be a lattice with a lower bound $\perp$ and let ∇ be an ideal in $\mathcal{K}$. The congruence $\approx_\nabla$ generated by ∇ is the least congruence in $\mathfrak{Trm}(\tau)$ identifying terms equal in $\mathcal{K}$ and which contains all the pairs $(*)\quad \langle\delta,\perp\rangle,\ \delta\in\nabla$.

Instead of writing ${}^{\mathcal{K}}/_{\approx_\nabla}$ and $[a]_{\approx_\nabla}$, we shall write ${}^{\mathcal{K}}/_{\nabla}$ and $[a]_\nabla$, respectively.

2. Hypergraphs and lattices generated by them

All the notions cited below can be found in [3].

A *hypergraph* is a structure $\langle\mathcal{V},\mathcal{E}\rangle$, where $\mathcal{V}$ is any set and $\mathcal{E}$ is a family of nonempty subsets of $\mathcal{V}$. The elements of $\mathcal{V}$ are called *vertices* and the elements of $\mathcal{E}$ - *edges* of $\mathcal{H}$.

Let $\mathcal{H}=\langle\mathcal{V},\mathcal{E}\rangle$ be a hypergraph. Then by $\mathfrak{K}^\circ(\mathcal{H})$ we denote the free distributive lattice with bounds $\perp$ and $\top$ freely generated by the set $\mathcal{V}$. Moreover, let ∇ be the ideal generated by all elements of the form $\prod e$ with $e\in\mathcal{E}$ and let $\mathfrak{K}(\mathcal{H})={}^{\mathfrak{K}^\circ(\mathcal{H})}/_{\nabla}$. It is shown in [3] that if $\mathcal{H}$ is *locally finite*, i.e. meets of infinite number of edges are empty, then $\mathfrak{K}(\mathcal{H})$ is a Heyting algebra. We say that $\mathfrak{K}(\mathcal{H})$ is *generated* by $\mathcal{H}$.

In the sequel the following lemma will be usefull:

Lemma 2.1. *Let $\mathcal{H}$ be a hypergraph and let $\mathcal{A}$ be a lattice with bounds $\perp$ and $\top$. Let, moreover, $v:\mathcal{V}\to\mathcal{A}$ be such that $\prod v[e]=\perp$ for every $e\in\mathcal{E}$. Then there exists the unique homomorphism $\mathfrak{h}^v:\mathfrak{K}(\mathcal{H})\to\mathcal{A}$ extending $\kappa\circ v$, where $\kappa:\mathfrak{K}^\circ(\mathcal{H})\to\mathfrak{K}(\mathcal{H})$ is defined as $\kappa(a)=[a]_\nabla$, $a\in\mathfrak{K}^\circ(\mathcal{H})$.*

Proof. Since $\mathfrak{K}^\circ(\mathcal{H})$ is free in the class of distributive bounded lattices and the set $\mathcal{V}$ is the set of its free generators, there exists the unique homomorphism $\mathfrak{h}^\circ:\mathfrak{K}^\circ\to\mathcal{A}$ extending v. Let us consider the relation:

$$E=\{\langle\alpha,\beta\rangle:\ \mathfrak{h}^\circ(\alpha)=\mathfrak{h}^\circ(\beta)\}\,.$$

One easily shows that E is a congruence of the algebra of lattice terms which contains all pairs of terms equal in $\mathfrak{K}^\circ$ and all pairs of the form $\langle \prod e, \perp \rangle$, $e \in \mathcal{E}$. Since $\approx_\nabla$ is the least congruence satisfying these conditions, $\mathcal{E}$ must contain $\approx_\nabla$. Hence $[\alpha]_\nabla = [\beta]_\nabla$ implies $\mathfrak{h}^\circ(\alpha) = \mathfrak{h}^\circ(\beta)$ for all terms α, β. This proves that $\mathfrak{h} : \mathfrak{K}(\mathcal{H}) \to \mathcal{A}$ given by the condition $\mathfrak{h}([\alpha]_\nabla) = \mathfrak{h}^\circ(\alpha)$ is well defined.

Uniqueness of $\mathfrak{h}$ is obvious. ∎

3. Grabowski lattices

Let A, B be any sets, and let $\mathsf{PFin}(A, B)$ be the set of all finite families of finite partial functions from A to B.

Moreover, let $\mu : \mathsf{PFin}(A, B) \to \mathsf{PFin}(A, B)$ be given by

$$\mu(\mathcal{C}) = \{ f \in \mathcal{C} \ : \ \forall_{g \in \mathcal{C}} (g \subseteq f \to f = g) \} .$$

Moreover let

$$\mathcal{C} \star \mathcal{D} = \{ f \cup g |\ f \in \mathcal{C},\ g \in \mathcal{D},\ f \approx g \} ,$$

$$\mathcal{C} \sqcap \mathcal{D} = \mu(\mathcal{C} \star \mathcal{D}), \qquad \mathcal{C} \sqcup \mathcal{D} = \mu(\mathcal{C} \cup \mathcal{D}),$$

where $f \approx g$ means that $f \cup g$ is also a function (we say then that f and g are *compatible*).

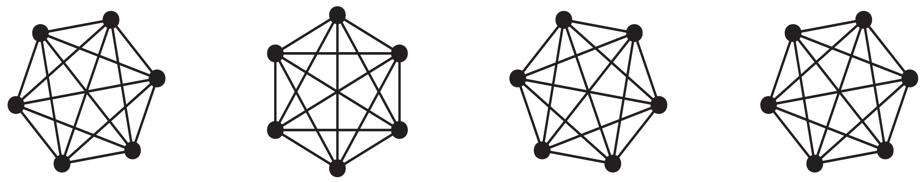
Let $\mathcal{U} = \mu[\mathsf{PFin}(A, B)]$. Grabowski shows in [1] that the structure $\mathfrak{Gr} = \langle \mathcal{U}, \sqcap, \sqcup \rangle$ is a distributive lattice. In [2] it is shown that it is a Heyting algebra. The elements $\emptyset$ and $\{\emptyset\}$ are the lower and the upper bounds of $\mathfrak{Gr}$, respectively.

4. The main result

In this section we will prove that Grabowski's lattices are isomorphic to lattices of certain hypergraphs.

Let A and B be sets, and let $\mathfrak{G} = \mathfrak{Gr}(A, B)$. Let, moreover, $\mathcal{H} = \langle \mathcal{V}, \mathcal{E} \rangle$, where $\mathcal{V} = A \times B$ and let $\mathcal{E} = \{ \{ \langle a, b_1 \rangle, \langle a, b_2 \rangle \} : \ b_1 \neq b_2 \}$. We will prove that $\mathfrak{K} = \mathfrak{K}(\mathcal{H})$ is isomorphic to $\mathfrak{G}$. Because for finite B the lattice $\mathfrak{K} = \mathfrak{K}(\mathcal{H})$ is locally finite, this implies that Grabowski lattices are Heyting algebras. In fact they are Heyting algebras if and only if the set B is finite.

It seems also interesting to figure out graphs generating Grabowski lattices. Let A and B be finite. Then the graph generating the lattice $\mathfrak{Gr}(A, B)$ consists of $\mathsf{Card}(A)$ copies the full graph with $\mathsf{Card}(B)$ elements. The following picture depicts that in the case of $\mathsf{Card}(A) = 4$ and $\mathsf{Card}(B) = 6$:



Let $\mathfrak{K}^\circ = \mathfrak{K}^\circ(\mathcal{H})$ and let $\Phi^\circ : \mathsf{PFin}(A, B) \to \mathfrak{K}^\circ$ be defined as $\Phi^\circ(\mathcal{C}) = \sum \{\prod f : f \in \mathcal{C}\}$, $\mathcal{C} \in \mathsf{PFin}(A, B)$. Moreover, let $\Phi = \kappa \circ \Phi^\circ$, where κ is the same as in Lemma 2.1. We will show that:

Theorem 4.1. $\Phi\lceil\mathfrak{G}$ *is an isomorphism of the lattices* $\mathfrak{G}$ *and* $\mathfrak{K}$.

We will prove this in three steps:

(1) $\Phi\lceil\mathfrak{G}$ is a homomorphism, i.e.

$$\Phi(\mathcal{C} \sqcup \mathcal{D}) = \Phi(\mathcal{C}) + \Phi(\mathcal{D}), \quad \Phi(\mathcal{C} \sqcap \mathcal{D}) = \Phi(\mathcal{C}) \cdot \Phi(\mathcal{D}),$$
$$\Phi(\emptyset) = \top, \quad \text{and} \quad \Phi(\{\emptyset\}) = \bot, \quad \mathcal{C}, \mathcal{D} \in \mathfrak{G}.$$

(2) Φ takes all elements of $\mathfrak{K}$ as its values.

(3) If $\Phi(\mathcal{C}) = \Phi(\mathcal{D})$, then $\mathcal{C} = \mathcal{D}$.

This, of course, will complete the proof of the theorem.

Lemma 4.2. $\Phi^\circ(\mathcal{C}) = \Phi^\circ(\mu(\mathcal{C})))$, $\qquad \mathcal{C} \in \mathsf{PFin}(A, B)$.

Proof. Let $\mathcal{C} \in \mathsf{PFin}(A, B)$. We have:

$$\begin{aligned}\Phi^\circ(\mathcal{C}) &= \sum\left\{\prod f:\ f \in \mathcal{C}\right\}\\ &= \sum\left\{\sum\left\{\prod f:\ g \subseteq f,\ f \in \mathcal{C}\right\}:\ g \in \mu(\mathcal{C})\right\}\\ &= \sum\left\{\prod g:\ g \in \mu(\mathcal{C})\right\},\end{aligned}$$

because $\prod f \leq \prod g$ if $g \subseteq f$. ∎

(1) We have:

$$\begin{aligned}\Phi^\circ(\mathcal{C} \sqcup \mathcal{D}) &= \Phi^\circ(\mu(\mathcal{C} \cup \mathcal{D})) = \Phi^\circ(\mathcal{C} \cup \mathcal{D}) =\\ &= \sum\left\{\prod f:\ f \in \mathcal{C} \cup \mathcal{D}\right\}\\ &= \sum\left\{\prod f:\ f \in \mathcal{C}\right\} + \sum\left\{\prod f:\ f \in \mathcal{D}\right\}\\ &= \Phi^\circ(\mathcal{C}) + \Phi^\circ(\mathcal{D}).\end{aligned}$$

We have:

$$\begin{aligned}\Phi^\circ(\mathcal{C} \sqcap \mathcal{D}) &= \Phi^\circ(\mu(\mathcal{C} \star \mathcal{D})) = \Phi^\circ(\mathcal{C} \star \mathcal{D})\\ &= \Phi^\circ(\{f \cup g:\ f \in \mathcal{C}, g \in \mathcal{D},\ f \approx g\})\\ &= \sum\left\{\prod(f \cup g):\ f \in \mathcal{C}, g \in \mathcal{D},\ f \approx g\right\}\\ &\approx_\nabla \sum\left\{\prod(f \cup g):\ f \in \mathcal{C}, g \in \mathcal{D},\ f \approx g\right\}\\ &\qquad + \sum\left\{\prod(f \cup g):\ f \not\approx g,\ f \in \mathcal{C}, g \in \mathcal{D}\right\}\\ &= \sum\left\{\prod(f \cup g):\ f \in \mathcal{C}, g \in \mathcal{D}\right\}\\ &= \sum\left\{\prod f \cdot \prod g:\ f \in \mathcal{C}, g \in \mathcal{D}\right\}\\ &= \sum\left\{\prod f:\ f \in \mathcal{C}\right\} \cdot \sum\left\{\prod g:\ g \in \mathcal{D}\right\} = \Phi^\circ(\mathcal{C}) \cdot \Phi^\circ(\mathcal{D}).\end{aligned}$$

The equivalence $\approx_\nabla$ above follows from that $\prod (f \cup g) = \bot$ for $f \cup g$ not being a function.

We have:

$$\Phi^\circ(\emptyset) = \sum\left\{\prod f : f \in \emptyset\right\} = \sum \emptyset = \bot$$

and

$$\Phi^\circ(\{\emptyset\}) = \sum\left\{\prod f : f \in \{\emptyset\}\right\} = \sum\left\{\prod \emptyset\right\} = \top.$$

(2) Since $\mathfrak{K}^\star$ is a distributive bounded lattice, every $a \in \mathfrak{K}^\circ$ is either equal to $\bot$, either to $\top$ or is of the form $\sum\{\prod f : f \in \mathcal{C}\}$ for some $\mathcal{C} \in \mathsf{PFin}(A, B)$. This means that $a = \Phi(\mathcal{C})^\circ = \Phi^\circ(\mu(\mathcal{C}))$. Since $\mu(\mathcal{C}) \in \mathfrak{G}$, this proves that Φ is onto $\mathfrak{K}$.

(3) Let $\mathcal{C}, \mathcal{D} \in \mathfrak{G}$ and let $\Phi(\mathcal{C}) = \Phi(\mathcal{D})$. I.e., let:

$$\left[\sum\left\{\prod f : f \in \mathcal{C}\right\}\right]_\nabla = \left[\sum\left\{\prod g : g \in \mathcal{D}\right\}\right]_\nabla.$$

We will prove that $\mathcal{C} \subseteq \mathcal{D}$. Let us fix $f_0 \in \mathcal{C}$ and let us suppose that $g \setminus f_0 \neq \emptyset$, for each $g \in \mathcal{D}$. Then, let

$$v(\langle a, b\rangle) = \begin{cases} 1, & \langle a, b\rangle \in f_0, \\ 0, & \text{otherwise}. \end{cases}$$

By Lemma 2.1, there exists a unique homomorphism $\mathfrak{h}$ from $\mathfrak{K}$ to the two-element lattice $Ł_2 = \{\bot, \top\}$ extending $\kappa \circ v$. Then, however,

$$\mathfrak{h}\left(\left[\sum\left\{\prod f : f \in \mathcal{C}\right\}\right]_\nabla\right) \geq \mathfrak{h}\left(\prod f_0\right) = \top$$

but

$$\mathfrak{h}\left(\left[\sum\left\{\prod g : g \in \mathcal{D}\right\}\right]_\nabla\right) \leq \mathfrak{h}\left(\left[\sum\left\{\prod (g \setminus f_0) : g \in \mathcal{D}\right]_\nabla\right\}\right) = \bot,$$

which yields $\bot = \top$. Contradiction. Hence, for every $f_0 \in \mathcal{C}$ there exists $g \in \mathcal{D}$ with $g \subseteq f_0$. By the symmetry, we also prove that for every $g_0 \in \mathcal{D}$ there must exist $f \in \mathcal{C}$ satisfying $f \subseteq g_0$. Now, let $f_1 \in \mathcal{C}$. By the above, there exists $g \in \mathcal{D}$ with $g \subseteq f_1$. Also, by the same argument, there must exist $f_2 \in \mathcal{C}$ with $f_2 \subseteq g$. This means that $f_2 \subseteq g \subseteq f_1$. Since

$f_1, f_2 \in \mathcal{C} \in \mathfrak{G}$, this imples that $f_1 = g = f_2$. Hence $f_1 \in \mathcal{D}$, for every $f_1 \in \mathcal{C}$, i.e. $\mathcal{C} \subseteq \mathcal{D}$. Analogously, we prove that $\mathcal{D} \subseteq \mathcal{C}$. ■

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