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ON THE PROBLEM OF R. E. TAX

A b s t r a c t. We prove that the purely equivalential fragment of intuitionitic logic is not axiomatizable by a finite number of axiom schemata and the modus ponens rule for equivalence as the only rule of inference.

The first finite axiomatization of the purely equivalential fragment of intuitionistic logic is due to R. E. Tax [9]. The calculus of Tax was based on a single – rather complicated – axiom schema and two structural rules of inference, one of which was the rule of modus ponens. The question whether it is possible to axiomatize this fragment by a finite number of axiom schemata and the rule of modus ponens alone has been posed by Tax himself in the above cited paper. At the first glance, the question did not look very difficult and a promising reformulation of it was soon found by the second author in the course of his early work on so called equivalential algebras. To get the result, however, he had to overcome a crucial obstacle which proved too hard for him. Thus, the question remained unanswered until the breakthrough has been achieved by the first author who managed to remove the obstacle and settled Tax's problem.

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We consider the purely equivalential fragment of intuitionistic logic denoted by $INT_{\leftrightarrow}$. Let $a, b, c \ldots$ be variables and let Greek letters $\alpha, \beta, \ldots$ stands for formulas of our propositional language. To enhance readability we adopt the convention of associating to the left and we denote the equivalence connective $\leftrightarrow$ by juxtaposition. For instance, the formula:

$$(((a \leftrightarrow b) \leftrightarrow b) \leftrightarrow (c \leftrightarrow c)) \leftrightarrow d$$

can be written simply as: $abb(cc)d$. Let us note the following paraphrase of the result of Tax [11]:

Theorem 1. *The following axiom schemata and rules of inference axiomatize the structural consequence relation of intuitionistic logic restricted to the purely equivalential language*

(A0) $\alpha\alpha$,

(A1) $(\alpha\beta)(\beta\alpha)$,

(A2) $(\alpha\beta\gamma\gamma)((\alpha\gamma)(\beta\gamma))$,

(A3) $(\alpha\beta)(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)$.

$$\text{(MP)}: \frac{\alpha\beta, \alpha}{\beta} \qquad\qquad \text{(T)}: \frac{\alpha}{\alpha\beta\beta}$$

The symbol $\vdash$ will now be reserved for the consequence relation determined by (A0), ... ,(A3), (MP) and (T). Thus, by the above theorem, $\alpha \in INT_{\leftrightarrow}$ iff $\vdash \alpha$ and it is easy to check that $INT_{\leftrightarrow}$ contains all formulas of the following list:

(A4) $(\alpha\beta\gamma\gamma)((\alpha\gamma\gamma)(\beta\gamma\gamma))$,

(A5) $(\alpha\alpha)(\beta\beta)$,

(A6) $(\alpha\beta\beta\beta)(\alpha\beta)$,

(A7) $\beta\beta\alpha\alpha$,

(A8) $(\alpha\beta\beta)(\beta\alpha\alpha)(\alpha\beta)$,

(A9) $(\alpha\beta\beta\gamma\gamma)(\alpha\gamma\gamma\beta\beta)$.

The algebraic counterpart of $INT_{\leftrightarrow}$ is the class of so called *equivalential algebras* see [8]. It can be characterized as the class of $\leftrightarrow$-subreducts of Brouwerian semilattices with $\leftrightarrow$ defined as $(a \to b) \wedge (b \to a)$. Equivalential algebras constitute a variety determined by the following identities (see [8]):

(i) $aab = b$,

(ii) $abcc = (ac)(bc)$,

(iii) $ab(acc)(acc) = ab$.

We would like to use this opportunity to stress the importance of equivalential algebras, which was brought to light during the recent study of so called *Fregean varieties* (see [5]). An important ingredient of Fregeanity is the following notion of *congruence-orderability*:

A variety $\mathbb{K}$ with a distinguished constant $\mathbf{1}$ is said to be *congruence-orderable* if for every $a, b \in \mathfrak{A} \in \mathbb{K}$, $\Theta_{\mathfrak{A}}(\mathbf{1}, a) = \Theta_{\mathfrak{A}}(\mathbf{1}, b)$ implies $a = b$.

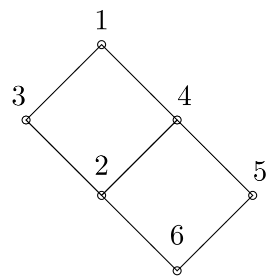
The above condition was introduced by Büchi and Owens in [1] where algebras obeying it were called *fission-free*. It forces the natural mapping $a \longmapsto \Theta_{\mathfrak{A}}(\mathbf{1}, a)$ of the underlying set of the algebra $\mathfrak{A}$ into its own congruence lattice to be 1–1 and thereby, it provides a natural ordering of the underlying set of $\mathfrak{A}$ by obvious stipulation: $a \leq b$ iff $\Theta_{\mathfrak{A}}(\mathbf{1}, a) \subseteq \Theta_{\mathfrak{A}}(\mathbf{1}, b)$.

The following result of Idziak, Słomczyńska and Wroński [5] shows the importance of the equivalential structure in the context of universal algebra

Theorem 2. *Every congruence-orderable variety with permutable congruences has a binary term* $\mathsf{e}(x, y)$ *that turns its members into equivalential algebras.*

Equivalential algebras are locally finite, congruence permutable but not congruence distributive (see [6]). The important congruence extension property which is known to hold for Brouwerian semilattices fails to hold for equivalential algebras. An elegant representation theory of finite equivalential algebras has been developed by Słomczyńska [9, 10] who proved – among other things – that every quasivariety of equivalential algebras is a variety.

Surprisingly, the pleasant image of equivalential algebras may change radically if their discourse language is endowed with an additional constant (see [3, 4]). For example, if $\mathfrak{A} = \langle \{1, \dots, 6\}, \leftrightarrow \rangle$ is the $\leftrightarrow$-reduct of the Brouwerian semilattice of the following diagram:



then the quasivariety generated by $\mathfrak{A}$ is just the variety of a 3-element chain, which has only two subvarieties and is finitely axiomatizable. However, the structure $\mathfrak{A}_c = \langle \{1, \dots, 6\}, \leftrightarrow, c \rangle$ where the new constant c is interpreted as 2 generates a quasivariety which is not finitely axiomatizable because it is the intersection of an infinite descending sequence of subquasivarieties of the quasivariety generated by a 3-element chain the middle element of which is denoted by the constant c (see [3, 4]).

The structure $\mathfrak{A}$ can also be used to show that it is impossible to axiomatize the consequence relation $\vdash$ by a set of axiom schemata and the rule (MP) alone. Indeed, let $\vdash_{\mathcal{A}}$ be the inference relation of the matrix $\mathcal{A} = \langle \mathfrak{A}, \{1, 2\} \rangle$. Then $\vdash \alpha$ implies $\vdash_{\mathcal{A}} \alpha$ and $\alpha, \alpha\beta \vdash_{\mathcal{A}} \beta$, but still $\alpha \nvdash_{\mathcal{A}} \alpha\beta\beta$ because $(2 \leftrightarrow 5) \leftrightarrow 5 = 3 \notin \{1, 2\}$.

We cannot remove the rule (T) from the considered axiomatization of $\vdash$. But we can replace it with other rules.

Theorem 3. *If a set X of formulas in $\{\leftrightarrow\}$ is closed under* (MP) *and contains all instances of* (A0)-(A3), *then X is closed under the rule* (T) *iff X is closed under one of the following rules*

$$(TR): \ \frac{\alpha\beta}{(\alpha\gamma)(\beta\gamma)} \qquad\qquad (EXT): \ \frac{\alpha\beta, \gamma\delta}{(\alpha\gamma)(\beta\delta)}$$

Proof. Note that by use of (TR) (together with (MP) and (A1)) one immediately gets the usual transitivity of equivalence:

$$\alpha\beta, \beta\gamma \vdash \alpha\gamma.$$

Next, we can derive (A7) using the rule (MP) and the axioms (A0)-(A2). Indeed, we get $(\beta\alpha)(\beta\alpha)$ by (A0). Then

$$(\beta\alpha)(\beta\alpha)(\beta\beta\alpha\alpha)$$

by (A2) and (A1). Hence $\beta\beta\alpha\alpha$ by (MP)

(TR). Assume $\alpha\beta$. Using (T), one gets $\alpha\beta\gamma\gamma$ and hence $(\alpha\gamma)(\beta\gamma)$ by (A2) and the rule (MP).

(EXT). Assume $\alpha\beta$ and $\gamma\delta$. Using (TR) one gets $(\alpha\gamma)(\beta\gamma)$ and $(\gamma\beta)(\delta\beta)$. Then, by (A1) and transitivity of $\leftrightarrow$, one gets $(\alpha\gamma)(\beta\delta)$.

(T). As (TR) is an immediate consequence of (EXT) it suffices to derive (T) using (TR) only. Let us assume α. By (A1), (A7) and (MP), one gets $\beta\beta\alpha$. Then $\alpha(\beta\beta)$ by (A1) and (MP). Hence $\alpha\beta(\beta\beta\beta)$ by (TR). Since $\beta\beta\beta\beta$ by (A7), we get $\alpha\beta\beta$ by transitivity of $\leftrightarrow$. ■

In the above proof it has been noted that the transitivity of equivalence in its usual form

$$(\mathrm{TR}^*) : \ \frac{\alpha\beta, \beta\gamma}{\alpha\gamma}$$

immediately follows from (TR). Let as add that (TR*) is strictly weaker than (TR). In fact, (TR*) can be even derived using (MP) and some intuitionistically valid schemata.

Although the consequence relation $\vdash$ cannot be axiomatized by (MP) as the only rule we do have a simple axiomatization of this kind for the set $INT_{\leftrightarrow}$. Namely, let us consider all schemata of the form:

$$(\&_m) \qquad \Delta\delta_1\delta_1 \dots \delta_m\delta_m$$

where $m = 0, 1, \dots$ and Δ falls under (A0)-(A4). Let $\vdash^\infty$ and $\vdash^n$ be the consequence relations determined by the rule (MP) and all instances of $(\&_m)$ or all instances of $(\&_m)$ with $m \leq n$, respectively.

Theorem 4. *For every formula* α

$$\vdash \alpha \quad \textit{iff} \quad \vdash^\infty \alpha \quad \textit{iff} \quad \vdash^n \alpha \ \textit{ for some } n$$

Proof. Since $\vdash^n$ is an increasing sequence of finitistic consequence relations, we get $\vdash^\infty \alpha$ iff $\vdash^n \alpha$ for some n. It is also clear that $\vdash^\infty \alpha$ yields

$\vdash \alpha$. Thus, there remains to prove that $\{\alpha : \vdash^\infty \alpha\}$ is closed under the rule (T).

Suppose that $\vdash^\infty \alpha$. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be a formal derivation of the formula α from the axioms ($\&_m$) by use of the rule (MP). We prove by induction on $i \leq r$ that $\vdash^\infty \alpha_i\beta\beta$ for every β.

The above holds if α_i falls under ($\&_m$) for some m. Suppose it holds for α_i and $\alpha_i\alpha_j$ (occurring in the derivation of α). By induction hypothesis, we get $\vdash^\infty \alpha_i\beta\beta$ and $\vdash^\infty \alpha_i\alpha_j\beta\beta$. Then, using (MP) and (A4), we get $\vdash^\infty \alpha_j\beta\beta$ which completes our argument. ■

Using a well-know result, called Tarski's test on finite axiomatizability, we can reduce the problem of R. E. Tax to the following:

Question: *Is there a number n such that for every α*

$$\vdash \alpha \qquad \textit{iff} \qquad \vdash^n \alpha.$$

Below, we settle this question in negative. But to do so we need certain auxiliary concepts and results.

The concept of a subformula is standard. If the variable a occurs in a formula Ψ exactly once, we write $\Psi[a]$. Suppose that a does not occur in a formula Φ. One easily shows that a formula ϕ is a subformula of Φ iff $\Phi = \Psi[\phi]$ for some $\Psi[a]$. Clearly, different formulas $\Psi[a]$, with $\Phi = \Psi[\phi]$, coincide with different occurrences of the formula ϕ in Φ. For any ϕ and any $\Psi[a]$, let $S_\phi(\Psi)$ be a finite set of formulas defined as follows

(i) $S_\phi(a)=\{\phi\}$;
(ii) $S_\phi(\Psi\Sigma)=S_\phi(\Sigma\Psi) = S_\phi(\Psi) \cup \{\Sigma\}$.

One can say that $S_\phi(\Psi)$ is a minimal set containing ϕ and subformulas of $\Psi[a]$ which do not contain the variable a such that the formula $\Psi[\phi]$ can be built up from its elements. Clearly,

Lemma 5. *For every formula ϕ and every $\Psi[a]$:*

$$\phi \in S_\phi(\Psi) \quad \textit{and} \quad S_\phi(\Psi) \vdash \Psi[\phi]$$

Suppose that $X \approx Y$ means that the two sets of formulas are equivalent in intuitionistic logic, i. e. they have the same consequences with respect to the relation $\vdash$. Let $\mathrm{Rank}(\phi, \Psi[a])$ be the minimal number k such that

$$S_\phi(\Psi) \approx \{\alpha_1, \ldots, \alpha_k\} \quad \text{for some } \alpha_1, \ldots, \alpha_k$$

Informally, the rank says how deep the considered occurrence of the formula ϕ takes place in $\Psi[\phi]$. If $\mathrm{Rank}(\phi, \Psi[a]) = k$ and $\Phi = \Psi[\phi]$, then ϕ is said to have a k-occurrence in Φ.

It is known that any variable has an even number (zero is allowed) of occurrences in any $\{\leftrightarrow\}$-tautology. The following lemma extends this property to complex k-subformulas of some intuitionistically valid formulas.

Lemma 6. *If $\vdash^n \Phi$ and $k > n + 5$, then the number of k-occurrences of any formula ϕ in Φ is even.*

Proof. Our argument is inductive with respect to the length of a formal derivation of Φ. We may assume that the variable a does not occur in this derivation. First, suppose that Φ is a formula of the form

(A0) $$\alpha\alpha\ \delta_1\delta_1 \cdots \delta_m\delta_m \qquad \text{for some } \ m \leq n.$$

If $\phi = \alpha\alpha \cdots \delta_i$, then $\Psi[a]$ is either $a\delta_i\delta_{i+1} \cdots \delta_m\delta_m$ or $a\delta_{i+1}\delta_{i+1} \cdots \delta_m\delta_m$ for some i and hence

$$S_\phi(\Psi) \subseteq \{\phi, \delta_i, \ldots, \delta_m\}$$

It means that $\mathrm{Rank}(\phi, \Psi) \leq n+1 < k$. We conclude that any k-occurrence of the formula ϕ in Φ takes place either in α or in some δ_i.

Suppose that $\alpha = \Psi[\phi]$ for some $\Psi[a]$. Since α appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\Psi\alpha\delta_1 \cdots \delta_m) = S_\phi(\alpha\Psi\delta_1 \cdots \delta_m) = \{\alpha, \delta_1, \ldots, \delta_m\} \cup S_\phi(\Psi)$$

Hence the rank of the corresponding two occurrences of ϕ in Φ is the same.

Suppose that $\delta_i = \Psi[\phi]$ for some $\Psi[a]$. Since δ_i appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\alpha \cdots \Psi\delta_i \cdots) = S_\phi(\Psi) \cup \{\alpha\alpha \cdots \delta_{i-1}\delta_{i-1}, \delta_i, \ldots, \delta_m\}$$

$$S_\phi(\alpha\alpha \cdots \delta_i\Psi \cdots) = S_\phi(\Psi) \cup \{\alpha\alpha \cdots \delta_{i-1}\delta_i, \delta_{i+1}, \ldots, \delta_m\}$$

As $S_\phi(\Psi) \vdash \delta_i$ by Lemma 5, the two sets are equivalent (with respect to $\approx$) which means that the rank of the corresponding two occurrences of ϕ in Φ is the same. Thus, the number of all k-occurrences of ϕ in (A0) is even.

Let us assume that Φ is of the form

(A1) $$\alpha\beta(\beta\alpha)\delta_1\delta_1\cdots\delta_m\delta_m \qquad \text{for some} \quad m \leq n$$

Similarly as above we can easily get rid of the cases where ϕ is:

$$\alpha\beta \qquad\qquad \beta\alpha \qquad\qquad \alpha\beta(\beta\alpha)\cdots\delta_i\cdots$$

In each of them the set $S_\phi(\cdots)$ contains at most $m+2$ elements and hence the rank of the corresponding occurrence of ϕ does not exceed k.Hence all k-occurrences of ϕ in (A1) must take place in α, or β, or δ_i for some i.

Suppose that $\alpha = \Psi[\phi]$ for some $\Psi[a]$. Since α appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\Psi\beta(\beta\alpha)\cdots\delta_i\cdots) = S_\phi(\Psi) \cup \{\beta, \beta\alpha, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\beta\Psi)\cdots\delta_i\cdots) = S_\phi(\Psi) \cup \{\beta, \alpha\beta, \delta_1, \ldots, \delta_m\}$$

Denote the above sets by T_1 and T_2, respectively. Clearly, $T_1 \approx T_2$ and hence the rank of the corresponding two occurrences of ϕ in Φ is the same.

Suppose that $\beta = \Psi[\phi]$ for some $\Psi[a]$. Since β appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\Psi(\beta\alpha)\cdots\delta_i\cdots) = S_\phi(\Psi) \cup \{\alpha, \beta\alpha, \delta_1 \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\Psi\alpha)\cdots\delta_i\cdots) = S_\phi(\Psi) \cup \{\alpha, \alpha\beta, \delta_1 \ldots \delta_m\}$$

Similarly as above we get $T_1 \approx T_2$ (where T_1 and T_2 denote the above sets) and hence the rank of the two occurrences of ϕ in Φ is the same.

Suppose that $\delta_i = \Psi[\phi]$ for some $\Psi[a]$. Since δ_i appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$T_1 = S_\phi(\alpha\cdots\Psi\delta_i\cdots) = S_\phi(\Psi) \cup \{\alpha\cdots\delta_{i-1}\delta_{i-1}, \delta_i, \ldots, \delta_m\}$$

$$T_2 = S_\phi(\alpha\cdots\delta_i\Psi\cdots) = S_\phi(\Psi) \cup \{\alpha\cdots\delta_{i-1}\delta_i, \delta_{i+1}, \ldots, \delta_m\}$$

But $S_\phi(\Psi) \vdash \delta_i$ by Lemma 5. Hence $T_1 \approx T_2$. It means that the rank of the corresponding two occurrences of ϕ in Φ is the same. We conclude that the number of all k-occurrences of ϕ in (A1) is even.

Let us assume that Φ is of the form

(A2) $\alpha\beta\gamma\gamma((\alpha\gamma)(\beta\gamma))\delta_1\delta_1\cdots\delta_m\delta_m$ for some $m \le n$

If ϕ is one of the following formulas:

$$\alpha\beta \quad \alpha\beta\gamma \quad \alpha\beta\gamma\gamma \quad \alpha\gamma \quad \beta\gamma \quad \alpha\gamma(\beta\gamma) \quad \alpha\beta\gamma\gamma((\alpha\gamma)(\beta\gamma)\cdots\delta_i\cdots$$

then $S_\phi(\cdots)$ contains at most $m+3$ elements and hence the rank of the corresponding occurrence of ϕ does not exceed k. Thus, all k-occurrences of ϕ in (A2) must take place in α, or β, or γ, or δ_i for some i.

Suppose that $\alpha = \Psi[\phi]$ for some $\Psi[a]$. Since α appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\Psi\beta\gamma\gamma((\alpha\gamma)(\beta\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\beta,\gamma,\alpha\gamma(\beta\gamma),\delta_1,\ldots,\delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma((\Psi\gamma)(\beta\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\gamma,\beta\gamma,\alpha\beta\gamma\gamma,\delta_1,\ldots,\delta_m\}$$

Clearly, $T_1 \approx T_2$ and hence the rank of the two occurrences of ϕ is the same.

Suppose that $\beta = \Psi[\phi]$ for some $\Psi[a]$. Since β appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\Psi\gamma\gamma((\alpha\gamma)(\beta\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\alpha,\gamma,\alpha\gamma(\beta\gamma),\delta_1,\ldots,\delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma((\alpha\gamma)(\Psi\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\gamma,\alpha\gamma,\alpha\beta\gamma\gamma,\delta_1,\ldots,\delta_m\}$$

Clearly, $T_1 \approx T_2$ and hence the rank of the two occurrences of ϕ is the same.

Suppose that $\gamma = \Psi[\phi]$ for some $\Psi[a]$. Since γ appears in Φ four times, we get four corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\beta\Psi\gamma((\alpha\gamma)(\beta\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\alpha\beta,\gamma,\alpha\gamma(\beta\gamma),\delta_1,\ldots,\delta_m\}$$

$$S_\phi(\alpha\beta\gamma\Psi((\alpha\gamma)(\beta\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\alpha\beta\gamma,\alpha\gamma(\beta\gamma),\delta_1,\ldots,\delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma((\alpha\Psi)(\beta\gamma))\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\alpha,\beta\gamma,\alpha\beta\gamma\gamma,\delta_1,\ldots,\delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma((\alpha\gamma)(\beta\Psi)\cdots\delta_i\cdots) = S_\phi(\Psi)\cup\{\beta,\alpha\gamma,\alpha\beta\gamma\gamma,\delta_1,\ldots,\delta_m\}$$

Let us denote the above sets by T_1, T_2, T_3, T_4,respectively. As $S_\phi(\Psi) \vdash \gamma$ by Lemma 5, we get $T_1 \approx T_2$ and $T_3 \approx T_4$. Hence all considered occurrences of ϕ in (A2) can be divided into pairs having the same rank.

Suppose that $\delta_i = \Psi[\phi]$ for some $\Psi[a]$. Since δ_i appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$T_1 = S_\phi(\alpha \cdots \Psi\delta_i \cdots) = S_\phi(\Psi) \cup \{\alpha \cdots \delta_{i-1}\delta_{i-1}, \delta_i, \ldots, \delta_m\}$$

$$T_2 = S_\phi(\alpha \cdots \delta_i\Psi \cdots) = S_\phi(\Psi) \cup \{\alpha \cdots \delta_{i-1}\delta_i, \delta_{i+1}, \ldots, \delta_m\}$$

But $S_\phi(\Psi) \vdash \delta_i$ by Lemma 5 and hence $T_1 \approx T_2$. It means that the rank of the corresponding two occurrences of ϕ in Φ is the same. We conclude that the number of all k-occurrences of ϕ in (A2) is even.

Let us assume that Φ is of the form

(A3) $$\alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)\delta_1\delta_1 \cdots \delta_m\delta_m \qquad \text{for some } m \leq n.$$

If ϕ is one of the following formulas:

$$\alpha\beta \quad \alpha\gamma \quad \alpha\gamma\gamma \quad \alpha\beta(\alpha\gamma\gamma) \quad \alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)\cdots\delta_i$$

then $S_\phi(\cdots)$ contains at most $m+5$ elements and hence the rank of the occurrence of ϕ does not exceed k. All k-occurrences of ϕ in (A3) (if any) must take place in α, or β, or γ, or δ_i for some i.

Suppose that $\alpha = \Psi[\phi]$ for some $\Psi[a]$. Since α appears in Φ four times, we get four corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\Psi\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\beta, \alpha\gamma\gamma, \alpha\beta, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\Psi\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\gamma, \alpha\beta, \alpha\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\alpha\gamma\gamma)(\Psi\gamma\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\gamma, \alpha\beta(\alpha\gamma\gamma), \alpha\beta, \delta_1, \ldots, \delta_m\}$$

$=$

$$S_\phi(\alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\Psi\beta)\cdots) = S_\phi(\Psi) \cup \{\alpha\beta(\alpha\gamma\gamma)\alpha\gamma\gamma, \beta, \delta_1, \ldots, \delta_m\}$$

Let $T_1 - T_4$ denote the above sets. Clearly, $T_2 \approx T_3$. Since $S_\phi(\Psi) \vdash \alpha$ by Lemma 5 and $\alpha \vdash \alpha\gamma\gamma$, we also get $T_1 \approx T_4$. Thus, the considered occurrences of ϕ in (A3) can be divided into pairs having the same rank.

Suppose that $\beta = \Psi[\phi]$ for some $\Psi[a]$. Since β appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\Psi(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\alpha, \alpha\gamma\gamma, \alpha\beta, \delta_1, \ldots, \delta_m\}$$

$=$

$$S_\phi(\alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma)(\alpha\Psi)\cdots) = S_\phi(\Psi) \cup \{\alpha, \alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

Clearly, $T_1 \approx T_2$ and hence the rank of the two occurrences of ϕ is the same.

Suppose that $\gamma = \Psi[\phi]$ for some $\Psi[a]$. Since γ appears in Φ four times, we get four corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\beta(\alpha\Psi\gamma)(\alpha\gamma\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\alpha, \gamma, \alpha\beta, \alpha\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\alpha\gamma\Psi)(\alpha\gamma\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\alpha\gamma, \alpha\beta, \alpha\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\alpha\gamma\gamma)(\alpha\Psi\gamma)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\alpha, \gamma, \alpha\beta, \alpha\beta(\alpha\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta(\alpha\gamma\gamma)(\alpha\gamma\Psi)(\alpha\beta)\cdots) = S_\phi(\Psi) \cup \{\alpha\beta, \alpha\gamma, \alpha\beta(\alpha\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

But $S_\phi(\Psi) \vdash \gamma$ by Lemma 5 and hence the above sets are equivalent. It means that the rank of all considered occurrences of ϕ in (A2) is the same.

Suppose that $\delta_i = \Psi[\phi]$ for some $\Psi[a]$. Since δ_i appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$T_1 = S_\phi(\alpha\cdots\Psi\delta_i\cdots) = S_\phi(\Psi) \cup \{\alpha\cdots\delta_{i-1}\delta_{i-1}, \delta_i, \ldots, \delta_m\}$$

$$T_2 = S_\phi(\alpha\cdots\delta_i\Psi\cdots) = S_\phi(\Psi) \cup \{\alpha\cdots\delta_{i-1}\delta_i, \delta_{i+1}, \ldots, \delta_m\}$$

As $S_\phi(\Psi) \vdash \delta_i$ by Lemma 5 we get $T_1 \approx T_2$ and hence the rank of the corresponding two occurrences of ϕ in Φ is the same. We conclude that the number of all k-occurrences of ϕ in (A3) is even.

Let us assume that Φ is of the form

(A4) $\quad \alpha\beta\gamma\gamma(\alpha\gamma\gamma(\beta\gamma\gamma))\delta_1\delta_1\cdots\delta_m\delta_m \quad$ for some $m \leq n$.

If ϕ is one of the following formulas:

$$\alpha\beta \;,\; \alpha\beta\gamma \;,\; \alpha\beta\gamma\gamma \;,\; \alpha\gamma \;,\; \beta\gamma \;,\; \alpha\gamma\gamma \;,\; \beta\gamma\gamma \;,$$

$$\alpha\gamma\gamma(\beta\gamma\gamma) \;,\; \alpha\beta\gamma\gamma(\alpha\gamma\gamma(\beta\gamma\gamma))\cdots\delta_i$$

then $S_\phi(\cdots)$ contains at most $m+4$ elements and hence the rank of the corresponding occurrence of ϕ does not exceed k. All k-occurrences of ϕ in (A4) must take place in α, or β, or γ, or δ_i for some i.

Suppose that $\alpha = \Psi[\phi]$ for some $\Psi[a]$. Since α appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\Psi\beta\gamma\gamma(\alpha\gamma\gamma(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\beta, \gamma, \alpha\gamma\gamma(\beta\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma(\Psi\gamma\gamma(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\gamma, \beta\gamma\gamma, \alpha\beta\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

Clearly, the both sets are equivalent in intuitionistic logic and hence the rank of the two occurrences of ϕ in Φ is the same.

Suppose that $\beta = \Psi[\phi]$ for some $\Psi[a]$. Since β appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$= S_\phi(\alpha\Psi\gamma\gamma(\alpha\gamma\gamma(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\alpha, \gamma, \alpha\gamma\gamma(\beta\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

$$= S_\phi(\alpha\beta\gamma\gamma(\alpha\gamma\gamma(\Psi\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\gamma, \alpha\gamma\gamma, \alpha\beta\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

Clearly, the above sets are equivalent in intuitionistic logic and hence the rank of the corresponding two occurrences of ϕ in Φ is the same.

Suppose that $\gamma = \Psi[\phi]$ for some $\Psi[a]$. Since γ appears in Φ six times, we get six corresponding occurrences of ϕ in Φ. Moreover,

$$S_\phi(\alpha\beta\Psi\gamma(\alpha\gamma\gamma(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\alpha\beta, \gamma, \alpha\gamma\gamma(\beta\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta\gamma\Psi(\alpha\gamma\gamma(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\alpha\beta\gamma, \alpha\gamma\gamma(\beta\gamma\gamma), \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma(\alpha\Psi\gamma(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\alpha, \gamma, \beta\gamma\gamma, \alpha\beta\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma(\alpha\gamma\Psi(\beta\gamma\gamma))\cdots) = S_\phi(\Psi) \cup \{\alpha\gamma, \beta\gamma\gamma, \alpha\beta\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma(\alpha\gamma\gamma(\beta\Psi\gamma))\cdots) = S_\phi(\Psi) \cup \{\beta, \gamma, \alpha\gamma\gamma, \alpha\beta\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

$$S_\phi(\alpha\beta\gamma\gamma(\alpha\gamma\gamma(\beta\gamma\Psi))\cdots) = S_\phi(\Psi) \cup \{\beta\gamma, \alpha\gamma\gamma, \alpha\beta\gamma\gamma, \delta_1, \ldots, \delta_m\}$$

Denote the above sets by $T_1 - T_6$, respectively. Since $S_\phi(\Psi) \vdash \gamma$ by Lemma 5, we get $T_1 \approx T_2$ and $T_3 \approx T_4 \approx T_5 \approx T_6$. It means that the considered occurrences of ϕ can be divided into pairs having the same rank.

Suppose that $\delta_i = \Psi[\phi]$ for some $\Psi[a]$. Since δ_i appears in Φ twice, we get two corresponding occurrences of ϕ in Φ. Moreover,

$$T_1 = S_\phi(\alpha\cdots\Psi\delta_i\cdots) = S_\phi(\Psi) \cup \{\alpha\cdots\delta_{i-1}\delta_{i-1}, \delta_i, \ldots, \delta_m\}$$

$$T_2 = S_\phi(\alpha\cdots\delta_i\Psi\cdots) = S_\phi(\Psi) \cup \{\alpha\cdots\delta_{i-1}\delta_i, \delta_{i+1}, \ldots, \delta_m\}$$

As $S_\phi(\Psi) \vdash \delta_i$ by Lemma 5, we get $T_1 \approx T_2$ and hence the rank of the corresponding two occurrences of ϕ in Φ is the same.

We conclude that the number of k-occurrences (for $k > n + 5$) of any formula ϕ in any axiom of $\vdash^n$ is even. There remains to consider the case:

(MP) $\vdash^n \Phi_1$ and $\vdash^n \Phi_1\Phi_2$, and the number of all k-occurrences of the formula ϕ in Φ_1 is even, and the number of all k occurrences of ϕ in $\Phi_1\Phi_2$ is even.

Since $k > 5$, we exclude the possibility that ϕ is one of the Φ_i's or ϕ is $\Phi_1\Phi_2$. Next, each occurrence of ϕ in Φ_i corresponds to an occurrence of ϕ in $\Phi_1\Phi_2$. For instance, if $\Phi_1 = \Psi[\phi]$ for some $\Psi[a]$, then $\Phi_1\Phi_2 = \Psi[\phi]\Phi_2$. Since $\vdash \Phi_i$, the rank of any occurrence of ϕ in Φ_i is the same as the rank of the corresponding occurrence of ϕ in $\Phi_1\Phi_2$, e. g.

$$S_\phi(\Psi\Phi_2) = S_\phi(\Psi) \cup \{\Phi_2\} \approx S_\phi(\Psi).$$

The number of all k-occurrences of the formula ϕ in Φ_2 is even as the difference of two even numbers is even. ■

Let $\Phi_k = a_0a_0a_1a_1 \cdots a_ka_k$ for any $k \geq 0$. Clearly, there is only one occurrence of a_0a_0 in Φ_k and we can easily calculate its rank:

Lemma 7. $\mathrm{Rank}(a_0a_0, aa_1a_1 \cdots a_ka_k) = k$ *for any* $k \geq 0$.

Proof. We have $S_{a_0a_0}(aa_1a_1 \cdots a_ka_k) \approx \{a_1, \ldots, a_k\}$. Thus, it suffices to show that $\{a_1, \ldots, a_k\} \approx \{\alpha_1, \ldots, \alpha_m\}$ does not hold for any $\alpha_1, \ldots, \alpha_m$ with $m < k$. Indeed, if it were so then we would have:

$$\alpha_1, \ldots, \alpha_m \vdash_c a_i, \quad \text{for } i = 1, \ldots, k$$

where $\vdash_c$ is the consequence relation determined by the classical logic. We can assume, of course, that no other variable than $a_1, \ldots, a_k$ occurs in the formulas $\alpha_1, \ldots, \alpha_m$. Next, it would follow that the free k-generated Boolean group has an m-element generating set of equivalence classes of formulas $\alpha_1, \ldots, \alpha_m$. This, however, is impossible because we know from Jónsson and Tarski [7] (see also [2] Theorem 0.4.54 on p. 141) that a free k-generated algebra of any variety with a non-trivial finite member cannot be generated by fewer than k elements. ■

On the basis of Lemma 6 and Lemma 7 we conclude $\vdash^n \Phi_k$ does not have place if $k > n + 5$. Since Φ_k are intuitionistically valid, we get

Theorem 8. *It is not possible to axiomatize $INT_{\leftrightarrow}$ by a finite set of axiom schemes and the rule (MP) alone.*

Our proof of Lemma 6 simply extends to cover the case where (TR*) is one of the rules of $\vdash^n$.

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