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A CANONICAL MODEL CONSTRUCTION FOR SUBSTRUCTURAL LOGICS WITH STRONG NEGATION

A b s t r a c t. We introduce Kripke models for propositional substructural logics with strong negation, and show the completeness theorems for these logics using an extended Ishihara's canonical model construction method. The framework presented can deal with a broad range of substructural logics with strong negation, including a modified version of Nelson's logic N^{-}, Wansing's logic COSPL, and extended versions of Visser's basic propositional logic, positive relevant logics, Corsi's logics and Méndez's logics.

1. Introduction

Examples of substructural logics include positive relevant logics, which are studied in the field of philosophical logic, intuitionistic linear logic in computer science, BCK-logic related to the theory of BCK-algebra, and

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full Lambek logic, which is studied in computational linguistics. In 1972, Routley and Meyer introduced Kripke semantics for positive relevant logics, characterized by a ternary accessibility relation [2] [11]. In 1985, Ono and Komori established Kripke semantics for the FL family, including BCK-logic, intuitionistic linear logic and full Lambek logic FL. These semantics were characterized by a semilattice-ordered monoid [10]. In 1988, Došen introduced Kripke semantics, termed groupoid semantics, for a broad range of substructural logics, including the logics mentioned above [4] [5]. Došen's semantics are essentially the same as Ono and Komori's semantics, yet they were developed independently. In 2000, Ishihara introduced a canonical model construction method for the semantics of Ono and Komori [6]. This method can deal with a broad range of substructural logics, including certain Došen logics, Corsi's logic F and the standard substructural logics mentioned above. The class of logics which this method is applicable to called, by Ishihara, *normal substructural logic.*

Strong negation was first proposed by Nelson (1949) and independently by Markov (1950). The negation was introduced in order to express "constructible falsity" in recursive realizability [8]. Nelson's logics, such as N and N^-, have been studied by many logicians [1] [16]. These logics are, roughly speaking, several extensions or modifications of intuitionistic logic. The Kripke semantics for Nelson's logics differ from those given for the intuitionistic logic. These semantics have two kinds of valuation: $\models^+$ (corresponds to provability) and $\models^-$ (corresponds to refutability). In 1969, Thomason introduced Kripke semantics for a predicate logic with strong negation. The semantics was characterized by the binary accessibility relation which is the same binary relation as in Kripke semantics for the predicate intuitionistic logic [14]. In 1993, Wansing introduced a family of propositional substructural logics with strong negation. We call the family the COSPL family, which includes a modified version of Nelson's logic N^-. Roughly speaking, the COSPL family is obtained from the FL family by adding strong negation. Wansing gave Kripke semantics for these logics and the corresponding informational interpretation for the semantics [16]. Wansing's semantics are natural extentions of Došen's semantics.

In the present paper, we extend Ishihara's result to substructural logics with strong negation, allowing us to deal with a broad range of logics with strong negation, such as the modified version of Nelson's N^- and Wansing's COSPL. This result is therefore also an extension of Wansing's result. The semantic framework presented is based on Wansing's semantics.

This paper is organized as follows.

In section 2, we present Ishihara's basic normal substructural logic L and its extensions, and introduce new normal substructural logics, such as Visser's basic propositional logic BPL [15] (Hilbert-style systems for BPL and its neighbors are proposed in [9] [13] [12]), relevant logics RMO^+ and EMO^+ [7], Corsi's logics FT, FR and FRT [3], and Méndez's logics T°_+, R°_+ and RMO°_+ [7]. The completeness theorems for the new logics are proved using Ishihara's canonical model construction method. Then we introduce extended basic normal substructural logic $\mathrm{L}^\sim$ with strong negation axioms and its extensions. These extensions include a modified version of Nelson's logic N^- and Wansing's logic COSPL. In addition, we define a new family of normal substructural logics with strong negation: extended positive relevant logics $\mathrm{B}^\sim$, $\mathrm{T}^\sim$, $\mathrm{E}^\sim$, $\mathrm{R}^\sim$, $\mathrm{RMO}^\sim$ and $\mathrm{EMO}^\sim$, extended strict implication $\mathrm{S4}^\sim$, extended Corsi's logics $\mathrm{F}^\sim$, $\mathrm{FT}^\sim$, $\mathrm{FR}^\sim$ and $\mathrm{FRT}^\sim$, extended Visser's logic $\mathrm{BPL}^\sim$, and extended Méndez's logics $\mathrm{T}^\sim_\circ$, $\mathrm{R}^\sim_\circ$ and $\mathrm{RMO}^\sim_\circ$. This new family sheds new light on the study of logics with strong negation and their application to computer science, linguistics, philosophy and so on.

In section 3, we introduce Kripke semantics for the normal substructural logics with strong negation mentioned above, and present a soundness theorem and a correspondence between axiom schemes and conditions on frames.

In section 4, we prove the completeness theorems for the logics using an extended Ishihara's canonical model construction method. Then we show that the completeness theorem can be extended to any logic obtained from $\mathrm{L}^\sim$ by adding arbitrary combinations of the axiom schemes introduced in section 2.

2. Normal Substructural Logics with Strong Negation

Prior to the precise discussion, we introduce the language used in this paper. *Formulas* are constructed from propositional variables, the constants $\mathbf{1}$ (multiplicative), $\top$ (additive) and $\bot$ (additive), $\rightarrow$ (implication), $\wedge$ (conjunction), $*$ (fusion), $\vee$ (disjunction), $\sim$ (strong negation). Lower case letters $p, q, ...$ are used as metavariables for propositional variables, lower case Greek letters $\alpha, \beta, ...$ are used as metavariables for formulas. The symbol $\equiv$ means equality of sequences of symbols frequently. We adopt the convention of association to the right in order to omit parentheses. For example, $(\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma \equiv (\alpha\to(\beta\to\gamma))\to(\beta\to(\alpha\to\gamma))$.

Definition 2.1. (Ishihara's Basic Logic L) The axiom schemes and inference rules for the basic normal substructural logic L in *[6]* are as follows:

A1: $\mathbf{1}$,

A2: $\alpha\to\top$,

A3: $\bot\to\alpha$,

A4: $\alpha\to\alpha$,

A5: $\alpha\wedge\beta\to\alpha$,

A6: $\alpha\wedge\beta\to\beta$,

A7: $(\gamma\to\alpha)\wedge(\gamma\to\beta)\to\gamma\to\alpha\wedge\beta$,

A8: $\alpha\to\alpha\vee\beta$,

A9: $\beta\to\alpha\vee\beta$,

A10: $\alpha\to\beta\to\alpha*\beta$,

$$\frac{\alpha}{\mathbf{1}\to\alpha}\ (ness),$$

$$\frac{\alpha\to\beta \quad \alpha}{\beta}\ (mp), \qquad \frac{\beta\to\gamma}{(\alpha\to\beta)\to(\alpha\to\gamma)}\ (pref), \qquad \frac{\alpha\to\beta}{(\beta\to\gamma)\to(\alpha\to\gamma)}\ (suff),$$

$$\frac{\alpha \quad \beta}{\alpha\wedge\beta}\ (adj), \qquad \frac{\alpha\to\gamma \quad \beta\to\gamma}{\alpha\vee\beta\to\gamma}\ (or), \qquad \frac{\alpha\to\beta\to\gamma}{\alpha*\beta\to\gamma}\ (residu).$$

The following result is in [6].

Proposition 2.2. (A Family of Normal Substructural Logics) *We consider the following axiom schemes:*

B1: $(\alpha\to\gamma)\wedge(\beta\to\gamma)\to\alpha\vee\beta\to\gamma$,

B2: $\alpha\wedge(\beta\vee\gamma)\to(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$,

B3: $(\beta\to\gamma)\to(\alpha\to\beta)\to(\alpha\to\gamma)$,

B4: $(\alpha\to\beta)\to(\beta\to\gamma)\to(\alpha\to\gamma)$,

B5: $(\alpha\to\alpha\to\beta)\to\alpha\to\beta$, B6: $(\mathbf{1}\rightarrow\alpha)\rightarrow\alpha$,

B7: $\alpha\rightarrow\mathbf{1}$,

B8: $\alpha\rightarrow(\alpha\rightarrow\beta)\rightarrow\beta$,

B9: $\alpha\rightarrow\bot\rightarrow\beta$,

B10: $\alpha\rightarrow\mathbf{1}\rightarrow\alpha$,

B11: $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\alpha * \beta\rightarrow\gamma$,

B12: $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\beta\rightarrow\alpha\rightarrow\gamma$,

B13: $\alpha\rightarrow\beta\rightarrow\alpha$,

B14: $(\alpha\rightarrow\beta)\wedge(\beta\rightarrow\gamma)\rightarrow\alpha\rightarrow\gamma$.

The following holds.

1. *The positive fragment* B^+ *of basic relevant logic is obtained from* L *by adding the axiom schemes* B1 *and* B2.

2. *The positive relevant logics* T^+, E^+ *and* R^+ *are obtained from* B^+ *by adding the axiom schemes (*B3, B4, B5*), (*B3, B4, B5, B6*) and (*B3, B4, B5, B8*) respectively.*

3. *The positive fragment* $\mathrm{S4}^+$ *of strict implication is obtained from* E^+ *by adding* B7.

4. *Full Lambek logic* FL, $\mathrm{FL_e}$ *(intuitionistic linear logic),* $\mathrm{FL_{ew}}$ *(BCK-logic),* $\mathrm{FL_{ec}}$ *and* $\mathrm{FL_{ecw}}$ *are obtained from* L *by adding (*B1, B3, B6, B9, B10, B11*), (*B1, B3, B6, B9, B10, B11, B12*), (*B1, B3, B6, B9, B10, B11, B12, B13*), (*B1, B3, B6, B9, B10, B11, B12, B5*) and (*B1, B3, B6, B9, B10, B11, B12, B13, B5*) respectively.*

5. *Corsi's logic* F *is obtained from* B^+ *by adding* B7 *and* B14.

We call the logic, obtained from L *by adding arbitrary combinations of the axiom schemes introduced above,* normal substructural logic.

Further we can get the following new facts.

Proposition 2.3. (A New Family of Normal Substructural Logics) *If a formula* γ *is of the form* $\gamma_1\rightarrow\gamma_2$, *then* γ *is denoted by* $\overrightarrow{\gamma}$. *We consider the following axiom schemes:*

B15: $\alpha \to \beta \to \alpha \wedge \beta$,

B16: $\alpha \wedge (\alpha \to \beta) \to \beta$,

B17: $\alpha \to \alpha \to \alpha$,

B18: $\overrightarrow{\alpha} \to \overrightarrow{\alpha} \to \overrightarrow{\alpha}$,

B19: $(\alpha \to \beta \to \overrightarrow{\gamma}) \to (\beta \to \alpha \to \overrightarrow{\gamma})$,

B20: $\overrightarrow{\alpha} \to \beta \to \overrightarrow{\alpha}$,

B21: $(\alpha \to \alpha \to \overrightarrow{\beta}) \to \alpha \to \overrightarrow{\beta}$,

B22: $\alpha \to (\alpha \to \overrightarrow{\beta}) \to \overrightarrow{\beta}$.

The following holds.

1. *Visser's basic propositional logic* BPL *is obtained from Corsi's logic* F *by adding the axiom schemes* B15 *and* B13 *(see* [13] [9] [12]*).*
2. *Corsi's logics* FT, FR *and* FRT *are obtained from* F *by adding the axiom scheme(s) (*B20*), (*B16*) and (*B20, B16*) respectively (see* [3]*).*
3. *Relevant logic* RMO^{+} *(or* EMO^{+}*) is obtained from* R^{+} *(or* E^{+} *respectively) by adding* B17 *(or* B18 *respectively) (see* [7]*)* [1].
4. *Méndez's logic* T_{+}° *is obtained from* T^{+} *by replacing* B5 *by* B21. *Méndez's logics* R_{+}° *and* RMO_{+}° *are respectively obtained from* R^{+} *and* RMO^{+} *by replacing* B5 *and* B8 *by* B21 *and (*B22 *or* B19*) (see* [7]*)* [2].

We also call the logic, obtained from L *by adding arbitrary combinations of the axiom shemes* B1–B22*, normal substructural logic.*

By using the same method used in [6], we can derive the completeness results for these logics. The proof is similar to that in the sections 3 and 4 of the present paper (roughly speaking, the proof is included). Thus we omit the proof.

Further we remark the following fact: $\mathrm{L} \subseteq \mathrm{B}^{+} \subseteq \mathrm{F} \subseteq \mathrm{FT} \subseteq \mathrm{BPL}$ where $\subseteq$ denotes the inclusion between the sets of provable formulas. The fact is first pointed out by Ono in [9].

[1] Strictly speaking, these logics defined in this paper are not the same formalizations.

[2] Strictly speaking, these logics are minor modifications of the original versions of Méndez's logics. For example, the original versions have no fusion connective.

Definition 2.4. (The Logic L$^{\sim}$) The logic L$^{\sim}$ is obtained from L by adding the following axiom schemes:

S1: $\alpha \to \sim\sim\alpha$,

S2: $\sim\sim\alpha \to \alpha$,

S3: $\sim\mathbf{1} \to \alpha$,

S4: $\sim \top \to \alpha$,

S5: $\alpha \to \sim \bot$,

S6: $\sim (\alpha \to \beta) \to \alpha * \sim \beta$,

S7: $\alpha * \sim \beta \to \sim (\alpha \to \beta)$,

S8: $\sim (\alpha \wedge \beta) \to \sim \alpha \vee \sim \beta$,

S9: $\sim \alpha \vee \sim \beta \to \sim (\alpha \wedge \beta)$,

S10: $\sim (\alpha \vee \beta) \to \sim \alpha \wedge \sim \beta$,

S11: $\sim \alpha \wedge \sim \beta \to \sim (\alpha \vee \beta)$,

S12: $\sim (\alpha * \beta) \to \sim \alpha * \sim \beta$,

S13: $\sim \alpha * \sim \beta \to \sim (\alpha * \beta)$

Proposition 2.5. (Nelson's N$^-$, Wansing's COSPL and Wansing's N) *The following holds.*

1. *Nelson's* N$^-$ *is obtained from the* $\{\to, \wedge, \vee, \sim\}$*-fragment of* $\mathrm{FL}_{\mathrm{ecw}}$ *by adding* S1–S2, S8–S11 *and the axiom schemes:* $\sim (\alpha \to \beta) \to \alpha \wedge \sim \beta$, $\alpha \wedge \sim \beta \to \sim (\alpha \to \beta)$ *(see* [1]*).*
2. *Wansing's* COSPL *is obtained from* FL *by adding* S1–S13 *(see* [16]*).*
3. *Wansing's* N *is obtained from* $\mathrm{FL}_{\mathrm{ecw}}$ *by adding* S1–S13. *Wansing used the name* N *for the extended Nelson's* N$^-$ *with the connective fusion and the constants (see* [16]*).*

We remark that 2 and 3 in this proposition are guaranteed by the completeness theorems proved in section 4 and the results of Wansing in [16].

In the present paper's framework, the completeness theorem for the original version of Nelson's N$^-$ is unknown whether it holds or not. But we can deal with Wansing's N.

Further, we define new logics with strong negation.

Definition 2.6. (A New Family of Logics with Strong Negation) $B^{\sim}$, $E^{\sim}$, $R^{\sim}$, $S4^{\sim}$, $F^{\sim}$, $FT^{\sim}$, $FR^{\sim}$, $FRT^{\sim}$, $BPL^{\sim}$, $RMO^{\sim}$, $EMO^{\sim}$, $T^{\sim}_{\circ}$, $R^{\sim}_{\circ}$ and $RMO^{\sim}_{\circ}$ are respectively defined by adding the axiom schemes S1−S13 to B^+, T^+, E^+, R^+, $S4^+$, F, *FT*, FR, FRT, BPL, RMO^+, EMO^+, T°_{+}, R°_{+} and RMO°_{+}.

We call the logics, obtained from $L^{\sim}$ by adding arbitrary combinations of B1−B22, *normal substructural logics with strong negation.*

3. Kripke Models

Definition 3.1. A *Kripke frame* for normal substructural logics with strong negation is a structure $\langle M, \cap, \cdot, \varepsilon, \omega\rangle$ satisfying the following conditions:

1. $\langle M, \cap\rangle$ is a meet-semilattice with the greatest element ω,
2. $\cdot$ is a binary operation on M and $\varepsilon \in M$ such that

   C1: $\varepsilon \cdot \mathrm{x} = \mathrm{x}$,
   C2: $\omega \cdot \mathrm{x} = \omega$,
   C3: $\mathrm{x} \le \mathrm{y}$ implies $\mathrm{z} \cdot \mathrm{x} \le \mathrm{z} \cdot \mathrm{y}$ for all x,y,z $\in M$ (where the order relation $x \le y$ is defined by $x \cap y = x$),
   C4: $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$ for all x,y,z $\in M$.

We remark that the condition C3′: $x \le y$ implies $x \cdot z \le y \cdot z$ for all $x, y, z \in M$ on the frame is derived from the condition C4.

Definition 3.2. A *valuation* $\models^+$ (and a valuation $\models^-$) on a Kripke frame $\langle M, \cap, \cdot, \varepsilon, \omega\rangle$ is a mapping which assigns a *filter* of M (i.e., a non-empty subset X of M such that $x, y, \in X$ iff $x \cap y \in X$) to each propositional variable. We will write $x \models^+ p$ ($x \models^- p$) for $x \in \models^+ (p)$ ($x \in \models^- (p)$). Each valuation $\models^+$ (and $\models^-$) can be extended to a mapping from the set of all formulas to the power set of M by

1. $x \models^+ \mathbf{1}$ iff $\varepsilon \le x$,
2. $x \models^+ \top$ for all $x \in M$,
3. $x \models^+ \perp$ iff $x = \omega$,

4. $x \models^+ \alpha \rightarrow \beta$ iff $x \cdot y \leq z$ and $y \models^+ \alpha$ imply $z \models^+ \beta$ for all $y, z \in M$,
5. $x \models^+ \alpha \wedge \beta$ iff $x \models^+ \alpha$ and $x \models^+ \beta$,
6. $x \models^+ \alpha \vee \beta$ iff $y \models^+ \alpha$ or $y \models^+ \beta$, and $z \models^+ \alpha$ or $z \models^+ \beta$ for some $y, z \in M$ with $y \cap z \leq x$.
7. $x \models^+ \alpha * \beta$ iff $y \models^+ \alpha$ and $z \models^+ \beta$ for some $y, z \in M$ with $y \cdot z \leq x$,
8. $x \models^+ \sim \alpha$ iff $x \models^- \alpha$,
9. $x \models^- \sim \alpha$ iff $x \models^+ \alpha$,
10. $x \models^- \mathbf{1}$ iff $x = \omega$,
11. $x \models^- \top$ iff $x = \omega$,
12. $x \models^- \bot$ for all $x \in M$,
13. $x \models^- \alpha \rightarrow \beta$ iff $y \models^+ \alpha$ and $z \models^- \beta$ for some $y, z \in M$ with $y \cdot z \leq x$,
14. $x \models^- \alpha \wedge \beta$ iff $y \models^- \alpha$ or $y \models^- \beta$, and $z \models^- \alpha$ or $z \models^- \beta$ for some $y, z \in M$ with $y \cap z \leq x$,
15. $x \models^- \alpha \vee \beta$ iff $x \models^- \alpha$ and $x \models^- \beta$,
16. $x \models^- \alpha * \beta$ iff $y \models^- \alpha$ and $z \models^- \beta$ for some $y, z \in M$ with $y \cdot z \leq x$.

In this definition, the valuation conditions for $\models^-$ are due to Wansing [16].

Proposition 3.3. *Let $\models^+$ and $\models^-$ be valuations on a Kripke frame $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$. Then both $\models^+ (\alpha)$ and $\models^- (\alpha)$ are filters for any formula α.*

Proof. By simultaneous induction on the complexity of α. Here we show some cases.

(Case $\alpha \equiv \sim \beta$ for $\models^+ (\alpha)$): $x, y \in \models^+ (\sim \beta) := \{z \in M | z \models^+ \sim \beta\}$ iff ($x \models^+ \sim \beta$ and $y \models^+ \sim \beta$) iff ($x \models^- \beta$ and $y \models^- \beta$) iff $x \cap y \models^- \beta$ (by the induction hypothesis) iff $x \cap y \models^+ \sim \beta$ iff $x \cap y \in \models^+ (\sim \beta)$.

(Case $\alpha \equiv \beta \rightarrow \gamma$ for $\models^- (\alpha)$): We only show that $x, y \in \models^- (\beta \rightarrow \gamma)$ implies $x \cap y \in \models^- (\beta \rightarrow \gamma)$. (The converse is straightforward.) Suppose that

$x \models^- \beta{\to}\gamma$ and $y \models^- \beta{\to}\gamma$. Then we have $z' \models^+ \beta$ and $w' \models^- \gamma$ for some $z', w' \in M$ with $z' \cdot w' \leq x$. We also have that $z'' \models^+ \beta$ and $w'' \models^- \gamma$ for some $z'', w'' \in M$ with $z'' {\cdot} w'' \leq y$. Then we get $z' {\cap} z'' \models^+ \beta$ and $w' {\cap} w'' \models^- \gamma$ by the induction hypothesis. Further we have $(z' \cap z'') \cdot (w' \cap w'') \leq x \cap y$ by $(z' \cdot w') \cap (z'' \cdot w'') \leq x \cap y$ (by $z' \cdot w' \leq x$ and $z'' \cdot w'' \leq y$) and $(z' \cap z'') \cdot (w' \cap w'') \leq (z' \cdot w') \cap (z'' \cdot w'')$ (by the frame condition C4 and the fact $w' \cap w'' \leq w'$ and $w' \cap w'' \leq w''$). We showed that $z' \cap z'' \models^+ \beta$ and $w' \cdot w'' \models^- \gamma$ for some $z' \cap z''$, $w' \cap w'' \in M$ with $(z' \cap z'') \cdot (w' \cap w'') \leq x \cap y$. Therefore we have $x \cap y \models^- \beta{\to}\gamma$ and hence $x \cap y \in \models^- (\beta{\to}\gamma)$. ■

Definition 3.4. A *Kripke model* is a structure $\langle M, \cap, \cdot, \varepsilon, \omega, \models^+, \models^- \rangle$ such that

1. $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ is a Kripke frame,
2. both $\models^+$ and $\models^-$ are valuations on $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$.

A formula α is *true* in a Kripke model $\langle M, \cap, \cdot, \varepsilon, \omega, \models^+, \models^- \rangle$ if $\varepsilon \models^+ \alpha$, and *valid* in a Kripke frame $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$ if it is true for any valuations $\models^+$ and $\models^-$ on the Kripke frame.

Theorem 3.5. (Soundness for $L^{\sim}$) *Let C be a class of Kripke frames, $L := \{\gamma \mid \gamma$ is provable in $L^{\sim}\}$ and $L(C) := \{\gamma \mid \gamma$ is valid in all frames of $C\}$. Then $L \subseteq L(C)$.*

Proof. We prove this theorem by induction on the proof P of γ in $L^{\sim}$. We show only the case: $\gamma \equiv\, \sim (\alpha{\to}\beta){\to}\alpha {*} \sim \beta$ (the axiom scheme S6). Let $\langle M, \cap, \cdot, \varepsilon, \omega \rangle \in C$ and $\models^+$ (and $\models^-$) be a valuation on $\langle M, \cap, \cdot, \varepsilon, \omega \rangle$. We show $\varepsilon \models^+ \sim (\alpha{\to}\beta){\to}\alpha {*} \sim \beta$. We prove that $x \models^+ \sim (\alpha{\to}\beta)$ implies $x \models^+ \alpha {*} \sim \beta$ for any $x \in M$. Suppose that $x \models^+ \sim (\alpha{\to}\beta)$. Then there exist $y, z \in M$ such that $y \cdot z \leq x$, $y \models^+ \alpha$ and $z \models^- \beta$. $z \models^- \beta$ means $z \models^+ \sim \beta$. Therefore $x \models^+ \alpha {*} \sim \beta$. ■

Lemma 3.6. *Let $F := \langle M, \cap, \cdot, \varepsilon, \omega \rangle$ be a Kripke frame, let $x, y \in M$, and let $\models^+$ be a valuation on F such that $x \cdot y \models^+ \beta$ and $\models^+ (\alpha) = \uparrow y := \{z \in M \mid y \leq z\}$. Then $x \models^+ \alpha{\to}\beta$.*

By using this lemma, we can show the following.

Proposition 3.7. *Let* $F := \langle M, \cap, \cdot, \varepsilon, \omega\rangle$ *be a Kripke frame. Then*

1. B1 *is valid in* F *iff* $(x \cdot y) \cap (x \cdot z) \leq x \cdot (y \cap z)$ *for all* $x, y, z \in M$,
2. B2 *is valid in* F *iff for all* $x, y, z \in M$ *with* $x > z, y > z$ *and* $x \cap y \leq z$, *there exist* $u, v \in M$ *such that* $x \leq u$, $y \leq v$ *and* $u \cap v = z$ *(where* $>$ *is a strict partial order),*
3. B3 *is valid in* F *iff* $x \cdot (y \cdot z) \leq (x \cdot y) \cdot z$ *for all* $x, y, z \in M$,
4. B4 *is valid in* F *iff* $y \cdot (x \cdot z) \leq (x \cdot y) \cdot z$ *for all* $x, y, z \in M$,
5. B5 *is valid in* F *iff* $(x \cdot y) \cdot y \leq x \cdot y$ *for all* $x, y \in M$,
6. B6 *is valid in* F *iff* $x \cdot \varepsilon \leq x$ *for all* $x \in M$,
7. B7 *is valid in* F *iff* $\varepsilon \leq x$ *for all* $x \in M$,
8. B8 *is valid in* F *iff* $y \cdot x \leq x \cdot y$ *for all* $x, y \in M$,
9. B9 *is valid in* F *iff* $\omega \leq x \cdot \omega$ *for all* $x \in M$,
10. B10 *is valid in* F *iff* $x \leq x \cdot \varepsilon$ *for all* $x \in M$,
11. B11 *is valid in* F *iff* $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$ *for all* $x, y, z \in M$,
12. B12 *is valid in* F *iff* $(x \cdot z) \cdot y \leq (x \cdot y) \cdot z$ *for all* $x, y, z \in M$,
13. B13 *is valid in* F *iff* $x \leq x \cdot y$ *for all* $x, y \in M$,
14. B14 *is valid in* F *iff* $x \cdot (x \cdot y) \leq x \cdot y$ *for all* $x, y \in M$,
15. B15 *is valid in* F *iff* $x \leq x \cdot y$ *and* $y \leq x \cdot y$ *for all* $x, y \in M$,
16. B16 *is valid in* F *iff* $x \cdot x \leq x$ *for all* $x \in M$,
17. B17 *is valid in* F *iff* $x \cap y \leq x \cdot y$ *for all* $x, y \in M$,
18. B18 *is valid in* F *iff* $(x \cdot z) \cap (y \cdot z) \leq (x \cdot y) \cdot z$ *for all* $x, y, z \in M$,
19. B19 *is valid in* F *iff* $((x \cdot y) \cdot z) \cdot w \leq ((x \cdot z) \cdot y) \cdot w$ *for all* $x, y, z, w \in M$,
20. B20 *is valid in* F *iff* $x \cdot y \leq (x \cdot z) \cdot y$ *for all* $x, y, z \in M$,
21. B21 *is valid in* F *iff* $((x \cdot y) \cdot y) \cdot z \leq (x \cdot y) \cdot z$ *for all* $x, y, z \in M$,

22. B22 *is valid in* F *iff* $(y \cdot x) \cdot z \leq (x \cdot y) \cdot z$ *for all* $x, y, z \in M$.

Proof. We show only the proof for the cases 15—22. (We remark that, taking the valuation $\models$ in [6] for $\models^+$, the proof of the cases 1—14 is the same as that in [6].)

(15): Suppose that $x \leq x \cdot y$ and $y \leq x \cdot y$ for all $x, y \in M$, and let $\models^+$ be a valuation on F. Let $x, y \in M$ be such that $x \models^+ \alpha$ and $y \models^+ \beta$. Then we have $x \cdot y \models^+ \alpha$ and $x \cdot y \models^+ \beta$ by the conditions $x \leq x \cdot y$ and $y \leq x \cdot y$. This means $x \cdot y \models^+ \alpha \wedge \beta$. Conversely, suppose that B15 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\alpha) = \uparrow x$ and $\models^+ (\beta) = \uparrow y$. Then we have $x \models^+ \beta \rightarrow \alpha \wedge \beta$ and $x \cdot y \models^+ \alpha \wedge \beta$ since B15 is valid in F. Thus $x \cdot y \models^+ \alpha$ and $x \cdot y \models^+ \beta$ by $x \cdot y \models^+ \alpha \wedge \beta$. Therefore $x \leq x \cdot y$ and $y \leq x \cdot y$ by $\models^+ (\alpha) = \uparrow x$ and $\models^+ (\beta) = \uparrow y$.

(16): Suppose that $x \cdot x \leq x$ for all $x \in M$, and let $\models^+$ be a valuation on F. Let $x \in M$ be such that $x \models^+ \alpha \wedge (\alpha \rightarrow \beta)$. Then $x \models^+ \alpha$ and $x \models^+ \alpha \rightarrow \beta$ and hence $x \cdot y \models^+ \beta$ if $y \models^+ \alpha$. Thus we have $x \cdot x \models^+ \beta$. Therefore we get $x \models^+ \beta$ by $x \cdot x \leq x$. This means $\varepsilon \models^+ \alpha \wedge (\alpha \rightarrow \beta) \rightarrow \beta$. Conversely, suppose that B16 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\beta) = \uparrow (x \cdot x)$ and $\models^+ (\alpha) = \uparrow x$. We have $x \models^+ \alpha \rightarrow \beta$ by $x \cdot x \models^+ \beta$, $\models^+ (\alpha) = \uparrow x$ and Lemma 3.6. Then we get $x \models^+ \alpha \wedge (\alpha \rightarrow \beta)$ by $x \models^+ \alpha$ and $x \models^+ \alpha \rightarrow \beta$. We have $x \models^+ \beta$ since B16 is valid in F. Therefore $x \cdot x \leq x$.

(17): Suppose that $x \cap y \leq x \cdot y$ for all $x, y \in M$, and let $\models^+$ be a valuation on F. Let $x, y \in M$ be such that $x \models^+ \alpha$ and $y \models^+ \alpha$ Then we have $x \cap y \models^+ \alpha$. Therefore we have $x \cdot y \models^+ \alpha$ by $x \cap y \leq x \cdot y$. This means $\varepsilon \models^+ \alpha \rightarrow \alpha \rightarrow \alpha$. Conversely, suppose that B17 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\alpha) = \uparrow (x \cap y)$. Then we have $x \models^+ \alpha$ and $y \models^+ \alpha$. We get $x \cdot y \models^+ \alpha$ since B17 is valid in F. Therefore $x \cap y \leq x \cdot y$.

(18): Suppose that $(x \cdot z) \cap (y \cdot z) \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$ and let $\models^+$ be a valuation on F. We show that $(\alpha_1 \rightarrow \alpha_2) \rightarrow (\alpha_1 \rightarrow \alpha_2) \rightarrow (\alpha_1 \rightarrow \alpha_2)$ is valid in F. Let $x, y, z \in M$ be such that $x \models^+ \alpha_1 \rightarrow \alpha_2$, $y \models^+ \alpha_1 \rightarrow \alpha_2$ and $z \models^+ \alpha_1$. Then we have that $y \cdot z \models^+ \alpha_2$ and $x \cdot z \models^+ \alpha_2$ by $y \models^+ \alpha_1 \rightarrow \alpha_2$ and $x \models^+ \alpha_1 \rightarrow \alpha_2$ respectively. We have $(x \cdot z) \cap (y \cdot z) \models^+ \alpha_2$. Thus we get $(x \cdot y) \cdot z \models \alpha_2$ by $(x \cdot z) \cap (y \cdot z) \leq (x \cdot y) \cdot z$. Therefore $\varepsilon \models^+ (\alpha_1 \rightarrow \alpha_2) \rightarrow (\alpha_1 \rightarrow \alpha_2) \rightarrow (\alpha_1 \rightarrow \alpha_2)$. Conversely, suppose that B18 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\alpha_2) = \uparrow (x \cdot z) \cap (y \cdot z)$ and $\models^+ (\alpha_1) = \uparrow z$. Then we have $x \cdot z \models^+ \alpha_2$ and $y \cdot z \models^+ \alpha_2$. We get

$x \models^+ \alpha_1{\to}\alpha_2$ by $x \cdot z \models^+ \alpha_2$, $\models^+ (\alpha_1) = \uparrow z$ and Lemma 3.6, and get $y \models^+ \alpha_1{\to}\alpha_2$ by $y \cdot z \models^+ \alpha_2$, $\models^+ (\alpha_1) = \uparrow z$ and Lemma 3.6. Thus we have $(x \cdot y) \cdot z \models^+ \alpha_2$ by $x \models^+ \alpha_1{\to}\alpha_2$, $y \models^+ \alpha_1{\to}\alpha_2$, $z \models^+ \alpha_1$ and the hypothesis: B18 is valid in F. Therefore $(x \cdot z) \cap (y \cdot z) \leq (x \cdot y) \cdot z$.

(19): Suppose that $((x{\cdot}y){\cdot}z){\cdot}w \leq ((x{\cdot}z){\cdot}y){\cdot}w$ for all $x, y, z, w \in M$ and let $\models^+$ be a valuation on F. We show that $(\alpha{\to}\beta{\to}\gamma_1{\to}\gamma_2){\to}\beta{\to}\alpha{\to}\gamma_1{\to}\gamma_2$ is valid in F. Let $x, y, z, w \in M$ be such that $x \models^+ \alpha{\to}\beta{\to}\gamma_1{\to}\gamma_2$, $z \models^+ \beta$, $y \models^+ \alpha$ and $w \models^+ \gamma_1$. Then we have $((x \cdot y) \cdot z) \cdot w \models^+ \gamma_2$. Thus we get $((x \cdot z) \cdot y) \cdot w \models^+ \gamma_2$ by $((x \cdot y) \cdot z) \cdot w \leq ((x \cdot z) \cdot y) \cdot w$. Therefore $\varepsilon \models^+ (\alpha{\to}\beta{\to}\gamma_1{\to}\gamma_2){\to}\beta{\to}\alpha{\to}\gamma_1{\to}\gamma_2$. Conversely, suppose that B19 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\gamma_2) = \uparrow ((x{\cdot}y){\cdot}z){\cdot}w$, $\models^+ (\gamma_1) = \uparrow w$, $\models^+ (\beta) = \uparrow z$ and $\models^+ (\alpha) = \uparrow y$. We have $(x \cdot y) \cdot z \models^+ \gamma_1{\to}\gamma_2$ by $w \models^+ \gamma_1$, $((x \cdot y) \cdot z) \cdot w \models^+ \gamma_2$ and Lemma 3.6, and hence we have $x \cdot y \models^+ \beta{\to}\gamma_1{\to}\gamma_2$ by $z \models^+ \beta$, $(x \cdot y) \cdot z \models^+ \gamma_1{\to}\gamma_2$ and Lemma 3.6. We get $x \models^+ \alpha{\to}\beta{\to}\gamma_1{\to}\gamma_2$ by $y \models^+ \alpha$, $x \cdot y \models^+ \beta{\to}\gamma_1{\to}\gamma_2$ and Lemma 3.6. Thus we have $((x \cdot z) \cdot y) \cdot w \models^+ \gamma_2$ by $w \models^+ \gamma_1$, $y \models^+ \alpha$, $z \models^+ \beta$, $x \models^+ \alpha{\to}\beta{\to}\gamma_1{\to}\gamma_2$ and the hypothesis: $(\alpha{\to}\beta{\to}\gamma_1{\to}\gamma_2){\to}\beta{\to}\alpha{\to}\gamma_1{\to}\gamma_2$ is valid in F. Therefore $((x \cdot y) \cdot z) \cdot w \leq ((x \cdot z) \cdot y) \cdot w$.

(20): Suppose that $x \cdot y \leq (x \cdot z) \cdot y$ for all $x, y, z \in M$ and let $\models^+$ be a valuation on F. We show that $(\alpha_1{\to}\alpha_2){\to}\beta{\to}\alpha_1{\to}\alpha_2$ is valid in F. Let $x, y, z \in M$ be such that $x \models^+ \alpha_1{\to}\alpha_2$, $z \models^+ \beta$ and $y \models^+ \alpha_1$. Then we have $x \cdot y \models^+ \alpha_2$ by $x \models^+ \alpha_1{\to}\alpha_2$ and $y \models^+ \alpha_1$. Thus we get $(x \cdot z) \cdot y \models^+ \alpha_2$ by $x \cdot y \leq (x \cdot z) \cdot y$ and $x \cdot y \models^+ \alpha_2$. Conversely, suppose that B20 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\alpha_2) = \uparrow (x \cdot y)$, $\models^+ (\alpha_1) = \uparrow y$ and $\models^+ (\beta) = \uparrow z$. Then we have $x \models^+ \alpha_1{\to}\alpha_2$ by $x \cdot y \models^+ \alpha_2$, $\models^+ (\alpha_1) = \uparrow y$ and Lemma 3.6. Thus we have $(x \cdot z) \cdot y \models^+ \alpha_2$ by $x \models^+ \alpha_1{\to}\alpha_2$, $z \models^+ \beta$, $y \models^+ \alpha_1$ and the hypothesis: $(\alpha_1{\to}\alpha_2){\to}\beta{\to}\alpha_1{\to}\alpha_2$ is valid in F. Therefore $(x \cdot y) \leq (x \cdot z) \cdot y$.

(21): Suppose that $((x \cdot y) \cdot y) \cdot z \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$, and let $\models^+$ be a valuation on F. We show that $(\alpha{\to}\alpha{\to}\beta_1{\to}\beta_2){\to}\alpha{\to}\beta_1{\to}\beta_2$ is valid in F. Let $x, y, z \in M$ be such that $x \models^+ \alpha{\to}\alpha{\to}\beta_1{\to}\beta_2$, $y \models^+ \alpha$ and $z \models^+ \beta_1$. Then we have $((x \cdot y) \cdot y) \cdot z \models^+ \beta_2$. Thus we get $(x \cdot y) \cdot z \models^+ \beta_2$ by $((x \cdot y) \cdot y) \cdot z \leq (x \cdot y) \cdot z$. Conversely, suppose that B21 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\beta_2) = \uparrow ((x \cdot y) \cdot y) \cdot z$, $\models^+ (\beta_1) = \uparrow z$ and $\models^+ (\alpha) = \uparrow y$. Then we have $(x \cdot y) \cdot y \models^+ \beta_1{\to}\beta_2$ by $((x \cdot y) \cdot y) \cdot z \models^+ \beta_2$, $\models^+ (\beta_1) = \uparrow z$ and Lemma 3.6, and hence we have $x \cdot y \models^+ \alpha{\to}\beta_1{\to}\beta_2$ by $(x \cdot y) \cdot y \models^+ \beta_1{\to}\beta_2$, $\models^+ (\alpha) \uparrow y$ and Lemma

3.6. We get $x \models^+ \alpha{\to}\alpha{\to}\beta_1{\to}\beta_2$ by $x \cdot y \models^+ \alpha{\to}\beta_1{\to}\beta_2$, $\models^+ (\alpha) \uparrow y$ and Lemma 3.6. Thus we get $(x \cdot y) \cdot z \models^+ \beta_2$ by $x \models^+ \alpha{\to}\alpha{\to}\beta_1{\to}\beta_2$, $y \models^+ \alpha$, $z \models^+ \beta_1$ and the hypothesis: $(\alpha{\to}\alpha{\to}\beta_1{\to}\beta_2){\to}\alpha{\to}\beta_1{\to}\beta_2$ is valid in F. Therefore $((x \cdot y) \cdot y) \cdot z \leq (x \cdot y) \cdot z$.

(22): Suppose that $(y \cdot x) \cdot z \leq (x \cdot y) \cdot z$ for all $x, y, z \in M$, and let $\models^+$ be a valuation on F. Let $x, y, z \in M$ be such that $x \models^+ \alpha$, $y \models^+ \alpha{\to}\beta_1{\to}\beta_2$ and $z \models^+ \beta_1$. Then we have $(y \cdot x) \cdot z \models^+ \beta_2$ by $y \models^+ \alpha{\to}\beta_1{\to}\beta_2$, $x \models^+ \alpha$ and $z \models^+ \beta_1$. Thus we get $(x \cdot y) \cdot z \models^+ \beta_2$ by $(y \cdot x) \cdot z \leq (x \cdot y) \cdot z$. Conversely, suppose that B22 is valid in F. Let $\models^+$ be a valuation on F such that $\models^+ (\beta_2) = \uparrow (y \cdot x) \cdot z$, $\models^+ (\beta_1) = \uparrow z$, $\models^+ (\alpha) = \uparrow x$. Then we have $y \cdot x \models^+ \beta_1{\to}\beta_2$ by $(y \cdot x) \cdot z \models^+ \beta_2$, $\models^+ (\beta_1) = \uparrow z$ and Lemma 3.6, and hence we have $y \models^+ \alpha{\to}\beta_1{\to}\beta_2$ by $y \cdot x \models^+ \beta_1{\to}\beta_2$, $\models^+ (\alpha) \uparrow x$ and Lemma 3.6. Thus we get $(x \cdot y) \cdot z \models^+ \beta_2$ by $x \models^+ \alpha$, $y \models^+ \alpha{\to}\beta_1{\to}\beta_2$, $z \models^+ \beta_1$ and the hypothesis: $\alpha{\to}(\alpha{\to}\beta_1{\to}\beta_2){\to}\beta_1{\to}\beta_2$ is valid in F. Therefore $(y \cdot x) \cdot z \leq (x \cdot y) \cdot z$. ■

By using this proposition, we can show the soundness theorem for any logic obtained from $\mathrm{L}^{\sim}$ by adding arbitrary combinations of the axiom schemes B1−B22.

4. A Canonical Model Construction

Definition 4.1. **(L-pretheory)** Let $L := \{\alpha \mid \alpha$ is provable in a normal substructural logic with strong negation$\}$. An L-pretheory x is a subset of the set Φ of all formulas such that

1. $\top \in x$,
2. if $\alpha \in x$ and $\alpha{\to}\beta \in L$, then $\beta \in x$,
3. if $\alpha, \beta \in x$, then $\alpha \wedge \beta \in x$.

Lemma 4.2. *Let $L := \{\alpha \mid \alpha$ is provable in a normal substructural logic with strong negation$\}$. Then*

1. *if x and y are L-pretheories, then so is $x \cap y$,*
2. *if x and y are L-pretheories, then so is*

$$x \cdot y := \{\beta \mid \exists \alpha \in y\ (\alpha{\to}\beta \in x)\},$$

3. $L \cdot \{\alpha\}$ *is an* L*-pretheory,*

4. *if* x *is an* L*-pretheory, then* $L \cdot x = x$*,*

5. *if* x, y *and* z *are* L*-pretheories, then* $(x \cap y) \cdot z = (x \cdot z) \cap (y \cdot z)$*.*

Proposition 4.3. *Let* $L := \{\alpha \mid \alpha$ *is provable in a normal substructural logic with strong negation*$\}$*,* Φ *be the set of all formulas and* M_L *be the set of all* L*-pretheories. Then* $F_L := \langle M_L, \cap, \cdot, L, \Phi \rangle$ *is a Kripke frame.*

Proposition 4.4. *Let* $L := \{\alpha \mid \alpha$ *is provable in a normal substructural logic with strong negation*$\}$*. Then*

1. *if* B1 $\in L$*, then* $(x \cdot y) \cap (x \cdot z) \subseteq x \cdot (y \cap z)$ *for all* $x, y, z \in M_L$*,*

2. *if* B2 $\in L$*, then for all* $x, y, z \in M_L$ *with* $x \supset z, y \supset z$ *and* $x \cap y \subseteq z$*, there exist* $u, v \in M_L$ *such that* $x \subseteq u$*,* $y \subseteq v$ *and* $u \cap v = z$ *(where* $\supset$ *is a proper inclusion),*

3. *if* B3 $\in L$*, then* $x \cdot (y \cdot z) \subseteq (x \cdot y) \cdot z$ *for all* $x, y, z \in M_L$*,*

4. *if* B4 $\in L$*, then* $y \cdot (x \cdot z) \subseteq (x \cdot y) \cdot z$ *for all* $x, y, z \in M_L$*,*

5. *if* B5 $\in L$*, then* $(x \cdot y) \cdot y \subseteq x \cdot y$ *for all* $x, y \in M_L$*,*

6. *if* B6 $\in L$*, then* $x \cdot L \subseteq x$ *for all* $x \in M_L$*,*

7. *if* B7 $\in L$*, then* $L \subseteq x$ *for all* $x \in M_L$*,*

8. *if* B8 $\in L$*, then* $y \cdot x \subseteq x \cdot y$ *for all* $x, y \in M_L$*,*

9. *if* B9 $\in L$*, then* $\Phi \subseteq x \cdot \Phi$ *for all* $x \in M_L$*,*

10. *if* B10 $\in L$*, then* $x \subseteq x \cdot L$ *for all* $x \in M_L$*,*

11. *if* B11 $\in L$*, then* $(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$ *for all* $x, y, z \in M_L$*,*

12. *if* B12 $\in L$*, then* $(x \cdot z) \cdot y \subseteq (x \cdot y) \cdot z$ *for all* $x, y, z \in M_L$*,*

13. *if* B13 $\in L$*, then* $x \subseteq x \cdot y$ *for all* $x, y \in M_L$*,*

14. *if* B14 $\in L$*, then* $x \cdot (x \cdot y) \subseteq x \cdot y$ *for all* $x, y \in M_L$*,*

15. *if* B15 $\in L$*, then* $x \subseteq x \cdot y$ *and* $y \subseteq x \cdot y$ *for all* $x, y \in M_L$*,*

16. *if* B16 $\in L$, *then* $x \cdot x \subseteq x$ *for all* $x \in M_L$,

17. *if* B17 $\in L$, *then* $x \cap y \subseteq x \cdot y$ *for all* $x, y \in M_L$,

18. *if* B18 $\in L$, *then* $(x \cdot z) \cap (y \cdot z) \subseteq (x \cdot y) \cdot z$ *for all* $x, y, z \in M_L$,

19. *if* B19 $\in L$, *then* $((x \cdot y) \cdot z) \cdot w \subseteq ((x \cdot z) \cdot y) \cdot w$ *for all* $x, y, z, w \in M_L$,

20. *if* B20 $\in L$, *then* $x \cdot y \subseteq (x \cdot z) \cdot y$ *for all* $x, y, z \in M_L$,

21. *if* B21 $\in L$, *then* $((x \cdot y) \cdot y) \cdot z \subseteq (x \cdot y) \cdot z$ *for all* $x, y, z \in M_L$,

22. *if* B22 $\in L$, *then* $(y \cdot x) \cdot z \subseteq (x \cdot y) \cdot z$ *for all* $x, y, z \in M_L$.

Proof. We show only the proof for 15—22.

(15): Suppose that B15 $\in L$, $x, y \in M_L$, $\gamma \in x$ and $\delta \in y$. We show $\gamma \in x \cdot y$ and $\delta \in x \cdot y$. First we show $\gamma \in x \cdot y$. We have $\top \rightarrow \gamma \wedge \top \in x$ by $\gamma \in x$, $\gamma \rightarrow \top \rightarrow \gamma \wedge \top \in L$ (the axiom scheme B15) and $x \in M_L$. We also have $(\top \rightarrow \gamma \wedge \top) \rightarrow \top \rightarrow \gamma \in L$ by (the axiom scheme A5) $\gamma \wedge \top \rightarrow \gamma$ and (*pref*). Then we get $\top \rightarrow \gamma \in x$ by $\top \rightarrow \gamma \wedge \top \in x$, $(\top \rightarrow \gamma \wedge \top) \rightarrow \top \rightarrow \gamma \in L$ and $x \in M_L$. On the other hand, we have $\top \in y$ by $y \in M_L$. Therefore we have $\top \rightarrow \gamma \in x$ and $\top \in y$. This means $\gamma \in x \cdot y$. Second, we show $\delta \in x \cdot y$. We have $\top \in x$ by $x \in M_L$. Then we have $\delta \rightarrow \top \wedge \delta \in x$ by $\top \in x$, (the axiom scheme B15) $\top \rightarrow \delta \rightarrow \top \wedge \delta \in L$ and $x \in M_L$. We also have $(\delta \rightarrow \top \wedge \delta) \rightarrow \delta \rightarrow \delta \in L$ by (the axiom scheme A6) $(\top \wedge \delta) \rightarrow \delta$ and (*pref*). Then we get $\delta \rightarrow \delta \in x$. Therefore we have $\delta \in y$ and $\delta \rightarrow \delta \in x$. This means $\delta \in x \cdot y$.

(16): Suppose that B16 $\in L$, $\gamma \in x \cdot x$ and $x \in M_L$. then there exist $\delta \in x$ such that $\delta \rightarrow \gamma \in x$. We have $\delta \wedge (\delta \rightarrow \gamma) \in x$ by $x \in M_L$, and hence $\gamma \in x$ by $\delta \wedge (\delta \rightarrow \gamma) \in x$, (the axiom scheme B16) $\delta \wedge (\delta \rightarrow \gamma) \rightarrow \gamma \in L$ and $x \in M_L$.

(17): Suppose that B17 $\in L$, $\gamma \in x \cap y$ and $x, y \in M_L$. Then we have $\gamma \in x$ and $\gamma \in y$. We get $\gamma \rightarrow \gamma \in x$ by $\gamma \in x$, (the axiom scheme B17) $\gamma \rightarrow \gamma \rightarrow \gamma \in L$ and $x \in M_L$. Therefore we have $\gamma \in y$ and $\gamma \rightarrow \gamma \in x$. This means $\gamma \in x \cdot y$.

(18): Suppose that B18 $\in L$, $\gamma \in (x \cdot z) \cap (y \cdot z)$ and $x, y, z \in M_L$. Then we have $\gamma \in x \cdot z$ and $\gamma \in y \cdot z$, and hence there exists $\alpha \in z$ such that $\alpha \rightarrow \gamma \in x$ and there exists $\beta \in z$ such that $\beta \rightarrow \gamma \in y$. We consider the cases: (1) $y \subseteq x$ and (2) $x \subseteq y$. First we consider (1). We have $\beta \rightarrow \gamma \in x$ by (1) and have (the axiom scheme B18) $(\beta\to\gamma)\to(\beta\to\gamma)\to\beta\to\gamma \in L$. Then we get $(\beta\to\gamma)\to\beta\to\gamma \in x$ by $x \in M_L$. Thus we have $\beta \in z$, $\beta\to\gamma \in y$ and $(\beta\to\gamma)\to(\beta\to\gamma) \in x$. This means $\gamma \in (x \cdot y) \cdot z$. Next we consider (2). We have $\alpha\to\gamma \in y$ by (2). We also have $(\alpha\to\gamma)\to(\alpha\to\gamma) \in x$ by $\alpha\to\gamma \in x$, (the axiom scheme B18) $(\alpha\to\gamma)\to(\alpha\to\gamma)\to\alpha\to\gamma \in L$ and $x \in M_L$. Therefore we have $\alpha \in z$, $\alpha\to\gamma \in y$ and $(\alpha\to\gamma)\to(\alpha\to\gamma) \in x$. This means $\gamma \in (x \cdot y) \cdot z$.

(19): Suppose that B19 $\in L$, $\gamma \in ((x \cdot y) \cdot z) \cdot w$, and $x, y, z, w \in M_L$. Then there exist $\alpha \in w$, $\beta \in z$, $\delta \in y$ such that $\delta\to\beta\to\alpha\to\gamma \in x$. We have $\beta\to\delta\to\alpha\to\gamma \in x$ by $\delta\to\beta\to\alpha\to\gamma \in x$, (the axiom scheme B19) $(\delta\to\beta\to\alpha\to\gamma)\to\beta\to\delta\to\alpha\to\gamma \in L$ and $x \in M_L$. Therefore we have $\alpha \in w$, $\delta \in y$, $\beta \in z$ and $\beta\to\delta\to\alpha\to\gamma \in x$. This means $\gamma \in ((x \cdot z) \cdot y) \cdot w$.

(20): Suppose that B20 $\in L$, $\gamma \in x \cdot y$ and $x, y, z \in M_L$. Then there exists $\alpha \in y$ such that $\alpha\to\gamma \in x$. We have $\top\to\alpha\to\gamma \in x$ by $\alpha\to\gamma \in x$, (the axiom scheme B20) $(\alpha\to\gamma)\to\top\to(\alpha\to\gamma) \in L$ and $x \in M_L$. Therefore we have $\alpha \in y$, $\top \in z$ (because of $z \in M_L$), $\top\to\alpha\to\gamma \in x$. This means $\gamma \in (x \cdot z) \cdot y$.

(21): Suppose that B21 $\in L$, $\gamma \in ((x \cdot y) \cdot y) \cdot z$ and $x, y, z \in M_L$. Then there exist $\alpha \in z, \beta \in y, \delta \in y$ such that $\delta\to\beta\to\alpha\to\gamma \in x$. We have $\delta \wedge \beta \in y$ by $y \in M_L$. We get $\delta \wedge \beta\to\delta \wedge \beta\to\alpha\to\gamma \in x$ by $\delta\to\beta\to\alpha\to\gamma \in x$, $(\delta\to\beta\to\alpha\to\gamma)\to\delta \wedge \beta\to\delta \wedge \beta\to\alpha\to\gamma \in L$ (this will be proved later) and $x \in M_L$. Further we get $\delta \wedge \beta\to\alpha\to\gamma \in x$ by $\delta \wedge \beta\to\delta \wedge \beta\to\alpha\to\gamma \in x$, (the axiom scheme B21) $(\delta \wedge \beta\to\delta \wedge \beta\to\alpha\to\gamma)\to\delta \wedge \beta\to\alpha\to\gamma \in L$ and $x \in M_L$. Therefore we get $\alpha \in z$, $\delta \wedge \beta \in y$ and $\delta \wedge \beta\to\alpha\to\gamma \in x$. This means $\gamma \in (x \cdot y) \cdot z$. It remains to prove that $(\delta\to\beta\to\alpha\to\gamma)\to\delta \wedge \beta\to\delta \wedge \beta\to\alpha\to\gamma \in L$. Applying (*suff*) to (the axiom scheme A5) $\delta \wedge \beta\to\delta$, we get $(\delta\to\beta\to\alpha\to\gamma)\to\delta \wedge \beta\to\beta\to\alpha\to\gamma \in L$ (†). We can get $((\delta\to\beta\to\alpha\to\gamma)\to\delta \wedge \beta\to\beta\to\alpha\to\gamma)\to(\delta\to\beta\to\alpha\to\gamma)\to\delta \wedge \beta\to\delta \wedge \beta\to\alpha\to\gamma \in L$ (‡) by (*suff*), (*pref*) and (the axiom scheme A6) $\delta \wedge \beta\to\beta$. Therefore we have the claim by (†), (‡) and (*mp*).

(22): Suppose that B22 $\in L$, $\gamma \in (y \cdot x) \cdot z$ and $x, y, z \in M_L$. Then there exist $\alpha \in z, \delta \in x$ such that $\delta\to\alpha\to\gamma \in y$. We have $(\delta\to\alpha\to\gamma)\to\alpha\to\gamma \in x$ by $\delta \in x$, (the axiom scheme B22) $\delta\to(\delta\to\alpha\to\gamma)\to\alpha\to\gamma \in L$ and $x \in M_L$. Therefore we have $\alpha \in z$, $\delta\to\alpha\to\gamma \in y$ and $(\delta\to\alpha\to\gamma)\to\alpha\to\gamma \in x$. This means $\gamma \in (x \cdot y) \cdot z$. ∎

Lemma 4.5. *Let* $L := \{\alpha \mid \alpha$ *is provable in a normal substructural logic with strong negation*$\}$, *and* $x \in M_L$. *Then*

1. $\alpha \in x$ *iff* $L \cdot \{\alpha\} \subseteq x$,
2. $(L \cdot \{\alpha\}) \cap (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha \vee \beta\}$,
3. $(L \cdot \{\alpha\}) \cdot (L \cdot \{\beta\}) \subseteq L \cdot \{\alpha * \beta\}$,
4. $(L \cdot \{\sim \alpha\}) \cap (L \cdot \{\sim \beta\}) \subseteq L \cdot \{\sim (\alpha \wedge \beta)\}$,
5. $(L \cdot \{\sim \alpha\}) \cdot (L \cdot \{\sim \beta\}) \subseteq L \cdot \{\sim (\alpha * \beta)\}$,
6. $(L \cdot \{\alpha\}) \cdot (L \cdot \{\sim \beta\}) \subseteq L \cdot \{\sim (\alpha \rightarrow \beta)\}$.

Proof. We show 4, 5 and 6. The proof for the other cases is the same as that in [6].

(4): Suppose that $\gamma \in L \cdot \{\sim \alpha\} \cap L \cdot \{\sim \beta\}$ for any $\gamma \in M_L$. Then we have $\gamma \in L \cdot \{\sim \alpha\}$ and $\gamma \in L \cdot \{\sim \beta\}$, and hence $\sim \alpha \rightarrow \gamma \in L$ and $\sim \beta \rightarrow \gamma \in L$. We can get $\sim (\alpha \wedge \beta) \rightarrow \gamma \in L$ by (*or*), (*pref*), (*mp*), $\sim \alpha \rightarrow \gamma \in L$, $\sim \beta \rightarrow \gamma \in L$ and (the axiom scheme S8) $\sim (\alpha \wedge \beta) \rightarrow \sim \alpha \vee \sim \beta$. Therefore we get $\gamma \in L \cdot \{\sim (\alpha \wedge \beta)\}$.

(5): Suppose that $\theta \in (L \cdot \{\sim \alpha\}) \cdot (L \cdot \{\sim \beta\})$ for any $\theta \in M_L$. Then there exists $\delta \in L \cdot \{\sim \beta\}$ such that $\delta \rightarrow \theta \in L \cdot \{\sim \alpha\}$, and hence $\sim \beta \rightarrow \delta \in L$ and $\sim \alpha \rightarrow \delta \rightarrow \theta \in L$. We can get $\sim (\alpha * \beta) \rightarrow \theta \in L$ by $\sim \beta \rightarrow \delta \in L$, $\sim \alpha \rightarrow \delta \rightarrow \theta \in L$, (*suff*), (*pref*), (*mp*), (*residu*) and (the axiom scheme S12) $\sim (\alpha * \beta) \rightarrow \sim \alpha * \sim \beta$. Therefore $\theta \in L \cdot \{\sim (\alpha * \beta)\}$.

(6): Suppose that $\delta \in (L \cdot \{\alpha\}) \cdot (L \cdot \{\sim \beta\})$ for any $\delta \in M_L$. Then there exists $\rho \in L \cdot \{\sim \beta\}$ such that $\rho \rightarrow \delta \in L \cdot \{\alpha\}$, and hence we have $\sim \beta \rightarrow \rho \in L$ and $\alpha \rightarrow \rho \rightarrow \delta \in L$. We can get $\sim (\alpha \rightarrow \beta) \rightarrow \delta \in L$ by $\sim \beta \rightarrow \rho \in L$, $\alpha \rightarrow \rho \rightarrow \delta \in L$, (*suff*), (*pref*), (*residu*), (*mp*) and (the axiom scheme S6) $\sim (\alpha \rightarrow \beta) \rightarrow \alpha * \sim \beta$. Therefore $\delta \in L \cdot \{\sim (\alpha \rightarrow \beta)\}$. ■

By using this lemma, we can prove the following.

Lemma 4.6. *Let* $L := \{\alpha \mid \alpha$ *is provable in a normal substructural logic with strong negation*$\}$, $\models^+_L$ *be a mapping from the set of all propositional variables to the set of all subset of* M_L *defined by* $\models^+_L (p) := \{x \in M_L \mid p \in x\}$, *and* $\models^-_L$ *be a mapping from the set of all propositional variables to the set of all subset of* M_L *defined by* $\models^-_L (p) := \{x \in M_L \mid \sim p \in x\}$. *Then* $\mathbf{M}_L := \langle M_L, \cap, \cdot, L, \Phi, \models^+_L, \models^-_L \rangle$ *is a Kripke model such that* $\alpha \in L$ *iff* α *is true in* $\mathbf{M}_L$.

Proof. We can easily show that $\models^+_L$ (and $\models^-_L$) is a valuation on F_L. Thus $\mathbf{M}_L$ is a Kripke model. It remains to show that $\alpha \in L$ iff $L \models^+_L \alpha$ for any $\alpha \in \Phi$. To show the fact, we will prove that (1): ($\alpha \in x$ iff $x \models^+_L \alpha$) and (2): ($\sim \alpha \in x$ iff $x \models^-_L \alpha$) for all $\alpha \in \Phi$ and $x \in M_L$ by simultaneous induction on the complexity of α.

• The base step: Straightforward.

• The induction step for (1): We show only the case $\alpha \equiv \sim \beta$. (Taking $\models$ in [6] for $\models^+$, the proof for the other cases is the same as that in [6].)

(Case $\alpha \equiv \sim \beta$ for (1)): Suppose that $\sim \beta \in x$. Then $x \models^-_L \beta$ holds by the induction hypothesis. Thus we get $x \models^+_L \sim \beta$. Conversely, suppose that $x \models^+_L \sim \beta$. Then $x \models^-_L \beta$ holds and hence $\sim \beta \in x$ holds by the induction hypothesis.

• The induction step for (2):

(Case $\alpha \equiv \sim \beta$ for (2)): Suppose that $\sim\sim \beta \in x$. Then $\beta \in x$ holds since $\sim\sim \beta \rightarrow \beta \in L$ (the axiom scheme S2) and $x \in M_L$. Thus we have $x \models^+_L \beta$ by the induction hypothesis, and hence $x \models^-_L \sim \beta$. Conversely, suppose that $x \models^-_L \sim \beta$. Then $x \models^+_L \beta$ holds and hence $\beta \in x$ holds by the induction hypothesis. Thus we have $\sim\sim \beta \in x$ by $\beta \in x$, $\beta \rightarrow \sim\sim \beta \in L$ (the axiom scheme S1) and $x \in M_L$.

(Case $\alpha \equiv \beta \rightarrow \gamma$ for (2)): Suppose that $\sim (\beta \rightarrow \gamma) \in x$. Then we get $L \cdot \{\sim (\beta \rightarrow \gamma)\} \subseteq x$ by Lemma 4.5 (1), and get $(L \cdot \{\beta\}) \cdot (L \cdot \{\sim \gamma\}) \subseteq L \cdot \{\sim (\beta \rightarrow \gamma)\}$ by Lemma 4.5 (6). Thus we have $(L \cdot \{\beta\}) \cdot (L \cdot \{\sim \gamma\}) \subseteq x$ (†) and have both $\beta \in L \cdot \{\beta\}$ and $\sim \gamma \in L \cdot \{\sim \gamma\}$ (‡). Now we show that there exist $y, z \in M_L$ such that $y \cdot z \subseteq x$, $y \models^+_L \beta$ and $z \models^-_L \gamma$. We take $y = (L \cdot \{\beta\})$ and $z = (L \cdot \{\sim \gamma\})$. Then we can get $y \cdot z \subseteq x$ by (†), and get both $y \models^+_L \beta$ and $z \models^-_L \gamma$ by (‡) and the induction hypothesis. Therefore $x \models^-_L \beta \rightarrow \gamma$. Conversely, suppose that $x \models^-_L \beta \rightarrow \gamma$. Then there exist $y, z \in M_L$ such that $y \cdot z \subseteq x$, $y \models^+_L \beta$ and $z \models^-_L \gamma$. Hence we have $\beta \in y$ and $\sim \gamma \in z$ by the induction hypothesis. We have $\sim \gamma \rightarrow \beta * \sim \gamma \in y$ by $\beta \rightarrow \sim \gamma \rightarrow \beta * \sim \gamma \in L$ (the axiom scheme A10), $\beta \in y$ and $y \in M_L$. Further we have $\beta * \sim \gamma \in x$ by $\sim \gamma \in z$, $\sim \gamma \rightarrow \beta * \sim \gamma \in y$ and $\beta * \sim \gamma \in y \cdot z \subseteq x$. Thus we get $\sim (\beta \rightarrow \gamma) \in x$ by $\beta * \sim \gamma \in x$, $\beta * \sim \gamma \rightarrow \sim (\beta \rightarrow \gamma) \in L$ (axiom scheme S7) and $x \in M_L$.

(Case $\alpha \equiv \beta \wedge \gamma$ for (2)): Suppose that $\sim (\beta \wedge \gamma) \in x$. Then we have $L \cdot \{\sim \beta\} \cap L \cdot \{\sim \gamma\} \subseteq x$ (†), since we have $L \cdot \{\sim (\beta \wedge \gamma)\} \subseteq x$ by Lemma 4.5 (1) and have $L \cdot \{\sim \beta\} \cap L \cdot \{\sim \gamma\} \subseteq L \cdot \{\sim (\beta \wedge \gamma)\}$ by Lemma 4.5 (4). We show that there exist $u, v \in M_L$ such that $u \cap v \subseteq x$, ($u \models^-_L \beta$ or $u \models^-_L \gamma$)

and ($v \models^-_L \beta$ or $v \models^-_L \gamma$). We take $u = L \cdot \{\sim \beta\}$ and $v = L \cdot \{\sim \gamma\}$. Then we have $u, v \in M_L$ and have $u \cap v \subseteq x$ by (†). We get $u \models^-_L \beta$ and $v \models^-_L \gamma$ since $\sim \beta \in L \cdot \{\sim \beta\}$, $\sim \gamma \in L \cdot \{\sim \gamma\}$ and the induction hypothesis. Therefore $x \models^-_L \beta \wedge \gamma$. Conversely, suppose that $x \models^-_L \beta \wedge \gamma$. Then there exist $y, z \in M_L$ such that $y \cap z \subseteq x$, ($y \models^-_L \beta$ or $y \models^-_L \gamma$) and ($z \models^-_L \beta$ or $z \models^-_L \gamma$). We can consider the cases (1) $y \models^-_L \beta$ and $z \models^-_L \beta$, (2) $y \models^-_L \beta$ and $z \models^-_L \gamma$, (3) $y \models^-_L \gamma$ and $z \models^-_L \beta$ and (4) $y \models^-_L \gamma$ and $z \models^-_L \gamma$. Here we consider only the case (4). Then we have $\sim \gamma \in y$ and $\sim \gamma \in z$ by the induction hypothesis, and hence we get $\sim \gamma \in y \cap z$. Thus we get $\sim \beta \vee \sim \gamma \in y \cap z$ since we have $\sim \gamma \in y \cap z$, $\sim \gamma \rightarrow \sim \beta \vee \sim \gamma \in L$ (the axiom scheme A9) and $y \cap z \in M_L$. We see $\sim \beta \vee \sim \gamma \in x$ by $y \cap z \subseteq x$. We get $\sim (\beta \wedge \gamma) \in x$ since $\sim \beta \vee \sim \gamma \in x$, $\sim \beta \vee \sim \gamma \rightarrow \sim (\beta \wedge \gamma) \in L$ (the axiom scheme S9) and $x \in M_L$.

(Case $\alpha \equiv \beta * \gamma$ for (2)): Suppose that $\sim (\beta * \gamma) \in x$. Then we have $(L \cdot \{\sim \beta\}) \cdot (L \cdot \{\sim \gamma\}) \subseteq x$ (†) because $L \cdot \{\sim (\beta * \gamma)\} \subseteq x$ by Lemma 4.5 (1) and $(L \cdot \{\sim \beta\}) \cdot (L \cdot \{\sim \gamma\}) \subseteq L \cdot \{\sim (\beta * \gamma)\}$ by Lemma 4.5 (5). Now we show that there exist $y, z \in M_L$ such that $y \cdot z \subseteq x$, $y \models^-_L \beta$ and $z \models^-_L \gamma$. We take $y = L \cdot \{\sim \beta\}$ and $z = L \cdot \{\sim \gamma\}$. We have $y \cdot z \subseteq x$ by (†). We have $\sim \beta \in L \cdot \{\sim \beta\}$ and $\sim \gamma \in L \cdot \{\sim \gamma\}$ and hence we get $y \models^-_L \beta$ and $z \models^-_L \gamma$ by the induction hypothesis. Therefore $x \models^-_L \beta * \gamma$. Conversely, suppose that $x \models^-_L \beta * \gamma$. Then there exist $y, z \in M_L$ such that $y \cdot z \subseteq x$, $y \models^-_L \beta$ and $z \models^-_L \gamma$. We have $\sim \beta \in y$ and $\sim \gamma \in z$ by the induction hypothesis. We can get $\sim \beta \rightarrow \sim \gamma \rightarrow \sim (\beta * \gamma) \in L$ by (*pref*), (*mp*), (the axiom scheme A10) $\sim \beta \rightarrow \sim \gamma \rightarrow \sim \beta * \sim \gamma$ and (the axiom scheme S13) $\sim \beta * \sim \gamma \rightarrow \sim (\beta * \gamma)$. Then we have $\sim \gamma \rightarrow \sim (\beta * \gamma) \in y$ by $\sim \beta \in y$, $\sim \beta \rightarrow \sim \gamma \rightarrow \sim (\beta * \gamma) \in L$ and $y \in M_L$. Thus we get $\sim (\beta * \gamma) \in y \cdot z$ by $\sim \gamma \in z$ and $\sim \gamma \rightarrow \sim (\beta * \gamma) \in y$. We have $\sim (\beta * \gamma) \in x$ by $y \cdot z \subseteq x$.

(Case $\alpha \equiv \beta \vee \gamma$ for (2)): Suppose that $\sim (\beta \vee \gamma) \in x$. Then we can get $\sim (\beta \vee \gamma) \rightarrow \sim \beta \in L$ and $\sim (\beta \vee \gamma) \rightarrow \sim \gamma \in L$ by (*pref*), (*mp*), (the axiom scheme S10) $\sim (\beta \vee \gamma) \rightarrow \sim \beta \wedge \sim \gamma$ and (the axiom scheme A5) $\sim \beta \wedge \sim \gamma \rightarrow \sim \beta$. Then we get $\sim \beta \in x$ and $\sim \gamma \in x$ by $\sim (\beta \vee \gamma) \in x$, $\sim (\beta \vee \gamma) \rightarrow \sim \beta \in L$, $\sim (\beta \vee \gamma) \rightarrow \sim \gamma \in L$ and $x \in M_L$. Thus we have $x \models^-_L \beta$ and $x \models^-_L \gamma$ by the induction hypothesis. Therefore $x \models^-_L \beta \vee \gamma$. Conversely, suppose that $x \models^-_L \beta \vee \gamma$. Then we have $x \models^-_L \beta$ and $x \models^-_L \gamma$. Thus we have $\sim \beta \in x$ and $\sim \gamma \in x$ by the induction hypothesis. Then we get $\sim \beta \wedge \sim \gamma \in x$ by $x \in M_L$. Therefore we have $\sim (\beta \vee \gamma) \in x$ by $\sim \beta \wedge \sim \gamma \in x$ and $\sim \beta \wedge \sim \gamma \rightarrow \sim (\beta \vee \gamma) \in L$ (the axiom scheme S11). ■

By using this lemma, we can prove the following.

Theorem 4.7. (Completeness) *Let S be any logic obtained from* $\mathrm{L}^{\sim}$ *by adding arbitrary combinations of the axiom schemes B1—B22, and C be a class of Kripke frames for S, $L := \{\gamma \mid \gamma$ is provable in $S\}$ and $L(C) := \{\gamma \mid \gamma$ is valid in all frames of $C\}$. Then $L = L(C)$.*

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