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FULLY ADEQUATE GENTZEN SYSTEMS AND THE DEDUCTION THEOREM

A b s t r a c t. A deductive system over an arbitrary language type Λ is a finitary and substitution-invariant consequence relation over the formulas of Λ. A Gentzen system is a finitary and substitution-invariant consequence relation over the sequents of Λ. A matrix model of a deductive system $\boldsymbol{\mathcal{S}}$ is a pair $\langle \mathbf{A}, \mathbf{F} \rangle$ where $\mathbf{A}$ is a Λ-algebra and F is an $\boldsymbol{\mathcal{S}}$-filter on $\mathbf{A}$, i.e., a subset of A closed under all interpretations of the consequence relation of $\boldsymbol{\mathcal{S}}$ in $\mathbf{A}$. A generalized matrix is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ where $\mathcal{C}$ is an algebraic closed-set system over A; it is a model of a Gentzen system $\mathcal{G}$ if $\mathcal{C}$ is closed under all interpretations of the consequence relation of $\mathcal{G}$ in $\mathbf{A}$. A Gentzen system $\mathcal{G}$ is *fully adequate* for a deductive system $\boldsymbol{\mathcal{S}}$ if (roughly speaking) every reduced generalized matrix model of $\mathcal{G}$ is of the form $\langle \mathbf{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \mathbf{A} \rangle$, where $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \mathbf{A}$ is the set of all $\boldsymbol{\mathcal{S}}$-filters on $\mathbf{A}$.

The existence of a fully adequate Gentzen system for a given protoalgebraic deductive system $\boldsymbol{\mathcal{S}}$ is completely characterized in terms of the following variant of the standard deduction theorem of classical

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and intuitionistic logic.

An infinite sequence $\boldsymbol{\Delta} = \langle \Delta_n(x_0, \ldots, x_{n-1}, y, \bar{u}) : n < \omega \rangle$ of possibly infinite nonempty sets of formulas in $n+1$ variables $x_0, \ldots, x_{n-1}, y$ and a possibly infinite system of parameters $\bar{u}$ is a *parameterized graded deduction-detachment* (PGDD) *system* for a deductive system $\boldsymbol{\mathcal{S}}$ over an $\boldsymbol{\mathcal{S}}$-theory T if, for every $n < \omega$ and for all $\varphi_0, \ldots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$, $T, \varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi$ iff $T \vdash_{\boldsymbol{\mathcal{S}}} \Delta_n(\varphi_0, \ldots, \varphi_{n-1}, \psi, \bar{\vartheta})$ for every possible system of formulas $\bar{\vartheta}$. An $\boldsymbol{\mathcal{S}}$-theory is *Leibniz* if it is included in every $\boldsymbol{\mathcal{S}}$-theory with the same Leibniz congruence. A PGDD system $\boldsymbol{\Delta}$ is *Leibniz-generating* if the union of the $\Delta_n(x_0, \ldots, x_{n-1}, y, \bar{\vartheta})$ as $\bar{\vartheta}$ ranges over all systems of formulas generates a Leibniz theory. The main result of the paper is the following:

Theorem. A protoalgebraic deductive system has a fully adequate Gentzen system if and only if it has a Leibniz-generating PGDD system over all Leibniz theories.

Two corollaries:

(I) A weakly algebraizable deductive system has a fully adequate Gentzen system iff it has the multiterm deduction-detachment theorem.

(II) A finitely equivalential deductive system has a fully adequate Gentzen system iff it has a finite Leibniz-generating system for over all Leibniz $\boldsymbol{\mathcal{S}}$-filters.

Several different variants of the deduction theorem arise in the course of the paper showing that this familiar notion is only one manifestation of a surprisingly complex phenomenon.

Contents

1. Introduction

Recent attempts to extend the theory of algebraizability to (Hilbert-style) deductive systems that cannot be algebraized in the standard way has led to consideration of Gentzen-style systems that are *adequate* for the given deductive system. In this paper a *deductive system* $\mathcal{S}$ is determined by a (finitary and substitution-invariant) consequence relation $\vdash_{\mathcal{S}}$ between the formulas of some given language type. Gentzen systems on the other hand can be identified with consequence relations between *sequents*, where a sequent is thought of as a finite, nonempty sequence $\varphi_0, \dots, \varphi_n$, which we shall write in the form $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$; alternative, and possibly more standard notations for a sequent are $\varphi_0, \dots, \varphi_{n-1} \to \varphi_n$ and $\dfrac{\varphi_0, \dots, \varphi_{n-1}}{\varphi_n}$.

A Gentzen system $\mathcal{G}$ is said to be *adequate* for a deductive system $\mathcal{S}$ if the sequent $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$ is a theorem of $\mathcal{G}$ if and only if $\varphi_0, \dots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi_n$. It turns out that the most useful Gentzen system for this purpose, provided it exists, is the one whose generalized matrix models coincide with the full generalized models of $\mathcal{S}$. A *generalized matrix* is a pair $\mathcal{A} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ where $\boldsymbol{A}$ is an algebra and $\mathcal{C}$ is an algebraic closed-set system on the universe of $\boldsymbol{A}$. Generalized matrices can be used as the basis for a semantics for Gentzen systems in much the same way ordinary matrices serve as the basis for a semantics for deductive systems. The generalized matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a *basic full generalized model* of a deductive system $\mathcal{S}$ if $\mathcal{C}$ coincides with the set of all $\mathcal{S}$-filters on $\boldsymbol{B}$, and it is a *full generalized model* if there is a basic full generalized model $\langle \boldsymbol{B}, \mathcal{D} \rangle$ of $\mathcal{S}$ and a surjective homomorphism $h: \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ such that $\mathcal{C} = h^{-1}(\mathcal{D})$ $(= \{ h^{-1}(F) : F \in \mathcal{D} \})$. If $\mathcal{S}$ is protoalgebraic, a full generalized model is uniquely determined by its smallest $\mathcal{S}$-filter, which is necessarily nonempty if $\mathcal{S}$ is not almost inconsistent. These filters are called the *Leibniz filters* of $\mathcal{S}$.

A Gentzen system $\mathcal{G}$ is said to be *fully adequate* for a deductive system $\mathcal{S}$ if the models of $\mathcal{G}$ coincide with the full generalized models of $\mathcal{S}$. (This is the case if $\mathcal{S}$ has at least one theorem and hence in particular if $\mathcal{S}$ is protoalgebraic and not almost inconsistent. Otherwise the definition of full adequacy has to be modified slightly. Since we consider exclusively protoalgebraic deductive systems in this paper we ignore the distinction.) The notion of a fully adequate Gentzen system (under the name "strongly adequate") was introduced in [15] and is studied in some detail in [16], a companion paper to the present one.

From recent work in abstract algebraic logic we now have a pretty clear idea of what it means for a deductive system to be algebraizable in one of the standard ways. ([3, 8, 10, 18, 19]). There are however a number of deductive systems that are not algebraizable in any of the standard ways but do have a fully adequate Gentzen system that is algebraizable in a standard way ([21]). Moreover in these cases the equivalent quasivariety of this Gentzen system gives a natural algebraic semantics for the given deductive system. The paradigm for these deductive systems is the conjunction-disjunction fragment of the classical propositional logic. The equivalent algebraic semantics for its fully adequate Gentzen system is, as expected, the variety of distributive lattices ([17]; see also [15, p. 99]).

With regard to the standard methods of abstract algebraic logic, the algebraic properties of a deductive system $\boldsymbol{\mathcal{S}}$ are reflected in those congruences, on an arbitrary algebra, that are maximal with respect to being compatible with some $\boldsymbol{\mathcal{S}}$-filter. These are the so-called *Leibniz congruences* of $\boldsymbol{\mathcal{S}}$. It is shown in [3] that a deductive system $\boldsymbol{\mathcal{S}}$ is finitely algebraizable in the standard way when the correspondence between $\boldsymbol{\mathcal{S}}$-filters and Leibniz congruences is essentially as close as possible. One of the main results of [15] is that a similar correspondence holds for every deductive system between full generalized models and arbitrary intersections of Leibniz congruences. In the case of protoalgebraic systems this induces a one-one correspondence between Leibniz filters and Leibniz congruences. It is largely because of these correspondences that the fully adequate Gentzen systems have such a large role to play in the attempt to extend the standard theory of algebraizability.

The main result of the present paper is a complete characterization of those protoalgebraic deductive systems that have a fully adequate Gentzen system. The characterization takes the form of a variant of the standard deduction theorem that is not comparable to the original in the sense that the two notions are logically independent. These are only two of the many different variants of the deduction theorem that arise in the course of the paper. Altogether they show that the familiar notion of the deduction theorem, when carefully analyzed, is seen to be but a single manifestation of a surprisingly complex phenomenon. Let

$$\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle \tag{1}$$

be an infinite sequence of sets of formulas, where for each $n < \boldsymbol{\omega}$, $\Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ is a possibly infinite nonempty set of formulas in the $n + 1$ variables $x_0, \dots, x_{n-1}, y$ and a potentially infinite number of parameters $\bar{u} = u_0, u_1, \dots, u_k, \dots$ $(k < \boldsymbol{\omega})$.

Let $\mathcal{S}$ be a deductive system, and let T be an $\mathcal{S}$-theory. Then $\boldsymbol{\Delta}$ is said to be a *parameterized graded deduction-detachment* (PGDD) *system for* $\mathcal{S}$ *over* T if, for every $n < \boldsymbol{\omega}$ and for all $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$,

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad T \vdash_{\mathcal{S}} \forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}). \tag{2}$$

In this expression $T \vdash_{\mathcal{S}} \forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta})$ is intended to represent the potentially infinite set of entailments of the form

$$T \vdash_{\mathcal{S}} \delta(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}),$$

where $\delta(x_0, \dots, x_{n-1}, y, \bar{u}) \in \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ and $\bar{\vartheta} = \vartheta_0, \vartheta_1, \dots, \vartheta_k, \dots$ $(k < \boldsymbol{\omega})$ ranges over all $\boldsymbol{\omega}$-sequences of formulas. A PGDD system $\boldsymbol{\Delta}$ is said to be *Leibniz-generating* if $\forall\bar{\vartheta}\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{\vartheta})$ generates a Leibniz $\mathcal{S}$-theory, i.e., a theory that is Leibniz as an $\mathcal{S}$-filter.

$\boldsymbol{\Delta}$ is a *graded deduction-detachment* (GDD) *system for* $\mathcal{S}$ *over* T if it is a PGDD system in which the parameter set is empty, and $\boldsymbol{\Delta}$ is *finite* if Δ_n is finite for each $n < \boldsymbol{\omega}$. A deductive system is said to have the *graded deduction-detachment theorem* if it has a finite GDD system over the smallest theory, the set of theorems. Thus $\mathcal{S}$ has the graded DD theorem iff there is a finite set $\Delta_n(x_0, \dots, x_{n-1}, y)$ of formulas for each $n < \boldsymbol{\omega}$ such that, for all formulas $\varphi_0, \dots, \varphi_{n-1}, \psi$,

$$\varphi_0, \dots, \varphi_{n-1} \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \vdash_{\mathcal{S}} \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi).$$

The graded DD theorem is strictly weaker than the multiterm DD theorem, i.e., the standard DD theorem where the role of the implication connective is taken by a fixed but arbitrary finite set of formulas (Thm. 5.12). The graded DD theorem, in the special case Δ_n contains a single formula without parameters, has also been considered by Rybakov [22, p. 477] under the name "the general deduction theorem".

Let $\mathcal{S}$ be a protoalgebraic deductive system. In the main result of the paper we prove that a protoalgebraic deductive system $\mathcal{S}$ has a fully adequate Gentzen system if and only if it has a Leibniz-generating PGDD system $\boldsymbol{\Delta}$ over every Leibniz theory; see Theorems 4.8 and 4.10. Moreover, if (1) is a PGDD system for $\boldsymbol{\mathcal{S}}$ over every Leibniz theory, then the family of Gentzen-style rules

$$\frac{x_0, \dots, x_{n-1} \rhd y}{\rhd\, \delta(x_0, \dots, x_{n-1}, y, \bar{u})}$$

$$\text{for each} \quad \delta(x_0, \dots, x_{n-1}, y, \bar{u}) \in \varDelta_n(x_0, \dots, x_{n-1}, y, \bar{u}), \qquad (\mathrm{R}_{\varDelta_n})$$

where n ranges over all natural numbers, constitutes a base for a presentation of the fully adequate Gentzen system for $\boldsymbol{\mathcal{S}}$ relative to a base for $\boldsymbol{\mathcal{S}}$ itself.

These results can be refined in various ways. Every protoalgebraic deductive system that has a fully adequate Gentzen system has a finite GDD system $\boldsymbol{\varDelta}$ (without parameters, but possibly not Leibniz-generating) over all Leibniz theories (Thm. 5.4). From this we get that every protoalgebraic deductive system with a fully adequate Gentzen system has the graded DD theorem (Cor. 5.5).

Every equivalential deductive system (Def. 3.9) with a fully adequate Gentzen system has a finite Leibniz-generating GDD system over all Leibniz filters (Cor. 5.9). Thus, for equivalential deductive systems, having a fully adequate Gentzen system is equivalent to the existence of a finite Leibniz-generating GDD system over all Leibniz filters. An example is given of a finitely equivalential deductive system that has a fully adequate Gentzen system and hence the graded DD theorem, but fails to have the multiterm DD theorem (Example 5.10).

Finally, if $\boldsymbol{\mathcal{S}}$ weakly algebraizable (Def. 5.6), in particular, if $\boldsymbol{\mathcal{S}}$ is (finitely) algebraizable, then $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system iff it has the multiterm DD theorem (Cor. 5.7).

The motivation for the proof of the main result of the paper is the fundamental isomorphism theorem [15, Theorem 2.30] (see also [16, Theorem 1.14]). It establishes for every deductive system $\boldsymbol{\mathcal{S}}$ a bijection between the full generalized models of $\boldsymbol{\mathcal{S}}$ over a fixed algebra $\boldsymbol{A}$ and the intersections of $\boldsymbol{\mathcal{S}}$-Leibniz congruences of $\boldsymbol{A}$. In the case $\boldsymbol{\mathcal{S}}$ is protoalgebraic with a fully adequate Gentzen system $\boldsymbol{\mathcal{G}}$, this gives rise to an order-preserving bijection between $\boldsymbol{\mathcal{G}}$-theories and Leibniz $\boldsymbol{\mathcal{S}}$-theories. This bijection turns out to commute with surjective substitutions and to preserve arbitrary joins (in the lattices of $\boldsymbol{\mathcal{G}}$-theories and $\boldsymbol{\mathcal{S}}$-theories). The method of [3, Theorem 3.7] can then be applied to show that there is a faithful interpretation of the consequence relation of $\boldsymbol{\mathcal{G}}$ in the consequence relation of $\boldsymbol{\mathcal{S}}$. The existence of this interpretation leads directly to the existence of a Leibniz-generating PGDD system for $\boldsymbol{\mathcal{S}}$ over Leibniz theories.

Section 2 is preliminary. Its primary purpose is to develop the part of the theory of generalized matrices that is needed to obtain the main results of the paper. The central idea of the paper is that the theory that underlies the interpretation of the equational logic of a quasivariety in the logic of a deductive system, and which in turn underlies the notion of algebraic semantics in [3], can be carried over more-or-less intact to the interpretation of the logic of a Gentzen system in that of a deductive system. For this purpose a theory of generalized matrices as models of Gentzen systems is needed. A large part of Section 2 is taken up in this task.

Section 3 is devoted to presenting facts about Leibniz theories in protoalgebraic systems that are needed for the main results of the paper. The equivalence between having a fully adequate Gentzen system and the existence of a Leibniz-generating parameterized GDD system for Leibniz theories is presented in Section 4. The refinements of this basic equivalence, including in particular the results on GDD systems without parameters, are given in Section 5.

2. Deductive Systems, Gentzen Systems, and their Models

Let A be a nonempty set. By a *string over* A we mean a nonempty finite sequence $\langle a_0, \dots, a_n \rangle$ of elements of A; the set of all strings over A is denoted by $A^{(\boldsymbol{\omega})}$. Strings will be written in the form $a_0, \dots, a_{n-1} \rhd a_n$. If $n = 0$ then $a_0, \dots, a_{n-1}$ is the empty sequence, and hence $a_0, \dots, a_{n-1} \rhd a_n$ is the one-element string $\langle a_0 \rangle$. We write this as $\rhd a_0$, and as usual we identify it with a_0. Sets of strings over A are called *generalized subsets* (*g-subsets*) of A. Every function from A to B induces a function from $A^{(\boldsymbol{\omega})}$ to $B^{(\boldsymbol{\omega})}$, which we denote by the same symbol. If $h\colon A \to B$ and $a_0, \dots, a_{n-1} \rhd a_n \in A^{(\boldsymbol{\omega})}$, then $h(a_0, \dots, a_{n-1} \rhd a_n) := h(a_0), \dots, h(a_{n-1}) \rhd h(a_n)$.

Definition 2.1. Let $\mathbf{R}$ be a g-subset of a set A.

(i) $\mathbf{R}$ is *reflexive* if $a \rhd a$ for all $a \in A$;

(ii) $\mathbf{R}$ is *transitive* if $a_0, \dots, a_{k-1} \rhd a_k \in \mathbf{R}$ and $b_0, \dots, b_{l-1} \rhd a_i \in \mathbf{R}$ for some $i < k$, then $a_0, \dots, a_{i-1}, b_0, \dots, b_{l-1}, a_{i+1}, \dots, a_{k-1} \rhd a_k \in \mathbf{R}$;

(iii) $\mathbf{R}$ is *standard* if it is both reflexive and transitive;

(iv) $\mathbf{R}$ is *structural* if $a_0, \dots, a_{k-1} \rhd a_k \in \mathbf{R}$, then $b_0, \dots, b_{l-1} \rhd a_k \in \mathbf{R}$ whenever $\{a_0, \dots, a_{k-1}\} \subseteq \{b_0, \dots, b_{l-1}\}$.

A standard, structural g-subset of A is called a *finite closure relation* on A.

The closure operator associated with a given finite closure relation $\mathbf{R}$ is denoted by $\mathrm{Clo}_{\mathbf{R}}$. Thus $\mathrm{Clo}_{\mathbf{R}} : \mathcal{P}(A) \to \mathcal{P}(A)$, and, for each $X \subseteq A$,

$$\mathrm{Clo}_{\mathbf{R}}\, X = \{\, a_n \in A : a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{R} \text{ for some } a_0, \dots, a_{n-1} \in X \,\}.$$

The finiteness of $\mathbf{R}$ is reflected in the fact that $\mathrm{Clo}_{\mathbf{R}}\, X = \bigcup\{\, \mathrm{Clo}_{\mathbf{R}}\, X' : X' \subseteq_{\boldsymbol{\omega}} X \,\}$. ($X' \subseteq_{\boldsymbol{\omega}} X$ means that X' is a finite subset of X.)

Let A be a nonempty set. By an *algebraic closed-set system* $\mathcal{C}$ on A we mean a family of subsets of A that contains A and is closed under arbitrary intersections and under the unions of subfamilies upward-directed by inclusion. If $\mathcal{C}$ is an algebraic closed-set system and $F \in \mathcal{C}$, we define

$$[\, F\,)_{\mathcal{C}} := \{\, G : F \subseteq G \in \mathcal{C} \,\}.$$

$[\, F\,)_{\mathcal{C}}$ is also an algebraic closed-set system, called the *principal subsystem* of $\mathcal{C}$ *generated* by F.

If $\mathcal{C}$ is an algebraic closed-set system on A, then

$$\mathbf{Fcr}\,\mathcal{C} := \Big\{\, a_0, \dots, a_{n-1} \rhd a_n \in A^{(\boldsymbol{\omega})} : \text{ for every } F \in \mathcal{C},\ a_0, \dots, a_{n-1} \in F \text{ implies } a_n \in F \,\Big\}$$

is a finite closure relation on A. Conversely, if $\mathbf{R}$ is a finite closure relation on A, then

$$\mathfrak{Css}\,\mathbf{R} := \{\, F \subseteq A : \text{ for every } a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{R},\ a_0, \dots, a_{n-1} \in F \text{ implies } a_n \in F \,\}$$

is an algebraic closed-set system. $\mathbf{Fcr}$ is a one-one correspondence between the algebraic closed-set systems and the finite closure relations on A, and $\mathfrak{Css}$ is its inverse. The sets of finite closure relations and algebraic closed-set systems on A are both closed under arbitrary intersections and unions of upward-directed sets and thus form algebraic lattices under set-theoretic inclusion. **Fcr** and $\mathfrak{Css}$ are inverse dual isomorphisms between these two lattices.

We use upright boldface Roman symbols for finite closure relations and calligraphic symbols for algebraic closed-set systems. For every algebraic closed-set system $\mathcal{C}$ we take $\mathrm{Clo}_{\mathcal{C}} := \mathrm{Clo}_{\mathbf{Fcr}\,\mathcal{C}}$.

By a *matrix* we mean a pair $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra over a fixed but arbitrary language type Λ, and F is a subset of the universe A of $\boldsymbol{A}$.

A *generalized matrix* (*g-matrix*) is a pair $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$, where $\boldsymbol{A}$ is a Λ-algebra and $\mathcal{C}$ is an algebraic closed-set system on A. The pair $\langle \boldsymbol{A}, \mathbf{Fcr}\,\mathcal{C}\rangle$ will be called the *closure-relation form* of $\boldsymbol{\mathcal{A}}$, and $\langle \boldsymbol{A}, \mathcal{C}\rangle$ will be referred to as the *closed-set form* of $\boldsymbol{\mathcal{A}}$.

A g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ in closed-set form is *closed under inverse endomorphisms* if, for each endomorphism $h\colon \boldsymbol{A} \to \boldsymbol{A}$ of the underlying algebra, $h^{-1}(\mathcal{C}) \subseteq \mathcal{C}$, i.e., $h^{-1}(F) \in \mathcal{C}$ for each $F \in \mathcal{C}$. A g-matrix $\langle \boldsymbol{A}, \mathbf{R}\rangle$ in closure-relation form is said to be *endomorphism-invariant* if, for each endomorphism h of $\boldsymbol{A}$, $h(\mathbf{R}) \subseteq \mathbf{R}$, i.e., $a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{R}$ implies $h(a_0), \dots, h(a_{n-1}) \rhd h(a_n) \in \boldsymbol{R}$. It is easy to see that a g-matrix in closed-set form is closed under inverse endomorphisms iff its closure-relation form is endomorphism-invariant.

The following easy lemma and its corollary will be useful to have.

Lemma 2.2. *Let $\mathcal{C}$ be an algebraic closed-set system on a nonempty set A and let $F \in \mathcal{C}$. Then $a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{Fcr}[\,F\,)_{\mathcal{C}}$ iff $a_n \in \mathrm{Clo}_{\mathcal{C}}\big(F \cup \{a_0, \dots, a_{n-1}\}\big)$.*

Proof. $a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{Fcr}[\,F\,)_{\mathcal{C}}$ iff $a_n \in \mathrm{Clo}_{[\,F\,)_{\mathcal{C}}}\big(\{a_0, \dots, a_{n-1}\}\big)$. But it is easy to see that, for any $X \subseteq A$, $\mathrm{Clo}_{[\,F\,)_{\mathcal{C}}}(X) = \mathrm{Clo}_{\mathcal{C}}(F \cup X)$. Thus $a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{Fcr}[\,F\,)_{\mathcal{C}}$ iff $a_n \in \mathrm{Clo}_{\mathcal{C}}\big(F \cup \{a_0, \dots, a_{n-1}\}\big)$. ■

Corollary 2.3. *Let $\mathcal{C}$ be an algebraic closed-set system on a nonempty set A and let $F \in \mathcal{C}$. Then $F = \{\, a \in A : \ \rhd a \in \mathbf{Fcr}[\,F\,)_{\mathcal{C}}\,\}$. In particular,* $\mathrm{Clo}_{\mathcal{C}}(\emptyset) = \bigcap \mathcal{C} = \{\, a \in A : \rhd a \in \mathbf{Fcr}\,\mathcal{C}\,\}$. ■

Let $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ and $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathcal{D}\rangle$ be g-matrices over Λ in closed-set form. $\boldsymbol{\mathcal{B}}$ is a *submatrix* of $\boldsymbol{\mathcal{A}}$, in symbols $\boldsymbol{\mathcal{B}} \subseteq \boldsymbol{\mathcal{A}}$, if $\boldsymbol{B} \subseteq \boldsymbol{A}$ (i.e., $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$) and $\mathcal{D} = \{\, F \cap B : F \in \mathcal{C}\,\}$. $\boldsymbol{\mathcal{B}}$ is a *weak submatrix* of $\boldsymbol{\mathcal{A}}$, in symbols $\boldsymbol{\mathcal{B}} \subseteq^{\mathrm{w}} \boldsymbol{\mathcal{A}}$, if $\boldsymbol{B} \subseteq \boldsymbol{A}$ and $\mathcal{D} \supseteq \{\, F \cap B : F \in \mathcal{C}\,\}$. A submatrix or weak submatrix $\boldsymbol{\mathcal{B}}$ of $\boldsymbol{\mathcal{A}}$ is *finitely generated* if $\boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A}$, i.e., $\boldsymbol{B}$ is a finitely generated subalgebra of $\boldsymbol{A}$. A homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a *matrix homomorphism* between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ if $h^{-1}(\mathcal{D}) \subseteq \mathcal{C}$, i.e., $h^{-1}(F) \in \mathcal{C}$ for every $F \in \mathcal{D}$. Note that $\boldsymbol{\mathcal{A}}$ is closed under inverse endomorphisms iff every endomorphism of the underlying algebra is a matrix endomorphism. A matrix homomorphism h is *strict* if $h^{-1}(\mathcal{D}) = \mathcal{C}$. If in addition h is surjective, we say that $\boldsymbol{\mathcal{B}}$ is a *strict homomorphic image* of $\boldsymbol{\mathcal{A}}$, in symbols $\boldsymbol{\mathcal{B}} \preccurlyeq \boldsymbol{\mathcal{A}}$ or $\boldsymbol{\mathcal{A}} \succcurlyeq \boldsymbol{\mathcal{B}}$. A strict bijective matrix homomorphism is a *matrix isomorphism*; in this case we write $\boldsymbol{\mathcal{A}} \cong \boldsymbol{\mathcal{B}}$.

Now let $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathbf{R} \rangle$ and $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathbf{S} \rangle$ be g-matrices over Λ in closure-relation form. $\boldsymbol{\mathcal{B}}$ is a *submatrix* of $\boldsymbol{\mathcal{A}}$, in symbols $\boldsymbol{\mathcal{B}} \subseteq \boldsymbol{\mathcal{A}}$, if $\boldsymbol{B} \subseteq \boldsymbol{A}$ and $\mathbf{S} = \mathbf{R} \cap B^{(\boldsymbol{\omega})}$. $\boldsymbol{\mathcal{B}}$ is a *weak submatrix* of $\boldsymbol{\mathcal{A}}$, in symbols $\boldsymbol{\mathcal{B}} \subseteq^{\mathrm{w}} \boldsymbol{\mathcal{A}}$, if $\boldsymbol{B} \subseteq \boldsymbol{A}$ and $\mathbf{S} \subseteq \mathbf{R}$. A homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a *matrix homomorphism* between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ if $h(\mathbf{R}) \subseteq \mathbf{S}$ (equivalently, $\mathbf{R} \subseteq h^{-1}(\mathbf{S})$) i.e., $h(a_0), \dots, h(a_{n-1}) \rhd h(a_n) \in \mathbf{S}$ for all $a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{R}$. h is *strict* if $h^{-1}(\mathbf{S}) = \mathbf{R}$, i.e., $h(a_0), \dots, h(a_{n-1}) \rhd h(a_n) \in \mathbf{S}$ iff $a_0, \dots, a_{n-1} \rhd a_n \in \mathbf{R}$.

For g-matrices $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ in closed-set form, $\boldsymbol{\mathcal{B}}$ is a (weak) submatrix of $\boldsymbol{\mathcal{A}}$ iff the closure-relation form of $\boldsymbol{\mathcal{B}}$ is a (weak) submatrix of the closure-relation form of $\boldsymbol{\mathcal{A}}$. Similarly, h is a (strict) homomorphism between $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ iff it is a (strict) homomorphism between the corresponding closure-relation forms.

2.1. Deductive systems

A (*first-order*) *deductive system* over the language type Λ is defined to be a g-matrix in closed-set form that is closed under inverse endomorphisms and whose underlying algebra is $\mathbf{Fm}_\Lambda$, the algebra of formulas over Λ. The closed sets of a deductive system $\boldsymbol{\mathcal{S}}$ are called the *theories* of $\boldsymbol{\mathcal{S}}$, and the set of all $\boldsymbol{\mathcal{S}}$-theories is denoted by $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$. Thus $\boldsymbol{\mathcal{S}} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\boldsymbol{\mathcal{S}} \rangle$.

Strings over the set Fm_Λ of formulas of type Λ are called *sequents*. The set of all sequents is $\mathrm{Fm}_\Lambda^{(\boldsymbol{\omega})}$. The sequent $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$ is sometimes written in the form

$$\frac{\varphi_0, \dots, \varphi_{n-1}}{\varphi_n}.$$

The closure-relation form $\mathbf{Fcr}\,\mathrm{Th}\,\boldsymbol{\mathcal{S}}$ of $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$ is called the (*finite*) *consequence relation* of $\boldsymbol{\mathcal{S}}$ and is denoted by $\vdash_{\boldsymbol{\mathcal{S}}}$. Thus $\boldsymbol{\mathcal{S}} = \langle \mathbf{Fm}_\Lambda, \vdash_{\boldsymbol{\mathcal{S}}} \rangle$ in closure-relation form. Following standard practice we write $\varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n$

instead of $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n \in \vdash_{\boldsymbol{\mathcal{S}}}$. The corresponding closure operator is called the *consequence operator* of $\boldsymbol{\mathcal{S}}$ and written $\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}$. Thus $\mathrm{Cn}_{\boldsymbol{\mathcal{S}}} = \mathrm{Clo}_{\vdash_{\boldsymbol{\mathcal{S}}}} = \mathrm{Clo}_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$. A formula φ is a *theorem* of $\boldsymbol{\mathcal{S}}$ if $\vdash_{\boldsymbol{\mathcal{S}}} \varphi$ ($\rhd \varphi \in \vdash_{\boldsymbol{\mathcal{S}}}$), or equivalently if $\varphi \in \bigcap \mathrm{Th}\,\boldsymbol{\mathcal{S}}$. The set of all theorems of $\boldsymbol{\mathcal{S}}$ is denoted by $\mathrm{Thm}\,\boldsymbol{\mathcal{S}}$.

Endomorphisms of the formula algebra are called *substitutions* and consequently deductive systems are said to be *closed under inverse substitutions* or, when in closure-relation form, *substitution-invariant*. Expressed in terms of the algebraic closed-set system of $\boldsymbol{\mathcal{S}}$-theories, closure under inverse substitutions takes the form

$$\sigma^{-1}(T) \in \mathrm{Th}\,\boldsymbol{\mathcal{S}}, \quad \text{for every } \sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda \text{ and every } T \in \mathrm{Th}\,\boldsymbol{\mathcal{S}}.$$

Expressed in terms of the consequence relation substitution-invariance takes the form

$$\varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n \quad \text{implies} \quad \sigma(\varphi_0), \dots, \sigma(\varphi_{n-1}) \vdash_{\boldsymbol{\mathcal{S}}} \sigma(\varphi_n), \quad \text{for every } \sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda.$$

Following standard use of the turnstyle symbol $\vdash_{\boldsymbol{\mathcal{S}}}$ we also use it to represent the associated closure operation. Thus, for any set Γ of formulas, possibly infinite, $\Gamma, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n$ will mean $\varphi_n \in \mathrm{Clo}_{\boldsymbol{\mathcal{S}}}\big(\Gamma \cup \{\varphi_0, \dots, \varphi_{n-1}\}\big)$. Also, $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \Delta$ means $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \varphi$ for all $\varphi \in \Delta$. As a particular case of Corollary 2.3 we have that, if $\boldsymbol{\mathcal{S}}$ is a deductive system and $T \in \mathrm{Th}\,\boldsymbol{\mathcal{S}}$, then $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n \in \mathbf{Fcr}[T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$ iff $T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n$.

A sequent $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$ is called a *Hilbert-style* (or *first-order*) *rule* of $\boldsymbol{\mathcal{S}}$, an $\boldsymbol{\mathcal{S}}$*-rule* for short, if it is contained in $\vdash_{\boldsymbol{\mathcal{S}}}$, i.e., if $\varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n$.

The sequent $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$ is *valid* in a matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, F \rangle$, and $\boldsymbol{\mathcal{A}}$ is a *model* of $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$, if, for every assignment $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$,

$$h(\varphi_0), \dots, h(\varphi_{n-1}) \in F \quad \text{implies} \quad h(\varphi_n) \in F.$$

$\boldsymbol{\mathcal{A}}$ is a *model* of a deductive system $\boldsymbol{\mathcal{S}}$ if it is a model of every rule of $\boldsymbol{\mathcal{S}}$. The class of all models of $\boldsymbol{\mathcal{S}}$ is denoted by $\mathsf{Mod}\,\boldsymbol{\mathcal{S}}$.

A subset F of the underlying set A of an algebra $\boldsymbol{A}$ is said to be a *filter* of $\boldsymbol{\mathcal{S}}$ ($\boldsymbol{\mathcal{S}}$*-filter*) if $\langle \boldsymbol{A}, F \rangle$ is a model of $\boldsymbol{\mathcal{S}}$. The set of all $\boldsymbol{\mathcal{S}}$-filters on $\boldsymbol{A}$ is denoted by $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$. Note that $\mathrm{Th}\,\boldsymbol{\mathcal{S}} = \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\mathbf{Fm}_\Lambda$, and like $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$, $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$ is an algebraic closed-set system. Thus $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} \rangle$ is a g-matrix.

Let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be an arbitrary homomorphism. If $F \subseteq A$ is an $\boldsymbol{\mathcal{S}}$-filter, then so is $h^{-1}(F)$; if $F \subseteq A$ is an $\boldsymbol{\mathcal{S}}$-filter that is compatible with the relation-kernel of h (in the sense defined just below), then $h(F)$ is an $\boldsymbol{\mathcal{S}}$-filter.

Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$. A congruence Θ on A is *compatible* with F if $a \in F$ and $a \equiv b \pmod{\Theta}$ imply $b \in F$ (i.e., F is a union of equivalence classes of Θ). The largest congruence compatible with F (it always exists) is called the *Leibniz congruence of* F and is denoted by $\boldsymbol{\Omega}_{\boldsymbol{A}} F$. The mapping $\boldsymbol{\Omega}_{\boldsymbol{A}}$ from subsets of A to congruences on $\boldsymbol{A}$ is called the *Leibniz operator.*

Theorem 2.4 ([5, Lemma 5.4]). *Let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be a surjective homomorphism. Then for every $F \subseteq A$, $\boldsymbol{\Omega}_{\boldsymbol{A}} h^{-1}(F) = h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$.* ■

Definition 2.5 ([5, Definition 7.1]). A deductive system $\boldsymbol{\mathcal{S}}$ is *protoalgebraic* if, for each algebra $\boldsymbol{A}$, the Leibniz operator restricted to $\boldsymbol{\mathcal{S}}$-filters is monotonic, i.e., for all $F, G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$, $F \subseteq G$ implies $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} G$.

It is not difficult to show that, if $\boldsymbol{\mathcal{S}}$ is protoalgebraic, then $\boldsymbol{\Omega}_{\boldsymbol{A}}(F \cap G) = \boldsymbol{\Omega}_{\boldsymbol{A}} F \cap \boldsymbol{\Omega}_{\boldsymbol{A}} G$ for all $F, G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$.

Protoalgebraic deductive systems were introduced in [2] using a different but equivalent defining condition.

The following theorem gives a characteristic property of protoalgebraic systems that will play a key role in the paper. It is one version of the so-called *correspondence theorem* for protoalgebraic logic (see [2, p. 352]).

Theorem 2.6. *Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras and $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ a surjective homomorphism. Then, for each $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$, the mapping $G \mapsto h^{-1}(G)$ is an isomorphism between the lattices $[\,F\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}}$ and $[\,h^{-1}(F)\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}$.* ■

For each language type Λ there is a unique protoalgebraic deductive system with no theorems, called the *almost inconsistent system*; it is presented by the single inference rule $\dfrac{x}{y}$ (and no axioms). The almost inconsistent system has exactly two theories, $\emptyset$ and Fm_{Λ}. In this paper we assume that all protoalgebraic deductive systems have at least one theorem.

2.2. Gentzen systems

The definition of a *Gentzen system*, or *second-order deductive system*, over the language type Λ is similar to that of a first-order deductive system except that we now take an algebraic closed-set system on the set $\mathrm{Fm}_\Lambda^{(\boldsymbol{\omega})}$ of sequents that is closed under inverse substitutions, rather than on the set of formulas. The members of the closed-set system that defines a Gentzen system $\boldsymbol{\mathcal{G}}$ are called *theories of* $\boldsymbol{\mathcal{G}}$ ($\boldsymbol{\mathcal{G}}$*-theories* for short), and the set of all $\boldsymbol{\mathcal{G}}$-theories is denoted by $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$. Thus $\boldsymbol{\mathcal{G}} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\boldsymbol{\mathcal{G}}\rangle$. The closure-relation form $\mathbf{Fcr}\,\mathrm{Th}\,\boldsymbol{\mathcal{G}}$ of $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$ is called the *consequence relation of* $\boldsymbol{\mathcal{G}}$ and is denoted by $\vdash_{\boldsymbol{\mathcal{G}}}$. Thus $\boldsymbol{\mathcal{G}} = \langle \mathbf{Fm}_\Lambda, \vdash_{\boldsymbol{\mathcal{G}}}\rangle$ in closure-relation form. The corresponding closure operator is called the *consequence operator of* $\boldsymbol{\mathcal{G}}$ and written $\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}$. Thus $\mathrm{Cn}_{\boldsymbol{\mathcal{G}}} = \mathrm{Clo}_{\mathrm{Th}\,\boldsymbol{\mathcal{G}}} = \mathrm{Clo}_{\vdash_{\boldsymbol{\mathcal{G}}}}$. The turnstyle symbol $\vdash_{\boldsymbol{\mathcal{G}}}$ is used to represent the closure operator of $\boldsymbol{\mathcal{G}}$, as in the case of deductive systems.

A sequent $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$ is a *theorem* of $\boldsymbol{\mathcal{G}}$ if $\vdash_{\boldsymbol{\mathcal{G}}} \varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n$ or equivalently if $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n \in \bigcap \mathrm{Th}\,\boldsymbol{\mathcal{G}}$. The set of all theorems of $\boldsymbol{\mathcal{G}}$ is denoted by $\mathrm{Thm}\,\boldsymbol{\mathcal{G}}$. The theorems of a Gentzen system $\boldsymbol{\mathcal{G}}$ are called *derivable sequents of* $\boldsymbol{\mathcal{G}}$ in the usual terminology of Gentzen calculi.

Expressed in terms of the consequence relation, the substitution-invariance of $\boldsymbol{\mathcal{G}}$ takes the form

$$\{\, \varphi^i_0, \dots, \varphi^i_{n_i-1} \rhd \varphi^i_{n_i} : i < k \,\} \vdash_{\boldsymbol{\mathcal{G}}} \psi_0, \dots, \psi_{m-1} \rhd \psi_m$$

implies

$$\{\, \sigma(\varphi^i_0), \dots, \sigma(\varphi^i_{n_i-1}) \rhd \sigma(\varphi^i_{n_i}) : i < k \,\} \vdash_{\boldsymbol{\mathcal{G}}} \sigma(\psi_0), \dots, \sigma(\psi_{m-1}) \rhd \sigma(\psi_m),$$

for every $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$.

The elements of $\vdash_{\boldsymbol{\mathcal{G}}}$ are *generalized sequents* (*g-sequents*) that is, sequents of sequents. G-sequents are traditionally called *Gentzen-style rules* and written in the form

$$\frac{\varphi^0_0, \dots, \varphi^0_{n_0-1} \rhd \varphi^0_{n_0}\ ; \dots\ ;\ \varphi^{k-1}_0, \dots, \varphi^{k-1}_{n_{k-1}-1} \rhd \varphi^{k-1}_{n_{k-1}}}{\psi_0, \dots, \psi_{m-1} \rhd \psi_m}. \tag{3}$$

In this paper the term "rule" is only used in connection with a particular deductive system or Gentzen system, and is reserved exclusively for those sequents or g-sequents that are contained in the consequence relation of the given system.

The sequents $\varphi^0_0, \dots, \varphi^0_{n_0-1} \rhd \varphi^0_{n_0}\ ; \dots\ ;\ \varphi^{k-1}_0, \dots, \varphi^{k-1}_{n_{k-1}-1} \rhd \varphi^{k-1}_{n_{k-1}}$ are called the *antecedents* of the g-sequent (3) and $\psi_0, \dots, \psi_{m-1} \rhd \psi_m$ is its *consequent.* A g-sequent is *proper* if it has at least one antecedent. The g-sequent (3) is called a *rule of* $\boldsymbol{\mathcal{G}}$, a $\boldsymbol{\mathcal{G}}$-rule, if it is contained in $\vdash_{\boldsymbol{\mathcal{G}}}$, i.e., if

$$\{\,\varphi^i_0,\dots,\varphi^i_{n_i-1} \rhd \varphi^i_{n_i} : i<k\,\} \vdash_{\boldsymbol{\mathcal{G}}} \psi_0,\dots,\psi_{m-1} \rhd \psi_m.$$

The following g-sequents are of special importance.

(axiom) $$\frac{}{x \rhd x};$$

(cut) $$\frac{x_0,\dots,x_{k-1} \rhd x_k\ ;\ \ y_0,\dots,y_{l-1} \rhd x_i}{x_0,\dots,x_{i-1},y_0,\dots,y_{l-1},x_{i+1},\dots,x_{k-1} \rhd x_k}, \quad i<k<\boldsymbol{\omega};$$

(structure)

$$\frac{x_0,\dots,x_{k-1} \rhd x_k}{y_0,\dots,y_{l-1} \rhd x_k}, \quad k,l<\boldsymbol{\omega},\ \{x_0,\dots,x_{k-1}\} \subseteq \{y_0,\dots,y_{l-1}\}.$$

The theories of $\boldsymbol{\mathcal{G}}$ are sets of sequents, i.e., g-subsets of Fm_Λ; as such they are standard (in the sense of Def. 2.1) just in case all the g-sequents of (axiom) and (cut) are $\boldsymbol{\mathcal{G}}$-rules, and they are structural just in case all of the g-sequents of (structure) are $\boldsymbol{\mathcal{G}}$-rules. In this case $\boldsymbol{\mathcal{G}}$ is said to be *structural.* Thus $\boldsymbol{\mathcal{G}}$ is structural and has (axiom) and (cut) as rules iff each $\boldsymbol{\mathcal{G}}$-theory is a finite closure relation on the set of formulas; in this case $\langle \mathbf{Fm}_\Lambda, \mathbf{T}\rangle$ is a g-matrix in closure-relation form for each $\boldsymbol{\mathcal{G}}$-theory $\mathbf{T}$. All the Gentzen systems we consider in this paper are automatically assumed to be structural and have (axiom) and (cut) as rules.

Since $\boldsymbol{\mathcal{G}}$-theories are finite closure relations they can be represented as algebraic closed-set systems by taking their closed-set form. In the sequel we will find it convenient to pass back-and-forth between these two concepts of a $\boldsymbol{\mathcal{G}}$-theory as we do with g-matrices.

The g-sequent (3) is *valid* in a g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathbf{R}\rangle$, in closure-relation form, and the g-matrix is a *model* of the g-sequent, if for every assignment $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$,

$$h(\varphi^i_0),\dots,h(\varphi^i_{n_i-1}) \rhd h(\varphi^i_{n_i}) \in \mathbf{R} \text{ for all } i<k$$
$$\text{implies} \quad h(\psi_0),\dots,h(\psi_{m-1}) \rhd h(\psi_m) \in \mathbf{R}.$$

A g-matrix is a *model* of a Gentzen system if it is a model of all its rules. The class of all models of a Gentzen system $\boldsymbol{\mathcal{G}}$ is denoted by $\mathsf{Mod}\,\boldsymbol{\mathcal{G}}$.

The proof of the following theorem is straightforward.

Theorem 2.7. *Let $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ be g-matrices.*

(i) *If $\boldsymbol{\mathcal{B}} \subseteq \boldsymbol{\mathcal{A}}$, then every g-sequent that is valid in $\boldsymbol{\mathcal{A}}$ is also valid in $\boldsymbol{\mathcal{B}}$.*

(ii) *If $\boldsymbol{\mathcal{B}} \preccurlyeq \boldsymbol{\mathcal{A}}$, then a g-sequent is valid in $\boldsymbol{\mathcal{B}}$ iff it is valid in $\boldsymbol{\mathcal{A}}$.* ■

The following technical lemma of a similar kind will also be useful in the sequel.

Let K be a family of g-matrices that is upward-directed under the weak-submatrix ordering $\subseteq^{\mathrm{w}}$. Assuming the members of K are in closure-relation form we define

$$\bigcup \mathsf{K} := \Big\langle \bigcup \{\, \boldsymbol{A} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}, \bigcup \{\mathbf{R} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}\Big\rangle.$$

where $\bigcup\{\, \boldsymbol{A} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}$ is the algebra whose universe is the union of the universes of the underlying algebras of the g-matrices of K and each of whose fundamental operations, when viewed as a relation, is the union of the corresponding fundamental operations of the underlying algebras. Clearly, $\bigcup \mathsf{K}$ is a g-matrix in closure-relation form.

Lemma 2.8. *Let K be any upward-directed family of g-matrices such that, for each $\boldsymbol{\mathcal{B}} \in \mathsf{K}$, there exists a $\boldsymbol{\mathcal{C}} \in \mathsf{K}$ such that $\boldsymbol{\mathcal{B}} \subseteq^{\mathrm{w}} \boldsymbol{\mathcal{C}}$ and the g-sequent* (3) *is valid in $\boldsymbol{\mathcal{C}}$. Then* (3) *is valid in $\bigcup \mathsf{K}$.*

Proof. Let $h\colon \mathbf{Fm}_\Lambda \to \bigcup\{\, \boldsymbol{A} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}$ be an assignment in $\bigcup\{\, \boldsymbol{A} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}$ such that $h(\varphi^i_0), \dots, h(\varphi^i_{n_i-1}) \rhd h(\varphi^i_{n_i}) \in \bigcup\{\, \mathbf{R} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}$ for all $i < k$. Let $\boldsymbol{\mathcal{B}} = \langle \boldsymbol{B}, \mathbf{S}\rangle \in \mathsf{K}$ be such that B contains the h-image of each variable occurring in the g-sequent (3), and $h(\varphi^i_0), \dots, h(\varphi^i_{n_i-1}) \rhd h(\varphi^i_{n_i}) \in \mathbf{S}$ for each $i < k$. By hypothesis there is a $\boldsymbol{\mathcal{C}} = \langle \boldsymbol{C}, \mathbf{T}\rangle \in \mathsf{K}$ such that $\boldsymbol{\mathcal{B}} \subseteq^{\mathrm{w}} \boldsymbol{\mathcal{C}}$ and (3) is valid in $\boldsymbol{\mathcal{C}}$. So $h(\psi_0), \dots, h(\psi_{m-1}) \rhd h(\psi_m) \in \mathbf{T} \subseteq \bigcup\{\, \mathbf{R} : \langle \boldsymbol{A}, \mathbf{R}\rangle \in \mathsf{K}\,\}$. Hence (3) is valid in $\bigcup \mathsf{K}$. ■

Let $\boldsymbol{\mathcal{G}} = \langle \mathbf{Fm}_\Lambda, \vdash_{\boldsymbol{\mathcal{G}}}\rangle$ be a Gentzen system ($\boldsymbol{\mathcal{G}}$ is structural and has (axiom) and (cut) as rules), and let $\boldsymbol{A}$ be an algebra. A finite closure relation $\mathbf{R}$ on A is a *filter of* $\boldsymbol{\mathcal{G}}$ (a $\boldsymbol{\mathcal{G}}$*-filter*) if $\langle \boldsymbol{A}, \mathbf{R}\rangle$ is a model of $\boldsymbol{\mathcal{G}}$. The set of all $\boldsymbol{\mathcal{G}}$-filters on $\boldsymbol{A}$ is denoted by $\mathrm{Fi}_{\boldsymbol{\mathcal{G}}}\,\boldsymbol{A}$. The $\boldsymbol{\mathcal{G}}$-filters on $\mathbf{Fm}_\Lambda$ are the $\boldsymbol{\mathcal{G}}$-theories. $\mathrm{Fi}_{\boldsymbol{\mathcal{G}}}\,\boldsymbol{A}$ is an algebraic closed-set system over the set $A^{(\omega)}$ of strings on A; thus it is closed under arbitrary intersection and the union of upward-directed sets.

Let $\boldsymbol{A}$ be an algebra and $\mathbf{R} \subseteq A^{(\boldsymbol{\omega})}$, a set of strings over A. A congruence Θ on $\boldsymbol{A}$ is *compatible* with $\mathbf{R}$ if $a_0, \ldots, a_{n-1} \rhd a_n \in \mathbf{R}$ and $a_i \equiv b_i \pmod{\Theta}$ for all $i \leq n$ implies $b_0, \ldots, b_{n-1} \rhd b_n \in \mathbf{R}$. The largest congruence on $\boldsymbol{A}$ compatible with $\mathbf{R}$ is called the *Tarski congruence* of $\mathbf{R}$ and is denoted by $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{R}$.

Lemma 2.9. *Let $\boldsymbol{A}$ be a Λ-algebra and $\mathbf{R}, \mathbf{S} \subseteq A^{(\boldsymbol{\omega})}$. If $\mathbf{R}$ and $\mathbf{S}$ are finite closure relations, then $\mathbf{R} \subseteq \mathbf{S}$ implies $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{R} \subseteq \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{S}$.*

Proof. Assume $\mathbf{R}$ and $\mathbf{S}$ are finite closure relations and $\mathbf{R} \subseteq \mathbf{S}$. Let Θ be a congruence compatible with $\mathbf{R}$. It suffices to show it is also compatible with $\mathbf{S}$. Suppose $a_0, \ldots, a_{n-1} \rhd a_n \in \mathbf{S}$ and $b_i \equiv a_i \pmod{\Theta}$ for all $i \leq n$. $a_0, \ldots, a_{n-1} \rhd a_i \in \mathbf{R}$ for all $i < n$ by reflexivity and structurality. So $b_0, \ldots, b_{n-1} \rhd a_i \in \mathbf{R} \subseteq \mathbf{S}$ for all $i < n$ by the assumption Θ is compatible with $\mathbf{R}$. Thus $b_0, \ldots, b_{n-1} \rhd a_n \in \mathbf{S}$ by transitivity. Then, since $a_n \rhd a_n \in \mathbf{R}$ by reflexivity, and $a_n \equiv b_n \pmod{\Theta}$, $a_n \rhd b_n \in \mathbf{R} \subseteq \mathbf{S}$. Transitivity now gives $b_0, \ldots, b_{n-1} \rhd b_n \in \mathbf{S}$. ■

If $\mathcal{C}$ is an algebraic closed-set system on the underlying set of an algebra $\boldsymbol{A}$, we define $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} = \widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{Fcr}\, \mathcal{C}$. It is not difficult to see that $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} = \bigcap_{F \in \mathcal{C}} \boldsymbol{\Omega}_{\boldsymbol{A}} F$.

A g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathbf{R} \rangle$, in closure-relation form, is *reduced* if $\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{R} = \mathrm{Id}_A$, the identity congruence on $\boldsymbol{A}$. For an arbitrary g-matrix $\boldsymbol{\mathcal{A}}$, the quotient matrix $\boldsymbol{\mathcal{A}}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{R} := \langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{R},\, \mathbf{R}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathbf{R} \rangle$ is reduced. It is called the *reduction* of $\boldsymbol{\mathcal{A}}$ and is denoted by $\boldsymbol{\mathcal{A}}^*$. (For any congruence Θ we define $\mathbf{R}/\Theta = \{ a_0/\Theta, \ldots, a_{n-1}/\Theta \rhd a_n/\Theta : a_0, \ldots, a_{n-1} \rhd a_n \in \mathbf{R} \}$.) $\boldsymbol{\mathcal{A}}^* \preccurlyeq \boldsymbol{\mathcal{A}}$, and hence $\boldsymbol{\mathcal{A}}^*$ is a model of exactly the same g-sequents as $\boldsymbol{\mathcal{A}}$. If $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ is in closed-set form, then $\boldsymbol{\mathcal{A}}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} = \langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}},\, \mathcal{C}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} \rangle$ where $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} = \{ F/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} : F \in \mathcal{C} \}$.

2.3. Generalized models of a deductive system

A Gentzen system $\boldsymbol{\mathcal{G}}$ is said to be *adequate* for a (first-order) deductive system $\boldsymbol{\mathcal{S}}$ if the theorems (derivable sequents) of $\boldsymbol{\mathcal{G}}$ coincide with the rules of $\boldsymbol{\mathcal{S}}$, i.e., $\vdash_{\boldsymbol{\mathcal{G}}} \varphi_0, \ldots, \varphi_{n-1} \rhd \varphi_n$ iff $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n$. There are in general many different Gentzen systems adequate for a given deductive system $\boldsymbol{\mathcal{S}}$. The weakest one is presented by the rules of $\boldsymbol{\mathcal{S}}$ taken as initial sequents (together with the g-sequents of (axiom), (cut), and (structure)). Since any proof in a Hilbert-style system can be replicated in a Gentzen system having (cut) as a rule, it is enough to take only a base for the rules $\boldsymbol{\mathcal{S}}$ for the initial sequents.

The *generalized models* (*g-models*) of $\boldsymbol{\mathcal{S}}$ are defined to be the models of the weakest Gentzen system adequate for $\boldsymbol{\mathcal{S}}$. It is easy to check that a g-matrix $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C} \rangle$ in closed-set system form is a g-model of $\boldsymbol{\mathcal{S}}$ iff $\mathcal{C} \subseteq \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$. It is clear that, if $\boldsymbol{\mathcal{A}} \preccurlyeq \boldsymbol{\mathcal{B}}$, then $\boldsymbol{\mathcal{A}}$ is a g-model of $\boldsymbol{\mathcal{S}}$ iff $\boldsymbol{\mathcal{B}}$ is.

A finite closure relation $\mathbf{R}$ on a Λ-algebra $\boldsymbol{A}$ is said to be a *generalized $\boldsymbol{\mathcal{S}}$-filter* (*$\boldsymbol{\mathcal{S}}$-g-filter*) of $\boldsymbol{A}$ if $\langle \boldsymbol{A}, \mathfrak{C}\mathrm{ss}\, \mathbf{R} \rangle$ is a g-model of $\boldsymbol{\mathcal{S}}$. The set of all $\boldsymbol{\mathcal{S}}$-g-filters is denoted by $\mathrm{GFi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$; it is clearly an algebraic closed-set system on $A^{(\boldsymbol{\omega})}$.

The following definition, in slightly modified form, is given in [15, Definition 2.8].

Definition 2.10. Let $\boldsymbol{\mathcal{S}}$ be a deductive system.

(i) G-models of the form $\langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A} \rangle$ are called *basic full g-models* of $\boldsymbol{\mathcal{S}}$ in closed-set form.

(ii) The *full g-models* of $\boldsymbol{\mathcal{S}}$ are the g-models whose reduction is a basic full g-model, i.e., $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a full g-model of $\boldsymbol{\mathcal{S}}$ if $\mathcal{C}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C} = \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}} \mathcal{C})$.

The class of all full g-models of $\boldsymbol{\mathcal{S}}$ is denoted by $\mathsf{FGMod}\, \boldsymbol{\mathcal{S}}$.

By the following theorem every basic full g-model is also a full g-model; this is not an immediate consequence of the definitions.

Theorem 2.11 ([15, Propositions 2.10 and 2.11]). *Let $\boldsymbol{\mathcal{S}}$ be a deductive system.*

(i) For every algebra $\boldsymbol{A}$, $(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A})/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}) = \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}_{\boldsymbol{A}}(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}))$, i.e., every basic full g-model of $\boldsymbol{\mathcal{S}}$ is a full g-model.

(ii) Let $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ be g-matrices. If $\boldsymbol{\mathcal{A}} \preccurlyeq \boldsymbol{\mathcal{B}}$, then $\boldsymbol{\mathcal{A}} \in \mathsf{FGMod}\, \boldsymbol{\mathcal{S}}$ iff $\boldsymbol{\mathcal{B}} \in \mathsf{FGMod}\, \boldsymbol{\mathcal{S}}$. ■

As an immediate corollary of this result and the definition of full g-model we have that $\boldsymbol{\mathcal{A}}$ is a full g-model of $\boldsymbol{\mathcal{S}}$ iff $\boldsymbol{\mathcal{B}} \preccurlyeq \boldsymbol{\mathcal{A}}$ for some basic full g-model $\boldsymbol{\mathcal{B}}$ of $\boldsymbol{\mathcal{S}}$. Consequently, every reduced full g-model of $\boldsymbol{\mathcal{S}}$ is basic.

A finite closure relation $\mathbf{R}$ on a Λ-algebra $\boldsymbol{A}$ is said to be a *full generalized $\boldsymbol{\mathcal{S}}$-filter* (*full $\boldsymbol{\mathcal{S}}$-g-filter*) of $\boldsymbol{A}$ if $\langle \boldsymbol{A}, \mathfrak{C}\mathrm{ss}\, \mathbf{R} \rangle$ is a full g-model of $\boldsymbol{\mathcal{S}}$. The set of all full $\boldsymbol{\mathcal{S}}$-g-filters is denoted by $\mathrm{FGFi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$. Note that

$\mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} \subseteq \mathrm{GFi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$. $\mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$ is not in general an algebraic closed-set system on $A^{(\boldsymbol{\omega})}$, but it is a complete lattice under set-theoretic inclusion ([15, Corollary 2.32]).

Let $\boldsymbol{A}$ be any algebra, and consider the closure-relation form $\langle \boldsymbol{A}, \mathbf{Fcr}(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A})\rangle$ of the basic full g-model of $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{A}$. $\mathbf{Fcr}(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A})$ is a full g-filter of $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{A}$. It is the smallest full $\boldsymbol{\mathcal{S}}$-g-filter and in fact the smallest $\boldsymbol{\mathcal{S}}$-g-filter on $\boldsymbol{A}$. This follows from the observation that, for every g-model $\langle \boldsymbol{A}, \mathcal{C}\rangle$ of $\boldsymbol{\mathcal{S}}$, $\mathcal{C} \subseteq \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$, and hence $\mathbf{Fcr}(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}) \subseteq \mathbf{Fcr}\,\mathcal{C}$. $\mathbf{Fcr}(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A})$ is called the *basic g-filter* of $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{A}$ and is denoted by $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$. We note for future reference that $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} = \bigcap \mathrm{GFi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$, and that $\langle \boldsymbol{A}, \mathbf{R}\rangle$ is a full g-model of $\boldsymbol{\mathcal{S}}$ in closure-relation form iff $\mathbf{R} = h^{-1}(\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B})$ for some surjective homomorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$.

The following technical lemma and its corollary will be useful later.

Lemma 2.12. *Let $\boldsymbol{\mathcal{S}}$ be a deductive system and $\boldsymbol{A}$ an algebra. Let*

$$\mathsf{K} = \{\, \langle \boldsymbol{B}, \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B}\rangle : \boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A} \,\}.$$

Then K is upward-directed and $\bigcup \mathsf{K} = \langle \boldsymbol{A}, \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}\rangle$, the (unique) basic full g-model of $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{A}$.

Proof. Clearly $\{\, \boldsymbol{B} : \boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A} \,\}$ is an upward-directed family of algebras. If $\boldsymbol{B}, \boldsymbol{C} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A}$ and $\boldsymbol{B} \subseteq \boldsymbol{C}$, then $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B} \subseteq \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{C}$ by the minimality of $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B}$ as an $\boldsymbol{\mathcal{S}}$-g-filter. So $\{\, \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B} : \boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A} \,\}$ is upward-directed, and hence K is upward-directed. Since it is obvious that $\boldsymbol{A} = \bigcup\{\, \boldsymbol{B} : \boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A} \,\}$, to show that $\bigcup \mathsf{K}$ is the basic full g-model of $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{A}$ it suffices to show that

$$\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} = \bigcup\{\, \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B} : \boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A} \,\}.$$

The inclusion from left-to-right follows immediately from the minimality of $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$ as an $\boldsymbol{\mathcal{S}}$-g-filter because it is clear that the union of all the $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B}$ is an $\boldsymbol{\mathcal{S}}$-g-filter. For the opposite inclusion we simply note that for each $\boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A}$, $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B} \subseteq \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} \cap B^{(\boldsymbol{\omega})}$ by the minimality of $\mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B}$. ■

Theorem 2.13 ([15, Theorem 3.4]). *Let $\boldsymbol{\mathcal{S}}$ be a deductive system. $\boldsymbol{\mathcal{S}}$ is protoalgebraic iff, for every algebra $\boldsymbol{A}$,* $\mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} \subseteq \big\{\, \mathbf{Fcr}\big([\,F\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big) : F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} \,\big\}$. ■

The full g-models of both the classical and intutionistic propositional calculi are definable in the sense that they are exactly the models of a

Gentzen system. The following definition identifies this important property of deductive systems.

Definition 2.14 ([15, Definition 4.10]). Let $\boldsymbol{\mathcal{S}}$ be a deductive system with at least one theorem. A Gentzen system $\boldsymbol{\mathcal{G}}$ is said to be *fully adequate* for $\boldsymbol{\mathcal{S}}$ if $\mathsf{Mod}\,\boldsymbol{\mathcal{G}} = \mathsf{FGMod}\,\boldsymbol{\mathcal{S}}$.

This definition must be modified slightly in case $\boldsymbol{\mathcal{S}}$ has no theorems. We will be concerned exclusively with protoalgebraic systems that are not almost inconsistent, and all such systems have at least one theorem. Since a Gentzen system, like a deductive system, is completely determined by its models, if a fully adequate Gentzen system exists it is unique.

Expressed in terms of filters, $\boldsymbol{\mathcal{G}}$ is fully adequate for $\boldsymbol{\mathcal{S}}$ if $\mathrm{Fi}_{\boldsymbol{\mathcal{G}}}\,\boldsymbol{A} = \mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$ for every algebra $\boldsymbol{A}$. The main goal of the present paper is to find useful conditions that guarantee the existence of a fully adequate Gentzen system.

3. Leibniz Filters

In a protoalgebraic deductive system $\boldsymbol{\mathcal{S}}$ every full g-filter of $\boldsymbol{\mathcal{S}}$ is a principal order-filter in the lattice $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$ of all $\boldsymbol{\mathcal{S}}$-filters (Thm. 2.13). But not every $\boldsymbol{\mathcal{S}}$-filter generates a full $\boldsymbol{\mathcal{S}}$-g-filter. In this section we characterize those that do as Leibniz filters and develop some of their properties. The close correspondence between Leibniz filters and full g-filters in the protoalgebraic case is the key to the proof of the main results of the paper that are given in the next section.

Definition 3.1 ([14]). Let $\boldsymbol{\mathcal{S}}$ be a deductive system and $\boldsymbol{A}$ an algebra. An $\boldsymbol{\mathcal{S}}$-filter F on $\boldsymbol{A}$ is a *Leibniz $\boldsymbol{\mathcal{S}}$-filter* if it is the smallest $\boldsymbol{\mathcal{S}}$-filter among all those with the same Leibniz congruence, i.e., $F = \bigcap\{\, G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A} : \boldsymbol{\Omega}_{\boldsymbol{A}}\, G = \boldsymbol{\Omega}_{\boldsymbol{A}}\, F \,\}$. The set of all Leibniz $\boldsymbol{\mathcal{S}}$-filters on $\boldsymbol{A}$ is denoted by $\mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$. The set of all *Leibniz $\boldsymbol{\mathcal{S}}$-theories* (Leibniz $\boldsymbol{\mathcal{S}}$-filters on $\mathbf{Fm}_{\Lambda}$) is denoted by $\mathrm{Th}^{+}\,\boldsymbol{\mathcal{S}}$.

Let $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$. If the set of all filters with the same Leibniz congruence as F has a smallest member G, i.e., if there exists a $G \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}}\, G = \boldsymbol{\Omega}_{\boldsymbol{A}}\, F$, then it is obviously unique and we denote it by F^{+}.

Lemma 3.2 ([14, Theorem 8]). *Let $\boldsymbol{\mathcal{S}}$ be any deductive system. Let $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ be a surjective homomorphism. Then $h^{-1}(\mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{B}) \subseteq \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}$, i.e.,*

$h^{-1}(F) \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ *for every* $F \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$. *In particular, for every* $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$ *such that* F^{+} *exists,* $h^{-1}(F^{+}) = h^{-1}(F)^{+}$.

Proof. Consider any $F \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$ and let $G = h^{-1}(F)$. G is an $\boldsymbol{\mathcal{S}}$-filter on $\boldsymbol{A}$. By Thm. 2.4 $\boldsymbol{\Omega}_{\boldsymbol{A}} G = h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$. Suppose $H \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ and $\boldsymbol{\Omega}_{\boldsymbol{A}} H = \boldsymbol{\Omega}_{\boldsymbol{A}} G$ $(= h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F))$. Then $h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$, and hence also the relation kernel of h, are compatible with H. We thus have $h(H) \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$ since h is surjective. $h^{-1}\big(\boldsymbol{\Omega}_{\boldsymbol{B}} h(H)\big) = \boldsymbol{\Omega}_{\boldsymbol{A}} h^{-1}h(H) = \boldsymbol{\Omega}_{\boldsymbol{A}} H = \boldsymbol{\Omega}_{\boldsymbol{A}} G = h^{-1}(\boldsymbol{\Omega}_{\boldsymbol{B}} F)$. Thus $\boldsymbol{\Omega}_{\boldsymbol{B}} h(H) = \boldsymbol{\Omega}_{\boldsymbol{B}} F$, and hence $h(H) \supseteq F$ since F is Leibniz. Then, because H is compatible with the relation kernel, $H = h^{-1}h(H) \supseteq h^{-1}(F) = G$. So $h^{-1}(F) = G \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$.

Consider any $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$ such that F^{+} exists. Then $h^{-1}(F^{+}) \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ for each $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{B}$. Using Thm. 2.4 we have

$$\boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F^{+})\big) = h^{-1}\big(\boldsymbol{\Omega}_{\boldsymbol{B}}(F^{+})\big) = h^{-1}\big(\boldsymbol{\Omega}_{\boldsymbol{B}}(F)\big) = \boldsymbol{\Omega}_{\boldsymbol{A}}\big(h^{-1}(F)\big).$$

Since $h^{-1}(F^{+})$ is Leibniz we have $h^{-1}(F^{+}) = h^{-1}(F)^{+}$. ■

Lemma 3.3. *If* $\boldsymbol{\mathcal{S}}$ *is protoalgebraic, then* F^{+} *exists for every algebra* $\boldsymbol{A}$ *and every* $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$. *In fact,* $F^{+} = \bigcap\{\, G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A} : \boldsymbol{\Omega}_{\boldsymbol{A}} G = \boldsymbol{\Omega}_{\boldsymbol{A}} F \,\}$.

Proof. Let $H = \bigcap\{\, G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A} : \boldsymbol{\Omega}_{\boldsymbol{A}} G = \boldsymbol{\Omega}_{\boldsymbol{A}} F \,\}$. $H \subseteq F$, so $\boldsymbol{\Omega}_{\boldsymbol{A}} H \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} F$ by protoalgebraicity. On the other hand, $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is compatible with each $G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}} G = \boldsymbol{\Omega}_{\boldsymbol{A}} F$. Hence it is compatible with their intersection H. So $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} H$ since $\boldsymbol{\Omega}_{\boldsymbol{A}} H$ is the largest congruence compatible with F. ■

If $\boldsymbol{\mathcal{S}}$ is protoalgebraic, then F^{+} exists for every $\boldsymbol{\mathcal{S}}$-filter F, and hence ${}^{+}\colon \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A} \to \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ it a total function for every algebra $\boldsymbol{A}$.

Lemma 3.4. *Let* $\boldsymbol{\mathcal{S}}$ *be a protoalgebraic deductive system.* ${}^{+}\colon \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A} \to \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ *is monotonic for every algebra* $\boldsymbol{A}$*, i.e., for every pair* F, G *of* $\boldsymbol{\mathcal{S}}$*-filters on* $\boldsymbol{A}$*,* $F \subseteq G$ *implies* $F^{+} \subseteq G^{+}$.

Proof. If $F \subseteq G$, then, since $\boldsymbol{\mathcal{S}}$ is protoalgebraic, $\boldsymbol{\Omega}_{\boldsymbol{A}}(F \cap G^{+}) = \boldsymbol{\Omega}_{\boldsymbol{A}} F \cap \boldsymbol{\Omega}_{\boldsymbol{A}} G^{+} = \boldsymbol{\Omega}_{\boldsymbol{A}} F \cap \boldsymbol{\Omega}_{\boldsymbol{A}} G = \boldsymbol{\Omega}_{\boldsymbol{A}} F$. Therefore, $F^{+} \subseteq F \cap G^{+} \subseteq G^{+}$. ■

The most significant aspect of Leibniz filters for our purposes is their close connection with full g-models.

Theorem 3.5 ([15, Proposition 3.6]). *Assume* $\boldsymbol{\mathcal{S}}$ *is a protoalgebraic deductive system. Then, for every algebra* $\boldsymbol{A}$ *and every* $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$, $\mathbf{Fcr}\big([\,F\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big) \in \mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$ *iff* F *is a Leibniz filter.* ■

Corollary 3.6. *Assume* $\boldsymbol{\mathcal{S}}$ *is a protoalgebraic deductive system. Then, for every algebra* $\boldsymbol{A}$, $\mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A} = \big\{\, \mathbf{Fcr}\big([\,F\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big) : F \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A} \,\big\}$*. In particular* $\mathrm{FGFi}_{\boldsymbol{\mathcal{S}}}\mathbf{Fm}_{\Lambda} = \big\{\, \mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big) : T \in \mathrm{Th}^{+}\boldsymbol{\mathcal{S}} \,\big\}$.

Proof. By Thms. 2.13 and 3.5. ■

Thus a Gentzen system $\boldsymbol{\mathcal{G}}$ is fully adequate for a protoalgebraic deductive system $\boldsymbol{\mathcal{S}}$ iff

$$\mathrm{Fi}_{\boldsymbol{\mathcal{G}}}\boldsymbol{A} = \big\{\, \mathbf{Fcr}\big([\,F\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big) : F \in \mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A} \,\big\} \quad \text{for every algebra } \boldsymbol{A}.$$

Lemma 3.7 ([14, Proposition 13]). *If* $\boldsymbol{\mathcal{S}}$ *is protoalgebraic, then* $\mathrm{Fi}^{+}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$ *is closed under* $\bigvee^{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}$*. In particular,* $\mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$ *is closed under* $\bigvee^{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$.

Proof. Let $F = \bigvee_{i}^{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}} G_i$. By Lem. 3.4 $G_i \subseteq F^{+}$ for all $i \in I$. Thus $F \subseteq F^{+}$. So F is Leibniz. ■

In the following, $\bar{u}$ denotes a normally infinite sequence $u_0, u_1, u_2, \dots$, without repetitions, of variables different from x and y. Let

$$E(x, y, \bar{u}) = \{\, \varepsilon_i(x, y, \bar{u}) : i \in I \,\}$$

be a possibly infinite system of formulas over Λ in two variables, x and y, and an arbitrary number of variables from the list $\bar{u}$; the latter variables are called *parameters*. Of course, each individual formula $\varepsilon_i(x, y, \bar{u})$ actually contains only a finite number of parameters, but the set of parameters that occur in at least one of the members of $E(x, y, \bar{u})$ may be infinite and normally is. For any algebra $\boldsymbol{A}$ and all $a, b \in A$, let $\forall\bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c})$ stand for the set of all elements $h\big(\varepsilon_i(x, y, \bar{u})\big)$ in $\boldsymbol{A}$, where i ranges over all of I and h ranges over all homomorphisms $h\colon \mathbf{Fm}_{\Lambda} \to \boldsymbol{A}$ such that $h(x) = a$ and $h(y) = b$; i.e., $\forall\bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) := \big\{\, \varepsilon_i^{\boldsymbol{A}}(a, b, \bar{c}) : i \in I, \bar{c} \in A^{\boldsymbol{\omega}} \,\big\}$. In particular, taking $\boldsymbol{A} = \mathbf{Fm}_{\Lambda}$ we have for all $\varphi, \psi \in \mathrm{Fm}_{\Lambda}$,

$$\forall\bar{\vartheta}\, E(\varphi, \psi, \bar{\vartheta}) := \big\{\, \varepsilon_i(\varphi, \psi, \bar{\vartheta}) : i \in I, \bar{\vartheta} \in \mathrm{Fm}_{\Lambda}^{\boldsymbol{\omega}} \,\big\}.$$

$E(x, y, \bar{u})$ is said to be an *equivalence system with parameters* for a deductive system $\boldsymbol{\mathcal{S}}$ if it defines the Leibniz congruences of $\boldsymbol{\mathcal{S}}$ in the following sense. For every algebra $\boldsymbol{A}$ and every $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$,

$$\boldsymbol{\Omega}_{\boldsymbol{A}} F = \big\{\, \langle a, b\rangle \in A^2 : \ \forall\bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) \subseteq F \,\big\}.$$

Theorem 3.8 ([5, Theorem 3.10]). *A deductive system is protoalgebraic iff it has an equivalence system with parameters.* ∎

The equivalential deductive systems form a natural subclass of the class of protoalgebraic systems. They were introduced in [20] and studied extensively in [6]. The defining condition we use in the following definition is known to be equivalent to the original one (see [5, Theorem 13.5]).

Definition 3.9. A deductive system is (*finitely*) *equivalential* if it has a (finite) equivalence system without parameters.

Lemma 3.10 ([10, Lemma 3.4]). *Assume $\boldsymbol{\mathcal{S}}$ is protoalgebraic and $E(x, y, \bar{u})$ is an equivalence system with parameters for $\boldsymbol{\mathcal{S}}$. Let $\boldsymbol{A}$ be an algebra and $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$. Then*

$$F^+ = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup\{\, \forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) : a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}\,\}\big).$$

Proof. Let

$$\widetilde{F} = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup\{\, \forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) : a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}\,\}\big).$$

Consider any $G \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ such that $\boldsymbol{\Omega}_{\boldsymbol{A}} G = \boldsymbol{\Omega}_{\boldsymbol{A}} F$. Then, for all $a, b \in A$, $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$ iff $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} G}$ iff $\forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) \subseteq G$. So $\widetilde{F} \subseteq F^+$, and hence $\boldsymbol{\Omega}_{\boldsymbol{A}} \widetilde{F} \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} F^+$. On the other hand, $\forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) \subseteq \widetilde{F}$ for all a, b such that $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$. This gives $\boldsymbol{\Omega}_{\boldsymbol{A}} F^+ = \boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} \widetilde{F}$. So $\boldsymbol{\Omega}_{\boldsymbol{A}} \widetilde{F} = \boldsymbol{\Omega}_{\boldsymbol{A}} F^+$ and hence $F^+ = \widetilde{F}$. ∎

Corollary 3.11. *Assume $\boldsymbol{\mathcal{S}}$ is protoalgebraic. Let $\boldsymbol{A}$ be an algebra and $F \in \mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$. Then $F \in \mathrm{Fi}^+_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ iff there exists a $X \subseteq A^2$ such that*

$$F = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup\{\, \forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) : \langle a, b\rangle \in X\,\}\big).$$

Proof. The implication from left to right is a trivial consequence of the lemma taking $X = \boldsymbol{\Omega}_{\boldsymbol{A}} F$.

For the implication in the other direction suppose

$$F = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup\{\, \forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) : \langle a, b\rangle \in X\,\}\big)$$

for some $X \subseteq A^2$. Then $X \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} F$. So

$$F \subseteq \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup\{\, \forall \bar{c}\, E^{\boldsymbol{A}}(a, b, \bar{c}) : a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} F}\,\}\big) = F^+$$

by the lemma. Thus $F = F^+$. ∎

We note that Lemma 3.7 is an easy corollary of Lem. 3.10 and Cor. 3.11. To see this assume $\{ G_i : i \in I \} \subseteq \mathrm{Fi}^+_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$, so that

$$G_i = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup\{ \forall \bar{c}\, E^{\boldsymbol{A}}(a,b,\bar{c}) : a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}}\, G_i} \}\big)$$

for each $i \in I$. Then

$$\bigvee_i^{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}} G_i = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}}\big(\bigcup_i \bigcup\{ \forall \bar{c}\, E^{\boldsymbol{A}}(a,b,\bar{c}) : a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}}\, G_i} \}\big).$$

Hence $\bigvee_i^{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}} G_i \in \mathrm{Fi}^+_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$ by Cor. 3.11.

The main purpose of the above characterization of Leibniz filters is to show that the set of Leibniz theories is invariant under surjective substitutions in the following technical sense. For any $\boldsymbol{\mathcal{S}}$-theory T and any substitution $\sigma : \mathbf{Fm}_\Lambda \to \mathbf{Fm}_\Lambda$, let

$$\sigma_{\boldsymbol{\mathcal{S}}}(T) := \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\sigma(T)\big) = \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\{ \sigma(\varphi) : \varphi \in T \}\big).$$

Lemma 3.12. *Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system. If $T \in \mathrm{Th}^+ \boldsymbol{\mathcal{S}}$ and $\sigma : \mathbf{Fm}_\Lambda \twoheadrightarrow \mathbf{Fm}_\Lambda$ is a surjective substitution, then $\sigma_{\boldsymbol{\mathcal{S}}}(T) \in \mathrm{Th}^+ \boldsymbol{\mathcal{S}}$.*

Proof. We have $T = \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\bigcup\{ \forall \bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta}) : \langle \varphi,\psi \rangle \in \boldsymbol{\Omega}\, T \}\big)$ by Lem. 3.10, taking $\mathbf{Fm}_\Lambda$ for $\boldsymbol{A}$ and $\mathrm{Th}\, \boldsymbol{\mathcal{S}}$ for $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}} \boldsymbol{A}$. By the substitution-invariance of $\boldsymbol{\mathcal{S}}$, $\sigma(T) \subseteq \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\bigcup\{\sigma(\forall \bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta})) : \langle \varphi,\psi \rangle \in \boldsymbol{\Omega}\, T \}\big)$. So

$$\sigma_{\boldsymbol{\mathcal{S}}}(T) \subseteq \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\bigcup\{\sigma(\forall \bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta})) : \langle \varphi,\psi \rangle \in \boldsymbol{\Omega}\, T \}\big).$$

The inclusion in the opposite direction is obvious. Since σ is surjective, $\sigma\big(\forall \bar{\vartheta}\, E(\varphi,\psi,\bar{\vartheta})\big) = \forall \bar{\vartheta}\, E(\sigma(\varphi),\sigma(\psi),\bar{\vartheta})$. So $\sigma_{\boldsymbol{\mathcal{S}}}(T) \in \mathrm{Th}^+ \boldsymbol{\mathcal{S}}$ by Cor. 3.11. ■

4. The Graded Deduction-Detachment Theorem with Parameters

Let $y, x_0, x_1, \dots, x_k, \dots$ and $\bar{u} = u_0, u_1, \dots, u_k, \dots$ $(k < \boldsymbol{\omega})$ be two disjoint infinite sequences of variables without repetitions. Let

$$\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle$$

be an infinite sequence of sets of formulas, where for each $n < \boldsymbol{\omega}$, $\Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ is a nonempty, possibly infinite, set of formulas in the $n + 1$ variables $x_0, \dots, x_{n-1}, y$ and a possibly infinite number of parameters from $\bar{u}$.

Definition 4.1. Let $\boldsymbol{\mathcal{S}}$ be a deductive system and let T be an $\boldsymbol{\mathcal{S}}$-theory. $\boldsymbol{\Delta}$ is said to be a *parameterized graded deduction-detachment* (PGDD) *system for* $\boldsymbol{\mathcal{S}}$ *over* T if, for every $n < \boldsymbol{\omega}$ and for all $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$,

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}).$$

The implication from $T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi$ to $T \vdash_{\boldsymbol{\mathcal{S}}} \forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta})$ will be referred to as the *graded deduction property* (*over* T) and the implication in the other direction as the *graded detachment property* (*over* T).

For each $n < \boldsymbol{\omega}$, consider the following (generally infinite) family of g-sequents.

$$\frac{x_0, \dots, x_{n-1} \rhd y}{\rhd\, \delta(x_0, \dots, x_{n-1}, y, \bar{u})} \qquad (\mathrm{R}_{\Delta_n})$$

for each $\delta(x_0, \dots, x_{n-1}, y, \bar{u}) \in \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$.

This entire family of g-sequents is abbreviated $\dfrac{x_0, \dots, x_{n-1} \rhd y}{\rhd\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})}$.

Lemma 4.2. *Let* $\boldsymbol{\Delta}$ *be a* PGDD *system for a deductive system* $\boldsymbol{\mathcal{S}}$ *over an* $\boldsymbol{\mathcal{S}}$*-theory* T. *Then every g-sequent in* (R_{Δ_n}) *is valid in* $\langle \mathbf{Fm}_\Lambda, [\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}} \rangle$.

Proof. Suppose $\varphi_0, \dots, \varphi_{n-1} \rhd \psi \in \mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$. Then by Lem. 2.2 $\psi \in \mathrm{Clo}_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big(T \cup \{\varphi_0, \dots, \varphi_{n-1}\}\big)$, i.e., $T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi$. Thus, by the graded deduction property, $\forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}) \subseteq T$, that is, $\delta(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}) \in T$, and hence

$$\rhd\, \delta(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}) \in \mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$$

for all $\delta(x_0, \dots, x_{n-1}, y, \bar{u}) \in \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ and all $\bar{\vartheta} \in \mathrm{Fm}_\Lambda^{\boldsymbol{\omega}}$. ■

Lemma 4.3. *Let* $\boldsymbol{\mathcal{S}}$ *be a protoalgebraic deductive system and assume that* $\boldsymbol{\Delta}$ *is a* PGDD *system for* $\boldsymbol{\mathcal{S}}$ *over every Leibniz* $\boldsymbol{\mathcal{S}}$*-theory. Then every full g-model of* $\boldsymbol{\mathcal{S}}$ *is a model of the g-sequents of* (R_{Δ_n}) *for all* $n < \boldsymbol{\omega}$.

Proof. In view of Thm. 2.7(ii), to prove every full g-model of $\boldsymbol{\mathcal{S}}$ is a model of (R_{Δ_n}) it suffices to prove that every basic full g-model is a model of (R_{Δ_n}). Let $\boldsymbol{\mathcal{A}}$ be a basic full g-model of $\boldsymbol{\mathcal{S}}$. Then, by Lem. 2.12, $\boldsymbol{\mathcal{A}} =$

$\bigcup\{\langle \boldsymbol{B}, \mathbf{Bf}_{\boldsymbol{\mathcal{S}}}\boldsymbol{B}\rangle : \boldsymbol{B} \subseteq_{\boldsymbol{\omega}} \boldsymbol{A}\}$. Consequently, by Lem. 2.8, to show that (R_{Δ_n}) is valid in $\boldsymbol{\mathcal{A}}$ it suffices to show that it is valid in every finitely generated weak submatrix of $\boldsymbol{\mathcal{A}}$ that is a basic full g-model of $\boldsymbol{\mathcal{S}}$. So we can assume that $\boldsymbol{\mathcal{A}}$ itself is a finitely generated basic full g-model of $\boldsymbol{\mathcal{S}}$.

It is convenient in this proof to work with the closed-set form of $\boldsymbol{\mathcal{A}}$, so we assume that $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}\rangle$. Let $h\colon \mathbf{Fm}_\Lambda \twoheadrightarrow \boldsymbol{A}$ be a surjective homomorphism. Let $F_0 = \bigcap \mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$, the smallest $\boldsymbol{\mathcal{S}}$-filter on $\boldsymbol{A}$, and let $T_0 = h^{-1}(F_0)$. By the correspondence theorem for protoalgebraic systems (Thm. 2.6) we have $h^{-1}\big(\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}\big) = [\,T_0\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$. Since F_0 is obviously Leibniz, we have by Lem. 3.2 that $T_0 \in \mathrm{Th}^+\boldsymbol{\mathcal{S}}$. h is a strict surjective homomorphism from $\langle \mathbf{Fm}_\Lambda, [\,T_0\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\rangle$ onto $\boldsymbol{\mathcal{A}}$, i.e., $\langle \mathbf{Fm}_\Lambda, [\,T_0\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\rangle \succcurlyeq \boldsymbol{\mathcal{A}}$. So from Thm. 2.7(ii) and Lem. 4.2 it follows that, for every $n < \boldsymbol{\omega}$, each of the g-sequents of (R_{Δ_n}) is valid in $\boldsymbol{\mathcal{A}}$. ■

Although we will not use the fact here, it can be shown that, if every full g-model of a deductive system $\boldsymbol{\mathcal{S}}$ is a model of the g-sequents of (R_{Δ_n}) for all $n < \boldsymbol{\omega}$, then $\boldsymbol{\mathcal{S}}$ has the graded deduction property over every Leibniz theory.

Definition 4.4. Let $\boldsymbol{\mathcal{S}}$ be a deductive system. A set $\Delta(x_0, \dots, x_n, \bar{u})$ of formulas is said to be *Leibniz-generating over* $\boldsymbol{\mathcal{S}}$ if the $\boldsymbol{\mathcal{S}}$-theory generated by the set of all parameter-substitution instances of $\Delta(x_0, \dots, x_n, \bar{u})$, i.e., the set

$$\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\forall\bar{\vartheta}\,\Delta(x_0, \dots, x_n, \bar{\vartheta})\big),$$

is a Leibniz $\boldsymbol{\mathcal{S}}$-theory.

Lemma 4.5. *Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system and assume that $\Delta(x_0, \dots, x_n, \bar{u})$ is Leibniz-generating over $\boldsymbol{\mathcal{S}}$. Then for every algebra $\boldsymbol{A}$ and all $a_0, \dots, a_n \in A$, the set $\forall\bar{c}\,\Delta^{\boldsymbol{A}}(a_0, \dots, a_n, \bar{c})$ generates a Leibniz $\boldsymbol{\mathcal{S}}$-filter of $\boldsymbol{A}$.*

Proof. Since $\Delta(x_0, \dots, x_n, \bar{u})$ is Leibniz-generating, by Cor. 3.11 there is a $\Xi \subseteq \mathrm{Fm}^2_\Lambda$ such that

$$\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\forall\bar{\vartheta}\,\Delta(x_0, \dots, x_n, \bar{\vartheta})\big) = \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\bigcup\{\forall\bar{\vartheta}\,E(\varphi, \psi, \bar{\vartheta}) : \langle\varphi, \psi\rangle \in \Xi\}\big), \quad (4)$$

where $E(x, y, \bar{u})$ is an equivalence system with parameters for $\boldsymbol{\mathcal{S}}$. Let

$$X = \bigcup\{h(\Xi) : h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A} \text{ and } h(x_i) = a_i \text{ for all } i \le n\}.$$

We will prove that

$$\mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big(\forall\bar{c}\,\Delta^{\boldsymbol{A}}(a_0,\dots,a_n,\bar{c})\big) = \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\Big(\bigcup\{\,\forall\bar{c}\,E^{\boldsymbol{A}}(f,g,\bar{c}) : \langle f,g\rangle \in X\,\}\Big), \tag{5}$$

from which it follows by Cor. 3.11 that $\forall\bar{c}\,\Delta^{\boldsymbol{A}}(a_0,\dots,a_n,\bar{c})$ generates a Leibniz $\boldsymbol{\mathcal{S}}$-filter.

To prove the inclusion from left to right, let

$$d \in \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big(\forall\bar{c}\,\Delta^{\boldsymbol{A}}(a_0,\dots,a_n,\bar{c})\big).$$

Since $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$ is an algebraic closed-set system there is an $m < \boldsymbol{\omega}$ and, for each $i < m$, $\delta_i(x_0,\dots,x_n,u_0,\dots,u_{k-1}) \in \Delta$ and $c^i_0,\dots,c^i_{k-1} \in A$ such that

$$d \in \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big(\{\,\delta_i^{\boldsymbol{A}}(a_0,\dots,a_n,c^i_0,\dots,c^i_{k-1}) : i < m\,\}\big).$$

By (4) there is a finite subset Γ of $\bigcup\{\,\forall\bar{\vartheta}\,E(\varphi,\psi,\bar{\vartheta}) : \langle\varphi,\psi\rangle \in \Xi\,\}\big)$ such that, for every $i < m$, $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \delta_i(x_0,\dots,x_n,u_0,\dots,u_{k-1})$. It follows immediately that, for each $i < m$,

$$\delta_i^{\boldsymbol{A}}(a_0,\dots,a_n,c^i_0,\dots,c^i_{k-1}) \in \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\Big(\bigcup\{\,\forall\bar{c}\,E^{\boldsymbol{A}}(f,g,\bar{c}) : \langle f,g\rangle \in X\,\}\Big).$$

To prove the other inclusion of (5) let $d \in \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big(\bigcup\{\,\forall\bar{c}\,E^{\boldsymbol{A}}(f,g,\bar{c}) : \langle f,g\rangle \in X\,\}\big)$. As before, since $\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}$ is algebraic, there is an $m < \boldsymbol{\omega}$ and for each $i < m$ there are $\varepsilon_i(x,y,u_0,\dots,u_{k-1}) \in E(x,y,\bar{u})$, $\langle f_i,g_i\rangle \in X$, and $c^i_0,\dots,c^i_{k-1} \in A$ such that

$$d \in \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big(\{\,\varepsilon_i^{\boldsymbol{A}}(f_i,g_i,c^i_0,\dots,c^i_{k-1}) : i < m\,\}).$$

Let us fix $i < n$. Since $\langle f_i,g_i\rangle \in X$, there is a homomorphism $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$ and $\langle\varphi,\psi\rangle \in \Xi$ such that $h(x_0) = a_0,\dots,h(x_n) = a_n$ and $h(\varphi) = f_i$ and $h(\psi) = g_i$. We can assume without loss of generality that the variables $u_0,\dots,u_{k-1}$ do not occur in either φ or ψ, and moreover that $h(u_0) = c^i_0,\dots,h(u_{k-1}) = c^i_{k-1}$. Now, since by (4) $\forall\bar{\vartheta}\,\Delta(x_0,\dots,x_n,\bar{\vartheta}) \vdash_{\boldsymbol{\mathcal{S}}} \varepsilon_i(\varphi_i,\psi_i,u_0,\dots,u_{k-1})$, after applying h we obtain

$$\varepsilon_i^{\boldsymbol{A}}(f_i,g_i,c^i_0,\dots,c^i_{k-1}) \in \mathrm{Clo}_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\boldsymbol{A}}\big(\forall\bar{c}\,\Delta^{\boldsymbol{A}}(a_0,\dots,a_n,\bar{c})\big).$$

■

Note that in particular, if $\Delta(x_0, \dots, x_n, \bar{u})$ is Leibniz-generating, then, for all $\varphi_0, \dots, \varphi_n \in \mathrm{Fm}_\Lambda$, the set $\Delta(\varphi_0, \dots, \varphi_n, \bar{u})$ is also Leibniz-generating

We will be chiefly interested in PGDD systems over all Leibniz theories that are also Leibniz-generating. A PGDD system

$$\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle$$

is *Leibniz-generating* if $\Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ is Leibniz-generating for each $n < \boldsymbol{\omega}$. These systems have the following *graded modus ponens property.*

Lemma 4.6. *Let $\boldsymbol{\mathcal{S}}$ be a deductive system and assume*

$$\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle$$

is a Leibniz-generating PGDD *system for $\boldsymbol{\mathcal{S}}$ over every Leibniz theory. Then, for every $n < \boldsymbol{\omega}$ and all $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$,*

$$\forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}),\ \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi.$$

Proof. Let T be the $\boldsymbol{\mathcal{S}}$-theory generated by $\forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta})$. T is a Leibniz theory since $\boldsymbol{\Delta}$ is Leibniz-generating. Thus

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}).$$

But $T \vdash_{\boldsymbol{\mathcal{S}}} \forall\bar{\vartheta}\, \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta})$ holds trivially. ■

A set $\Delta(y, \bar{u})$ of formulas in one variable with parameters is said to *explicitly define Leibniz filters* over a deductive system $\boldsymbol{\mathcal{S}}$ if, for every algebra $\boldsymbol{A}$ and every $\boldsymbol{\mathcal{S}}$-filter F of $\boldsymbol{A}$, $F^+ = \{\, b \in A : \forall\bar{c}\, \Delta^{\boldsymbol{A}}(b, \bar{c}) \subseteq F \,\}$.

Theorem 4.7. *Assume $\boldsymbol{\mathcal{S}}$ is protoalgebraic and*

$$\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle$$

is a Leibniz-generating PGDD *system for $\boldsymbol{\mathcal{S}}$ over every Leibniz theory. Then the initial set $\Delta_0(y, \bar{u})$ of $\boldsymbol{\Delta}$ explicitly defines Leibniz filters over $\boldsymbol{\mathcal{S}}$.*

Proof. Assume $\forall\bar{c}\, \Delta_0^{\boldsymbol{A}}(a, \bar{c}) \subseteq F$. Let G be the $\boldsymbol{\mathcal{S}}$-filter of $\boldsymbol{A}$ generated by the set $\forall\bar{c}\, \Delta_0^{\boldsymbol{A}}(a, \bar{c})$. By Lem. 4.5 it is a Leibniz $\boldsymbol{\mathcal{S}}$-filter, and by Lem. 3.4 $G \subseteq F$ implies $G \subseteq F^+$. So $\forall\bar{c}\, \Delta_0^{\boldsymbol{A}}(a, \bar{c}) \subseteq F^+$, and hence by the graded modus ponens property (Lem. 4.6) $a \in F^+$.

Assume conversely that $a \in F^+$, i.e., $\rhd\ a \in \mathbf{Fcr}\big([\,F^+\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}}\big)$. By Thm. 3.5, the g-matrix $\big\langle \boldsymbol{A}, \mathbf{Fcr}\big([\,F^+\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}}\big\rangle$ is a full g-model of $\boldsymbol{\mathcal{S}}$, and hence by Lem. 4.3 it satisfies the g-sequents of (R_{Δ_0}). Therefore, for every sequence $\bar{c}$ of elements of $\boldsymbol{A}$, $\forall\bar{c}\,\Delta_0^{\boldsymbol{A}}(a,\bar{c}) \subseteq F^+$, that is, $\forall\bar{c}\,\Delta_0^{\boldsymbol{A}}(a,\bar{c}) \subseteq F^+ \subseteq F$. ■

Theorem 4.8. *Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system.*[1] *If $\boldsymbol{\mathcal{S}}$ has a Leibniz-generating* PGDD *system over all Leibniz theories, then $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system. More precisely, if $\boldsymbol{\Delta}$ is a Leibniz-generating* PGDD *system of $\boldsymbol{\mathcal{S}}$ over every Leibniz theory, then the fully adequate Gentzen system for $\boldsymbol{\mathcal{S}}$ has a presentation whose initial sequents are* (axiom) *and the rules of $\boldsymbol{\mathcal{S}}$, and whose proper rules are* (cut), (structure)*, and the g-sequents of* (R_{Δ_n}) *for $n < \boldsymbol{\omega}$.*

Proof. We prove that, under the hypothesis of the theorem, the Gentzen system presented by the initial sequents (axiom) and the rules of $\boldsymbol{\mathcal{S}}$, and the proper rules (cut), (structure), and the g-sequents of (R_{Δ_n}) for $n < \boldsymbol{\omega}$, is fully adequate for $\boldsymbol{\mathcal{S}}$. The g-matrices that are models of the rules of $\boldsymbol{\mathcal{S}}$ (considered as g-sequents with an empty set of antecedents) are all g-models of $\boldsymbol{\mathcal{S}}$. Hence, from Lem. 4.3 it follows that every full g-model of $\boldsymbol{\mathcal{S}}$ is a model of this Gentzen system.

Now let $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A}, \mathcal{C}\rangle$ be a g-model of $\boldsymbol{\mathcal{S}}$. We have to show that, if $\boldsymbol{\mathcal{A}}$ is a model of (R_{Δ_n}) for a $n < \boldsymbol{\omega}$, then $\boldsymbol{\mathcal{A}}$ is a full g-model of $\boldsymbol{\mathcal{S}}$. For the purposes of obtaining a contradiction, assume that the contrary holds, i.e., that $\boldsymbol{\mathcal{A}}$ is a g-model of $\boldsymbol{\mathcal{S}}$ that is also a model of (R_{Δ_n}) for all $n < \boldsymbol{\omega}$, but $\boldsymbol{\mathcal{A}}$ is not a full g-model of $\boldsymbol{\mathcal{S}}$. Let F_0 be the smallest $\boldsymbol{\mathcal{S}}$-filter in $\mathcal{C}$.Since $\boldsymbol{\mathcal{S}}$ is protoalgebraic and $\boldsymbol{\mathcal{A}}$ is not a full g-model of $\boldsymbol{\mathcal{S}}$, this implies that $\mathcal{C}$ is properly included in $[\,F_0^+\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}}$ by Thm. 3.5. Let $G \in [\,F_0^+\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}} \setminus \mathcal{C}$, and set $\bar{G} = \mathrm{Clo}_{\mathcal{C}}\, G$. Then $\bar{G} \in [\,F_0^+\,)_{\mathrm{Fi}_{\boldsymbol{\mathcal{S}}}\,\boldsymbol{A}}$ and $F_0^+ \subseteq G \subset \bar{G}$. Let $b \in \bar{G} \setminus G$. Then there is a finite subset $\{a_0, \dots, a_{n-1}\}$ of G such that $b \in \mathrm{Clo}_{\mathcal{C}}\{a_0, \dots, a_{n-1}\}$, i.e., $a_0, \dots, a_{n-1} \rhd b \in \mathbf{Fcr}\,\mathcal{C}$. Thus, since the g-sequents of (R_{Δ_n}) are valid in $\boldsymbol{\mathcal{A}}$, $\forall\bar{c}\,\Delta_n^{\boldsymbol{A}}(a_0, \dots, a_{n-1}, b, \bar{c}) \subseteq F_0$. We now use the assumption that $\boldsymbol{\Delta}$ is a Leibniz-generating PGDD system over all Leibniz theories to conclude by Lem. 4.5 that $\forall\bar{c}\,\Delta_n^{\boldsymbol{A}}(a_0, \dots, a_{n-1}, b, \bar{c})$ generates a Leibniz $\boldsymbol{\mathcal{S}}$-filter, and thus, by Lem. 3.4, that $\forall\bar{c}\,\Delta_n^{\boldsymbol{A}}(a_0, \dots, a_{n-1}, b, \bar{c}) \subseteq F_0^+$. Now by

[1]This theorem was originally proved under the assumption that the language type of $\boldsymbol{\mathcal{S}}$ is countable. The authors want to thank an anonymous referee for pointing out that this restriction can be eliminated.

the graded modus ponens property (Lem. 4.6) we get $b \in G$, and this is a contradiction. ■

We are now ready to state and prove the main result of the paper, Theorem 4.10, the converse of Thm. 4.8. Together the two theorems completely characterize the protoalgebraic deductive systems with a fully adequate Gentzen system as exactly those with a Leibniz-generating PGDD system over every Leibniz theory.

The proof emulates the proof of Theorem 3.7(i) in [3]. A quasivariety Q is said, in [3], to be an *algebraic semantics* (in contrast to being an equivalent algebraic semantics) for a deductive system $\mathcal{S}$ if there is a faithful interpretation of the consequence relation $\vdash_{\mathcal{S}}$ of $\mathcal{S}$ in the equational consequence relation $\models_{\mathsf{Q}}$ of Q (see [3, Definition 2.2] for details). A pseudo-lattice-theoretic criterion for the existence of such a faithful interpretation is established in Theorem 3.7 of [3]. There it is proved that a faithful interpretation of $\vdash_{\mathcal{S}}$ in $\models_{\mathsf{Q}}$ exists iff there is an isomorphism between the lattice of Q-theories and a complete join subsemilattice of the theory lattice of $\mathcal{S}$ that commutes with substitutions. In proving Theorem 4.10 we use the methods of the proof of [3, Theorem 3.7] to show that, under the assumption that a deductive system $\mathcal{S}$ has a fully adequate Gentzen system $\mathcal{G}$, there is a similar faithful interpretation of $\vdash_{\mathcal{G}}$ in $\vdash_{\mathcal{S}}$. We then show that this faithful interpretation leads directly to the existence of a Leibniz-generating PGDD system for $\mathcal{S}$ over every Leibniz theory.

Let $\mathcal{G}$ be a Gentzen system. For each surjective substitution $\sigma : \mathbf{Fm}_\Lambda \twoheadrightarrow \mathbf{Fm}_\Lambda$ we define a transformation $\sigma_{\mathcal{G}} : \mathrm{Th}\,\mathcal{G} \to \mathrm{Th}\,\mathcal{G}$ between $\mathcal{G}$-theories in complete analogy to the way the transformation $\sigma_{\mathcal{S}}$ between theories of the deductive system $\mathcal{S}$ was defined in Section 3. For each $\mathbf{T} \in \mathrm{Th}\,\mathcal{G}$, in closure-relation form, let

$$\sigma_{\mathcal{G}}(\mathbf{T}) := \mathrm{Cn}_{\mathcal{G}}\big(\sigma(\mathbf{T})\big),$$

where

$$\sigma(\mathbf{T}) = \big\{\, \sigma(\varphi_0), \dots, \sigma(\varphi_{n-1}) \rhd \sigma(\varphi_n) : \varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n \in \mathbf{T} \,\big\}.$$

Assume the protoalgebraic deductive system $\mathcal{S}$ has a fully adequate Gentzen system. Then by Cor. 3.6 there exists a Gentzen system $\mathcal{G}$ whose g-theories are exactly the $\mathcal{S}$-g-filters of $\mathbf{Fm}_\Lambda$ that have the closed-set form

$[\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$, with $T \in \mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$, i.e.,

$$\mathrm{Th}\,\boldsymbol{\mathcal{G}} = \{\,\mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big) : T \in \mathrm{Th}^{+}\boldsymbol{\mathcal{S}}\,\}.$$

Let $\varPhi$: $\mathrm{Th}\,\boldsymbol{\mathcal{G}} \to \mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$ be the bijection defined by $\varPhi\big(\mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)\big) = T$ for all $T \in \mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$.

Lemma 4.9. *Let $\boldsymbol{\mathcal{G}}$ be a fully adequate Gentzen system for the protoalgebraic deductive system $\boldsymbol{\mathcal{S}}$. Then the mapping $\varPhi$ defined above is an isomorphism between the lattice* $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$ *and the join-complete subsemilattice* $\mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$ *of* $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$.

Moreover, $\varPhi$ commutes with surjective substitutions in the following sense. If $\sigma : \mathbf{Fm}_{\Lambda} \twoheadrightarrow \mathbf{Fm}_{\Lambda}$ is a surjective substitution and $\mathbf{T} \in \mathrm{Th}\,\boldsymbol{\mathcal{G}}$, then

$$\varPhi(\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T})) = \sigma_{\boldsymbol{\mathcal{S}}}(\varPhi(\mathbf{T})). \tag{6}$$

Proof. As we already observed, $\varPhi$ is a bijection between $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$ and $\mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$. For every $\mathbf{T} \in \mathrm{Th}\,\boldsymbol{\mathcal{G}}$, $\mathcal{C}\mathrm{ss}\,\mathbf{T} = [\,\varPhi(\mathbf{T})\,)_{\mathbf{Th}\,\boldsymbol{\mathcal{S}}}$. Thus $\mathbf{T} = \mathbf{Fcr}\,\mathcal{C}\mathrm{ss}\,\mathbf{T} = \mathbf{Fcr}\big([\,\varPhi(\mathbf{T})\,)_{\mathbf{Th}\,\boldsymbol{\mathcal{S}}}\big)$. It is now clear that, for all $\mathbf{T}, \mathbf{T}' \in \mathrm{Th}\,\boldsymbol{\mathcal{G}}$, $\mathbf{T} \subseteq \mathbf{T}'$ iff $\varPhi(\mathbf{T}) \subseteq \varPhi(\mathbf{T}')$, and hence $\varPhi$ is an order-isomorphism between $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$ and $\mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$, which by Lem. 3.7 is a join-complete subsemilattice of $\mathrm{Th}\,\boldsymbol{\mathcal{S}}$.

To prove the second part of the lemma, let $\mathbf{T} \in \mathrm{Th}\,\boldsymbol{\mathcal{G}}$ and put $T = \varPhi(\mathbf{T}) \in \mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$, so that $\mathbf{T} = \mathbf{Fcr}\big([\,T\,)_{\mathbf{Th}\,\boldsymbol{\mathcal{S}}}\big)$. Now $\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T}) \in \mathrm{Th}\,\boldsymbol{\mathcal{G}}$. Thus, if we put $S = \varPhi(\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T})) \in \mathrm{Th}^{+}\boldsymbol{\mathcal{S}}$, then $\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T}) = \mathbf{Fcr}\big([\,S\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$, and hence $\varPhi(\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T})) = \varPhi\big(\mathbf{Fcr}\big([\,S\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)\big) = S$. So to show that $\varPhi(\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T})) = \sigma_{\boldsymbol{\mathcal{S}}}(\varPhi(\mathbf{T}))$ it is necessary and sufficient to show that $S = \sigma_{\boldsymbol{\mathcal{S}}}(T)$.

If $\varphi \in T$, then $\rhd\, \varphi \in \mathbf{T}$, so $\rhd\, \sigma(\varphi) \in \sigma(\mathbf{T}) \subseteq \sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T})$, that is, $\sigma(\varphi) \in S$. This shows that $\sigma(T) \subseteq S$, and hence that $\sigma_{\boldsymbol{\mathcal{S}}}(T) \subseteq S$ because S is an $\boldsymbol{\mathcal{S}}$-theory.

To show the opposite inclusion, i.e., $\sigma_{\boldsymbol{\mathcal{S}}}(T) \supseteq S$, we will actually show $[\,\sigma_{\boldsymbol{\mathcal{S}}}(T)\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}} \subseteq [\,S\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$, and it is easier to work with the corresponding closure-relation forms and show that

$$\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T}) = \mathbf{Fcr}\big([\,S\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big) \subseteq \mathbf{Fcr}\big([\,\sigma_{\boldsymbol{\mathcal{S}}}(T)\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big).$$

But $T \in \mathrm{Th}^{+}_{\boldsymbol{\mathcal{S}}}$ implies $\sigma_{\boldsymbol{\mathcal{S}}}(T) \in \mathrm{Th}^{+}_{\boldsymbol{\mathcal{S}}}$ by Lem. 3.12, and thus $\mathbf{Fcr}\big([\,\sigma_{\boldsymbol{\mathcal{S}}}(T)\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}$ is a $\boldsymbol{\mathcal{G}}$-theory, so it is enough to show that $\sigma(\mathbf{T}) \subseteq \mathbf{Fcr}\big([\,\sigma_{\boldsymbol{\mathcal{S}}}(T)\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$. Let $\varphi_0, \dots, \varphi_{n-1} \rhd \varphi_n \in \mathbf{T} = \mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$. By

Lem. 2.2, this means that $T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \varphi_n$, which by the substitution-invariance of $\boldsymbol{\mathcal{S}}$ gives $\sigma(T), \sigma(\varphi_0), \dots, \sigma(\varphi_{n-1}) \vdash_{\boldsymbol{\mathcal{S}}} \sigma(\varphi_n)$, and thus, a fortiori,

$$\sigma_{\boldsymbol{\mathcal{S}}}(T), \sigma(\varphi_0), \dots, \sigma(\varphi_{n-1}) \vdash_{\boldsymbol{\mathcal{S}}} \sigma(\varphi_n).$$

From this we get $\sigma(\varphi_0), \dots, \sigma(\varphi_{n-1}) \rhd \sigma(\varphi_n) \in \mathbf{Fcr}\big([\,\sigma_{\boldsymbol{\mathcal{S}}}(T)\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$, thus showing that $\sigma(\mathbf{T}) \subseteq \mathbf{Fcr}\big([\,\sigma_{\boldsymbol{\mathcal{S}}}(T)\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$, as required. ■

Theorem 4.10. *Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system. If $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system, then $\boldsymbol{\mathcal{S}}$ has a Leibniz-generating* PGDD *system over every Leibniz theory.*

Proof. Let $\boldsymbol{\mathcal{G}}$ be a Gentzen system fully adequate for $\boldsymbol{\mathcal{S}}$, and let $\varPhi$ be the mapping considered in Lemma 4.9. For each $n < \boldsymbol{\omega}$ let

$$\mathbf{T}_n := \mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\{x_0, \dots, x_{n-1} \rhd y\}\big),$$

where $x_0, \dots, x_{n-1}, y$ are distinct variables. Let $\varDelta_n(x_0, \dots, x_{n-1}, y, \bar{u}) \subseteq \mathrm{Fm}_\varLambda$ be a set of generators of the $\boldsymbol{\mathcal{S}}$-theory $\varPhi(\mathbf{T}_n)$, where $\bar{u}$ is a sequence without repetitions of all variables distinct from $x_0, \dots, x_{n-1}, y$. Set

$$\boldsymbol{\varDelta} := \langle\, \varDelta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle.$$

Note that $\varPhi(\mathbf{T}_n)$ is a Leibniz $\boldsymbol{\mathcal{S}}$-theory, and hence $\varDelta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ generates a Leibniz theory. We verify that $\boldsymbol{\varDelta}$ is a Leibniz-generating PGDD system for $\boldsymbol{\mathcal{S}}$ over every Leibniz $\boldsymbol{\mathcal{S}}$-theory.

Let T be a Leibniz $\boldsymbol{\mathcal{S}}$-theory, $n < \boldsymbol{\omega}$, and $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\varLambda$. We must show that

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \forall\bar{\vartheta}\, \varDelta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}), \tag{7}$$

and also that

$$\forall\bar{\vartheta}\, \varDelta_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}) \text{ generates a Leibniz theory.} \tag{8}$$

Let $\varSigma$ be the set of all surjective substitutions σ such that

$$\sigma(x_0) = \varphi_0, \dots \sigma(x_{n-1}) = \varphi_{n-1}, \sigma(y) = \psi.$$

Then, for each $\sigma \in \varSigma$,

$$
\begin{aligned}
&\varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{\varphi_0,\dots,\varphi_{n-1}\rhd\psi\}\big)\\
&\qquad= \varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{\sigma(x_0,\dots,x_{n-1}\rhd y)\}\big)\\
&\qquad= \varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\sigma(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1}\rhd y\})\big)\big),\\
&\qquad\qquad\qquad\qquad\text{by substitution-invariance of } \boldsymbol{\mathcal{G}}\\
&\qquad= \varPhi\big(\sigma_{\boldsymbol{\mathcal{G}}}(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1}\rhd y\})\big),\\
&\qquad\qquad\qquad\qquad\text{by definition of } \sigma_{\boldsymbol{\mathcal{G}}}\\
&\qquad= \sigma_{\boldsymbol{\mathcal{S}}}\big(\varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1}\rhd y\}\big)\big),\\
&\qquad\qquad\qquad\qquad\text{by Lem. 4.9}\\
&\qquad= \sigma_{\boldsymbol{\mathcal{S}}}\,\varPhi(\mathbf{T}_n)\\
&= \sigma_{\boldsymbol{\mathcal{S}}}\big(\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})\big)\big),\\
&\qquad\qquad\qquad\text{by definition of } \varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})\\
&= \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\sigma(\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u}))\big),\\
&\qquad\qquad\qquad\text{by substitution-invariance of } \boldsymbol{\mathcal{S}}.
\end{aligned}
$$

Thus we have

$$\varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{\varphi_0,\dots,\varphi_{n-1}\rhd\psi\}\big) = \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\sigma(\bar{u}))\big). \tag{9}$$

Recall that T is a Leibniz $\boldsymbol{\mathcal{S}}$-theory. Let $\mathbf{T} := \mathbf{Fcr}\big([\,T\,)_{\mathrm{Th}\,\boldsymbol{\mathcal{S}}}\big)$. $\mathbf{T}$ is a $\boldsymbol{\mathcal{G}}$-theory since T is Leibniz. Then

$$
\begin{aligned}
T,\varphi_0,\dots,\varphi_{n-1}\vdash_{\boldsymbol{\mathcal{S}}}\psi \quad&\text{iff}\quad \varphi_0,\dots,\varphi_{n-1}\rhd\psi\in\mathbf{T}\\
&\text{iff}\quad \mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{\varphi_0,\dots,\varphi_{n-1}\rhd\psi\}\subseteq\mathbf{T}\\
&\text{iff}\quad \varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{\varphi_0,\dots,\varphi_{n-1}\rhd\psi\}\big)\subseteq\varPhi(\mathbf{T})=T\\
&\text{iff}\quad \varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\sigma(\bar{u}))\subseteq T,\quad\text{by (9)}.
\end{aligned}
$$

Thus, for every $\sigma\in\varSigma$,

$$T,\varphi_0,\dots,\varphi_{n-1}\vdash_{\boldsymbol{\mathcal{S}}}\psi \quad\text{iff}\quad T\vdash_{\boldsymbol{\mathcal{S}}}\varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\sigma(\bar{u})).$$

Consider any $\delta(x_0,\dots,x_{n-1},y,\bar{u})\in\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})$ and any $\bar{\vartheta}\in\mathrm{Fm}^{\boldsymbol{\omega}}_{\varLambda}$. Since δ contains only a finite number of variables in the list $\bar{u}$, there is a $\sigma\in\varSigma$ such that $\delta(\varphi_0,\dots,\varphi_{n-1},\psi,\bar{\vartheta})=\delta(\varphi_0,\dots,\varphi_{n-1},\psi,\sigma(\bar{u}))$. Thus

$$\forall\bar{\vartheta}\,\varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\bar{\vartheta}) = \bigcup\big\{\,\varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\sigma(\bar{u})) : \sigma\in\varSigma\,\big\}.$$

This gives (7). Moreover, we have

$$\begin{aligned}
&\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\forall\bar\vartheta\, \varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\bar\vartheta)\big)\\
&\quad= \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\bigcup\{\, \varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\sigma(\bar u)) : \sigma\in\varSigma\,\}\big)\\
&\quad= \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\bigcup\{\, \sigma(\varDelta_n(x_0,\dots,x_{n-1},y,\bar u)) : \sigma\in\varSigma\,\}\big)\\
&\quad= \bigvee_{\sigma\in\varSigma}^{\mathrm{Th}\,\mathcal{S}} \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\sigma(\varDelta_n(x_0,\dots,x_{n-1},y,\bar u))\big)
\end{aligned}$$

$$= \bigvee_{\sigma\in\varSigma}^{\mathrm{Th}\,\mathcal{S}} \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\sigma\big(\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}(\varDelta_n(x_0,\dots,x_{n-1},y,\bar u))\big)\big),$$

by substitution-invariance of $\boldsymbol{\mathcal{S}}$

$$= \bigvee_{\sigma\in\varSigma}^{\mathrm{Th}\,\mathcal{S}} \sigma_{\boldsymbol{\mathcal{S}}}\big(\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}(\varDelta_n(x_0,\dots,x_{n-1},y,\bar u)\big),$$

by definition of $\sigma_{\boldsymbol{\mathcal{S}}}$

$$= \bigvee_{\sigma\in\varSigma}^{\mathrm{Th}\,\mathcal{S}} \sigma_{\boldsymbol{\mathcal{S}}}\,\varPhi(\mathbf{T}_n),$$

by definition of $\varDelta_n(x_0,\dots,x_{n-1},y,\bar u)$.

But $\varPhi(\mathbf{T}_n)$ is a Leibniz $\boldsymbol{\mathcal{S}}$-theory. So, for each $\sigma\in\varSigma$, $\sigma_{\boldsymbol{\mathcal{S}}}\,\varPhi(\mathbf{T}_n)$ is Leibniz by Lem. 3.12 and the assumption that σ is surjective. Finally, we get that

$$\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\forall\bar\vartheta\, \varDelta_n(\varphi_0,\dots,\varphi_{n-1},\psi,\bar\vartheta)\big) = \bigvee_{\sigma\in\varSigma}^{\mathrm{Th}\,\mathcal{S}} \sigma_{\boldsymbol{\mathcal{S}}}\,\varPhi(\mathbf{T}_n)$$

is a Leibniz $\boldsymbol{\mathcal{S}}$-theory by Lem. 3.7. This gives (8). ■

Corollary 4.11. *A protoalgebraic deductive system has a fully adequate Gentzen system iff it has a Leibniz-generating* PGDD *system over every Leibniz theory.*

Proof. By Thms. 4.8 and 4.10. ■

5. The Graded Deduction-Detachment Theorem without Parameters

In the last section we completely characterized the protoalgebraic deductive systems with a fully adequate Gentzen system by means of a Leibniz-generating PGDD system over Leibniz theories. In this section we investigate circumstances under which the parameters can be eliminated and the other two special conditions, Leibniz-generation and restriction to Leibniz theories, can be weakened.

$\boldsymbol{\Delta}$ is said to be simply a *graded deduction-detachment* (GDD) *system* for a deductive system $\boldsymbol{\mathcal{S}}$ over the $\boldsymbol{\mathcal{S}}$-theory T if it is a PGDD system for $\boldsymbol{\mathcal{S}}$ over T in which the set of parameters in each formula of $\boldsymbol{\Delta}$ is empty. In this case we write

$$\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y) : n < \boldsymbol{\omega} \,\rangle,$$

and the graded deduction and detachment properties become

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi).$$

We sometimes say that $\boldsymbol{\Delta}$ is a GDD system *without parameters* for emphasis. A GDD system $\boldsymbol{\Delta}$ is said to be *finite* if $\Delta_n(x_0, \dots, x_{n-1}, y)$ is finite for each $n < \boldsymbol{\omega}$.

It is interesting to compare the notion of a GDD system with the familiar deduction theorem from classical (and intuitionistic) logic. In the present context we consider a slight extension of the latter notion in which there are a finite number of formulas (not necessarily atomic) that collectively play the role of the implication connective in the classical deduction theorem. A deductive system $\boldsymbol{\mathcal{S}}$ is said to have the *multiterm deduction-detachment* (DD) *theorem* if there is a finite nonempty set $\Omega(x, y)$ of formulas in two variables with the property that, for all $\Gamma \subseteq \mathrm{Fm}_\Lambda$ and all $\varphi, \psi \in \mathrm{Fm}_\Lambda$,

$$\Gamma, \varphi \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad \Gamma \vdash_{\boldsymbol{\mathcal{S}}} \Omega(\varphi, \psi).$$

The set $\Omega(x, y)$ is called a *deduction-detachment* (DD) *system* for $\boldsymbol{\mathcal{S}}$. The classical deduction theorem corresponds to the special case where the DD system consists of a single formula, usually of the form $x \to y$; we refer to this as the *uniterm* DD theorem. The DD theorem (in both its multi- and uniterm forms) has been extensively studied in abstract algebraic logic, and several equivalent characterizations have been found for a protoalgebraic or an algebraizable deductive system to have the multiterm DD theorem; see [4, 1, 7, 9]. By the main result of [9] every deductive system with the multiterm DD theorem is protoalgebraic and has theorems.

The proof of the next lemma is straightforward (compare with the proof of Lemma 4.2).

Lemma 5.1. *A deductive system $\boldsymbol{\mathcal{S}}$ has the* DD *theorem with* DD *system*

$$\Omega(x, y) = \{\omega_0(x, y), \dots, \omega_{k-1}(x, y)\}$$

iff, as a g-matrix $\boldsymbol{\mathcal{S}} = \langle \mathbf{Fm}_\Lambda, \mathrm{Th}\,\boldsymbol{\mathcal{S}} \rangle$, it is a model of the g-sequents

$$\frac{x_0, \dots, x_n \rhd y}{x_0, \dots, x_{n-1} \rhd \omega_i(x_n, y)} \quad \textit{for all } i < k, \qquad (\mathrm{R}_{\Omega,n})$$

all $n < \boldsymbol{\omega}$, and of the following sequents (g-sequents without antecedents)

$$\overline{x, \omega_0(x, y), \dots, \omega_{k-1}(x, y) \rhd y}.$$

■

The following theorem spells out the exact connection between the multiterm DD theorem and GDD systems.

Theorem 5.2. *Let $\boldsymbol{\mathcal{S}}$ be a deductive system. $\boldsymbol{\mathcal{S}}$ has the multiterm DD theorem iff there is a finite* GDD *system for $\boldsymbol{\mathcal{S}}$ over every $\boldsymbol{\mathcal{S}}$-theory.*

Proof. Suppose $\Omega(x, y)$ is a multiterm DD system for $\boldsymbol{\mathcal{S}}$., i.e., $\Omega(x, y)$ is finite and $T, \varphi \vdash_{\boldsymbol{\mathcal{S}}} \psi$ iff $T \vdash_{\boldsymbol{\mathcal{S}}} \Omega(\varphi, \psi)$ for every $\boldsymbol{\mathcal{S}}$-theory T and all $\varphi, \psi \in \mathrm{Fm}_\Lambda$. Define $\Delta_n(x_0, \dots, x_{n-1}, y)$ for every $n < \boldsymbol{\omega}$ by recursion on n. Take $\Delta_0(y) = \{y\}$ and $\Delta_1(x_0, y) = \Omega(x_0, y)$. For each nonzero $n < \boldsymbol{\omega}$ take

$$\begin{aligned} &\Delta_{n+1}(x_0, \dots, x_n, y) \\ &\quad = \bigcup \{\, \Omega(x_0, \vartheta(x_1, \dots, x_n, y)) : \vartheta(x_1, \dots, x_n, y) \in \Delta_n(x_1, \dots, x_n, y) \,\}. \end{aligned}$$

It is easy to see that $\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y) : n < \boldsymbol{\omega} \,\rangle$ is a finite GDD system for $\boldsymbol{\mathcal{S}}$ over every $\boldsymbol{\mathcal{S}}$-theory T. Conversely, if $\boldsymbol{\Delta} = \langle\, \Delta_n(x_0, \dots, x_{n-1}, y) : n < \boldsymbol{\omega} \,\rangle$ is a finite GDD system for $\boldsymbol{\mathcal{S}}$ over every $\boldsymbol{\mathcal{S}}$-theory T, then $\Delta_1(x, y)$ is a multiterm DD system for $\boldsymbol{\mathcal{S}}$. ■

If one is chiefly interested in a graded deduction-detachment system for theorem proving, then probably the most interesting situation is when $\boldsymbol{\Delta}$ is a finite GDD system (without parameters) over the smallest theory, i.e., the set of theorems. This seems closest in spirit to the multiterm deduction-detachment theorem, and we define the graded deduction-detachment theorem in this way.

Definition 5.3. Let $\boldsymbol{\mathcal{S}}$ be a deductive system. $\boldsymbol{\mathcal{S}}$ is said to have the *graded deduction-detachment* (GDD) *theorem* if there is a finite GDD system $\boldsymbol{\Delta}$ (without parameters) for $\boldsymbol{\mathcal{S}}$ over the smallest $\boldsymbol{\mathcal{S}}$-theory, i.e., for all $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$,

$$\varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad \vdash_{\boldsymbol{\mathcal{S}}} \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi).$$

It is clear from Theorem 5.2 that every deductive system with the multi-term deduction-detachment theorem has the graded deduction-detachment theorem. The converse does not hold; see Theorems 5.11 and 5.12 below.

It turns out that if a protoalgebraic system $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system, then the parameters in the PGDD system for $\boldsymbol{\mathcal{S}}$ can be eliminated; this is the content of the next theorem. When the parameters are eliminated, the PGDD system can be reduced to a finite system. There is a price to pay for this simplification however: the finite parameterless GDD system we finally obtain need no longer be Leibniz-generating, and, as a consequence, it is no longer guaranteed to define the full g-models of $\boldsymbol{\mathcal{S}}$ in the sense of Thm. 4.8.

Theorem 5.4. *Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system. If $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system, then there is a finite* GDD *system for $\boldsymbol{\mathcal{S}}$* (*without parameters*) *over every Leibniz theory.*

Proof. Assume $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system, and let $\boldsymbol{\Gamma} = \langle \Gamma_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \rangle$ be a Leibniz-generating PGDD system for $\boldsymbol{\mathcal{S}}$ over every Leibniz theory. Then, for every $n < \boldsymbol{\omega}$, for all $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$, and for every Leibniz $\boldsymbol{\mathcal{S}}$-theory T,

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \forall \bar{\vartheta}\, \Gamma_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}); \tag{10}$$

moreover, $\forall \bar{\vartheta}\, \Gamma_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta})$ generates a Leibniz $\boldsymbol{\mathcal{S}}$-theory. In particular,

$$T, x_0, \dots, x_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} y \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \forall \bar{u}\, \Gamma_n(x_0, \dots, x_{n-1}, y, \bar{u}), \tag{11}$$

and $\forall \bar{u}\, \Gamma_n(x_0, \dots, x_{n-1}, y, \bar{u})$ generates a Leibniz $\boldsymbol{\mathcal{S}}$-theory. Thus, taking the Leibniz theory T in (11) to be $\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\forall \bar{u}\, \Gamma(x_0, \dots, x_{n-1}, y, \bar{u})\big)$, we have

$$\forall \bar{u}\, \Gamma_n(x_0, \dots, x_{n-1}, y, \bar{u}),\ x_0, \dots, x_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} y.$$

Since $\boldsymbol{\mathcal{S}}$ is finitary, there exists a $k < \boldsymbol{\omega}$,

$\gamma_0(x_0, \dots, x_{n-1}, y, \bar{u}), \dots, \gamma_{k-1}(x_0, \dots, x_{n-1}, y, \bar{u}) \in \Gamma_n(x_0, \dots, x_{n-1}, y, \bar{u})$,

and $\bar{\vartheta}_0, \dots, \bar{\vartheta}_{k-1} \in \mathrm{Fm}_{\Lambda}^{\boldsymbol{\omega}}$ such that

$$\big\{\, \gamma_i(x_0, \dots, x_{n-1}, y, \bar{\vartheta}_i) : i < k \,\big\},\ x_0, \dots, x_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} y.$$

Let σ be the (nonsurjective) substitution such that $\sigma(x_0) = x_0, \dots, \sigma(x_n) = x_n, \sigma(y) = y$ and $\sigma(z) = y$ for all $z \in \mathrm{Va} \setminus \{x_0, \dots, x_{n-1}, y\}$. Then, by the substitution invariance of $\boldsymbol{\mathcal{S}}$,

$$\big\{\, \gamma_i(x_0, \dots, x_{n-1}, y, \sigma(\bar{\vartheta}_i)) : i < k \,\big\},\ x_0, \dots, x_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} y.$$

Define

$$\Delta_n(x_0, \dots, x_{n-1}, y) := \big\{\, \gamma_i(x_0, \dots, x_{n-1}, y, \sigma(\bar{\vartheta}_i)) : i < k \,\big\}.$$

Then $\Delta_n(x_0, \dots, x_{n-1}, y)$, $x_0, \dots, x_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} y$, and, again by the substitution invariance of $\boldsymbol{\mathcal{S}}$ we get $\Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi)$, $\varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi$. Thus

$$T \vdash_{\boldsymbol{\mathcal{S}}} \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi) \quad \text{implies} \quad T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi. \tag{12}$$

Let τ be any substitution such that $\tau(x_0) = \varphi_0, \dots, \tau(x_n) = \varphi_n, \tau(y) = \psi$. Then

$$\begin{aligned} \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi) &= \tau\big(\Delta_n(x_0, \dots, x_{n-1}, y)\big) \\ &= \big\{\, \tau\big(\gamma_i(x_0, \dots, x_{n-1}, y, \sigma(\bar{\vartheta}_i))\big) : i < k \,\big\} \\ &= \big\{\, \gamma_i(\varphi_0, \dots, \varphi_{n-1}, \psi, (\tau \circ \sigma)(\bar{\vartheta}_i)) : i < k \,\big\} \\ &\subseteq \forall \bar{\vartheta}\, \Gamma_n(\varphi_0, \dots, \varphi_{n-1}, \psi, \bar{\vartheta}). \end{aligned}$$

Thus by (10) we have

$$T, \varphi_0, \dots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{implies} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \Delta_n(\varphi_0, \dots, \varphi_{n-1}, \psi).$$

Combining this with (12) we obtain the conclusion of the theorem. ■

We do not know for certain that the finite GDD system in this theorem cannot always be taken to be Leibniz-generating, i.e., we do not know if, for protoalgebraic systems, having a fully adequate Gentzen system is actually equivalent to the existence of a finite GDD system (without parameters) over every Leibniz theory. However, this seems highly unlikely. What makes it difficult to settle the question is the fact that this equivalence does hold for all finitely equivalential deductive systems. (See Cor. 5.9 below.)

Corollary 5.5. *Every protoalgebraic deductive system that has a fully adequate Gentzen system has the* GDD *theorem.*

Proof. If $\boldsymbol{\mathcal{S}}$ is protoalgebraic and has a fully adequate Gentzen system, then, by Thm. 5.4, there is a finite GDD system $\boldsymbol{\Delta}$ for $\boldsymbol{\mathcal{S}}$ over every Leibniz $\boldsymbol{\mathcal{S}}$-theory. The smallest $\boldsymbol{\mathcal{S}}$-theory T_0, the set of theorems of $\boldsymbol{\mathcal{S}}$, is obviously Leibniz. Thus $\boldsymbol{\Delta}$ is a finite GDD system for $\boldsymbol{\mathcal{S}}$ over T_0, i.e., $\boldsymbol{\mathcal{S}}$ has the GDD theorem. ■

Any deductive system with a fully adequate Gentzen system that does not have the multiterm DD theorem is an example of a deductive system with a graded DD system over all Leibniz theories but without the multiterm DD theorem. The deductive system of Example 5.10 has these properties, but it is specifically constructed for this purpose. The difficulty in finding examples of this kind already in the literature is not surprising. There is a wide class of deductive systems, including most of the known protoalgebraic systems, that have a fully adequate Gentzen system iff they have the multiterm DD theorem. (See Cor. 5.7 below.)

Definition 5.6 ([10]). A deductive system $\boldsymbol{\mathcal{S}}$ is *weakly algebraizable* if $\boldsymbol{\Omega}\colon \mathrm{Th}\,\boldsymbol{\mathcal{S}} \to \mathrm{Co}\,\mathbf{Fm}_\Lambda$ is order-preserving and injective.

It is easy to see that a protoalgebraic deductive system $\boldsymbol{\mathcal{S}}$ is weakly algebraizable iff every $\boldsymbol{\mathcal{S}}$-theory is Leibniz, i.e., $\mathrm{Th}^+\,\boldsymbol{\mathcal{S}} = \mathrm{Th}\,\boldsymbol{\mathcal{S}}$.

The weakly algebraizable deductive systems are all protoalgebraic and include all the algebraizable systems ([8, 10, 18, 19]), in particular all the finitely algebraizable systems ([3]). But they also include a number of important deductive systems that are not algebraizable, for instance any orthologic that is not orthomodular ([10]).

Corollary 5.7. *A weakly algebraizable deductive system $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system iff it has the multiterm deduction-detachment theorem. More precisely, if $\boldsymbol{\Omega}(x,y) = \{\omega_0(x,y), \dots, \omega_{k-1}(x,y)\}$ is a DD system for $\boldsymbol{\mathcal{S}}$, then the fully adequate Gentzen system for $\boldsymbol{\mathcal{S}}$ has a presentation whose initial sequents are* (axiom) *and the rules of $\boldsymbol{\mathcal{S}}$, and whose proper rules are* (cut), (structure), *and the g-sequents of* $(\mathrm{R}_{\Omega,n})$ *for $n < \boldsymbol{\omega}$.*

Proof. Assume $\boldsymbol{\mathcal{S}}$ is weakly algebraizable. Then $\boldsymbol{\mathcal{S}}$ is protoalgebraic and, since every $\boldsymbol{\mathcal{S}}$-theory is Leibniz, every GDD system over all Leibniz theories is Leibniz-generating. Thus, by Thms. 4.8 and 5.4, $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system iff it has a GDD system over all $\boldsymbol{\mathcal{S}}$-theories. Now apply Thm. 5.2.

That the fully adequate Gentzen system for $\boldsymbol{\mathcal{S}}$ has the presentation claimed follows without difficulty from Thm. 4.8 and the precise connection between DD systems for $\boldsymbol{\mathcal{S}}$ and finite GDD systems for $\boldsymbol{\mathcal{S}}$ over every $\boldsymbol{\mathcal{S}}$-theory that is spelled out in the proof of Thm. 5.2. ■

We have no general method of eliminating the parameters from a PGDD system without losing the Leibniz-generating property. However, such a method does exist for all the protoalgebraic deductive systems that satisfy the rather technical condition given in the next theorem. There are at least two important classes of deductive systems that satisfy the condition. One of them is the weakly algebraizable systems. Another is the equivalential systems that we discussed briefly in Sec. 3; see Def. 3.9.

Let $\boldsymbol{\mathcal{S}}$ be a protoalgebraic deductive system. The set of Leibniz $\boldsymbol{\mathcal{S}}$-filters on an arbitrary algebra is not in general closed under intersection, but they do form a complete lattice (under set-theoretic inclusion) in light of Lem. 3.7. And it is shown in [16] that, if $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system, then they are also closed under intersection.

Theorem 5.8. *Assume that $\boldsymbol{\mathcal{S}}$ is a protoalgebraic deductive system with the property that every Leibniz filter that is compact in the lattice of Leibniz $\boldsymbol{\mathcal{S}}$-filters is also compact in the lattice of all $\boldsymbol{\mathcal{S}}$-filters. If $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system, then it has a finite Leibniz-generating* GDD *system* (*without parameters*) *over all Leibniz theories.*

Proof. Assume $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system. Let Φ be the isomorphism between the lattice $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$ and the lattice of Leibniz $\boldsymbol{\mathcal{S}}$-theories investigated in Lem. 4.9. Recall that Φ commutes with surjective substitutions in the sense that

$$\Phi(\sigma_{\boldsymbol{\mathcal{G}}}(\mathbf{T})) = \sigma_{\boldsymbol{\mathcal{S}}}(\Phi(\mathbf{T})) \quad \text{for every surjective } \sigma : \mathbf{Fm}_\Lambda \twoheadrightarrow \mathbf{Fm}_\Lambda \text{ and every } \mathbf{T} \in \mathrm{Th}\,\boldsymbol{\mathcal{G}}. \tag{13}$$

Recall also that the Leibniz-generating PGDD system

$$\boldsymbol{\Delta} := \langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n < \boldsymbol{\omega} \,\rangle$$

for $\boldsymbol{\mathcal{S}}$ over all Leibniz theories that was constructed in the proof of Thm. 4.10 was defined as follows. $\Delta_n(x_0, \dots, x_{n-1}, y, \bar{u})$ is any set of generators of the

$\boldsymbol{\mathcal{S}}$-theory $\varPhi(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\{x_0,\dots,x_{n-1} \rhd y\}\big)$, where $\bar{u}$ is a sequence without repetitions of all variables distinct from $x_0,\dots,x_{n-1},y$. $\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\{x_0,\dots,x_{n-1} \rhd y\}\big)$ is finitely generated as a generalized $\boldsymbol{\mathcal{G}}$-filter, and hence it is a compact element in the algebraic lattice $\mathrm{Th}\,\boldsymbol{\mathcal{G}}$. Thus $\varPhi(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\{x_0,\dots,x_{n-1} \rhd y\}\big)$ is a compact element in the lattice of Leibniz theories of $\boldsymbol{\mathcal{S}}$. By hypothesis then $\varPhi(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\{x_0,\dots,x_{n-1} \rhd y\}\big)$ is a compact element in the lattice of all $\boldsymbol{\mathcal{S}}$-theories and hence finitely generated as an $\boldsymbol{\mathcal{S}}$-theory. So $\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})$ can be taken to be finite; in particular, only a finite number of the variables $\bar{u}$ can actually occur in $\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})$. So there exists a surjective substitution σ such that $\sigma(x_i) = x_i$ for $i < n$, $\sigma(y) = y$, and $\sigma(u) = y$ for each u different from $x_0,\dots,x_{n-1},y$ that actually occurs in $\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})$. Then

$$
\begin{aligned}
&\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})\big)\\
&\quad= \varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1} \rhd y\}\big)\\
&\quad= \varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{\sigma(x_0,\dots,x_{n-1} \rhd y)\}\big)\\
&\quad= \varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\big(\sigma(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1} \rhd y\})\big)\big),\\
&\qquad\qquad\qquad\text{by substitutional invariance of } \boldsymbol{\mathcal{G}}\\
&\quad= \varPhi\big(\sigma_{\boldsymbol{\mathcal{G}}}(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1} \rhd y\})\big),\\
&\qquad\qquad\qquad\text{by definition of } \sigma_{\boldsymbol{\mathcal{G}}}\\
&\quad= \sigma_{\boldsymbol{\mathcal{S}}}\big(\varPhi\big(\mathrm{Cn}_{\boldsymbol{\mathcal{G}}}\{x_0,\dots,x_{n-1} \rhd y\}\big)\big),\\
&\qquad\qquad\qquad\text{by (13)}\\
&\quad= \sigma_{\boldsymbol{\mathcal{S}}}\big(\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})\big)\big),\\
&\qquad\qquad\qquad\text{by definition of } \varDelta_n(x_0,\dots,x_{n-1},y,\bar{u})\\
&\quad= \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\sigma(\varDelta_n(x_0,\dots,x_{n-1},y,\bar{u}))\big),\\
&\qquad\qquad\qquad\text{by substitutional invariance of } \boldsymbol{\mathcal{S}}\\
&\quad= \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\varDelta_n(x_0,\dots,x_{n-1},y,\sigma(\bar{u}))\big).
\end{aligned}
$$

Thus $\varDelta_n$ can be taken to be a finite set of formulas in $x_0,\dots,x_{n-1},y$, without parameters. ■

Obviously every weakly algebraizable deductive system satisfies the condition in Theorem 5.8 because every filter is Leibniz. We shall now see that every finitely equivalential system also satisfies the condition.

Corollary 5.9. *A finitely equivalential deductive system $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system iff there is a finite Leibniz-generating* GDD *system for $\boldsymbol{\mathcal{S}}$ over all Leibniz $\boldsymbol{\mathcal{S}}$-theories.*

Proof. Let $E(x, y)$ be a finite equivalence system for $\boldsymbol{\mathcal{S}}$ without parameters. We verify that $\boldsymbol{\mathcal{S}}$ satisfies the condition of Thm. 5.8. Let F be a Leibniz $\boldsymbol{\mathcal{S}}$-filter on an algebra $\boldsymbol{A}$ that is compact in the lattice of Leibniz filters. But the lattice of Leibniz $\boldsymbol{\mathcal{S}}$-filters and the lattice of Leibniz congruences are isomorphic (under the Leibniz operator). Thus $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is compact in the lattice of Leibniz congruences on $\boldsymbol{A}$. Since $\boldsymbol{\mathcal{S}}$ is equivalential and hence protoalgebraic, the set of all Leibniz congruences on $\boldsymbol{A}$ is closed under intersection and thus forms a closed-set system (actually, an algebraic closed-set system because $\boldsymbol{\mathcal{S}}$ is equivalential, but this fact is not needed here). In a closed-set system every closed set that is compact in the lattice of closed sets is finitely generated. Thus we finally get that $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is finitely generated as a Leibniz congruence. Let $\langle a_0, b_0\rangle, \dots, \langle a_{n-1}, b_{n-1}\rangle$ be a finite set of generators of $\boldsymbol{\Omega}_{\boldsymbol{A}} F$. Then, for every $\boldsymbol{\mathcal{S}}$-filter G on $\boldsymbol{A}$, $F \subseteq G$ iff $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \boldsymbol{\Omega}_{\boldsymbol{A}} G$ iff $a_i \equiv b_i \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} G}$ for each $i < n$. But E, as an equivalence system, generates Leibniz congruences, i.e., for all $a, b \in A$, $a \equiv b \pmod{\boldsymbol{\Omega}_{\boldsymbol{A}} G}$ iff $E(a, b) \subseteq G$. Thus $F \subseteq G$ iff $E(a_i, b_i) \subseteq G$ for all $i < n$, i.e., F is generated as an $\boldsymbol{\mathcal{S}}$-filter by $\bigcup_{i<n} E(a_i, b_i)$. So F is finitely generated. Now apply Thm. 5.8. ■

In summary we have considered five putatively different deduction-detachment-like properties of an arbitrary protoalgebraic deductive system.

(I) $\boldsymbol{\mathcal{S}}$ has a Leibniz-generating parameterized GDD system over all Leibniz theories.

(II) $\boldsymbol{\mathcal{S}}$ has a finite Leibniz-generating GDD system (without parameters) over all Leibniz theories.

(III) $\boldsymbol{\mathcal{S}}$ has a finite (not necessarily Leibniz-generating) GDD system over all Leibniz theories.

(IV) $\boldsymbol{\mathcal{S}}$ has a finite GDD system over the smallest theory (i.e., $\boldsymbol{\mathcal{S}}$ has the GDD theorem).

(V) $\boldsymbol{\mathcal{S}}$ has the multiterm DD theorem.

The logical relationships between these five notions are given in the following diagram.

$$\begin{array}{ccccccc} \text{(II)} & \Rightarrow & \text{(I)} & \Rightarrow & \text{(III)} & \Rightarrow & \text{(IV)} \\ & & & & \Uparrow & & \\ & & & & \text{(V)} & & \end{array}$$

We have shown that (I) characterizes full adequacy in general and that full adequacy implies (III) in general, so (I) implies (III). For finitely equivalential systems (I) and (II) are equivalent, and for weakly algebraizable systems (I)–(III),(V) are all equivalent.

The following example shows that (II) does not imply (V), even under the assumption that $\boldsymbol{\mathcal{S}}$ is finitely equivalential; hence each of (I)–(IV) also fails to imply (V).

Example 5.10. Let $\Lambda = \{\leftrightarrow, \top, \gamma_0, \gamma_1, \dots, \gamma_n, \dots\}_{n<\boldsymbol{\omega}}$ be a language type where $\leftrightarrow$ is of rank 2, $\top$ of rank 0, and γ_k of rank $k+1$ for each $k < \boldsymbol{\omega}$. Let $\boldsymbol{\mathcal{G}}$ be the Gentzen system over Λ presented by the following axioms and rules of inference, in addition to those of (axiom), (cut), and (structure).

Equivalence axioms:

$$\rhd\, x \leftrightarrow x,$$

$$x \leftrightarrow y \rhd y \leftrightarrow x,$$

$$x \leftrightarrow y, y \leftrightarrow z \rhd x \leftrightarrow z,$$

$$x_0 \leftrightarrow y_0, x_1 \leftrightarrow y_1 \rhd (x_0 \leftrightarrow x_1) \leftrightarrow (y_0 \leftrightarrow y_1),$$

$$x, x \leftrightarrow y \rhd y,$$

and, for each $n < \boldsymbol{\omega}$,

$$\begin{aligned} &x_0 \leftrightarrow y_0, \dots, x_{n-1} \leftrightarrow y_{n-1}, z \leftrightarrow w \\ &\quad \rhd \gamma_n(x_0, \dots, x_{n-1}, z) \leftrightarrow \gamma_n(y_0, \dots, y_{n-1}, w). \end{aligned}$$

Graded detachment axioms: for every $n < \boldsymbol{\omega}$,

$$\gamma_n(x_0, \dots, x_{n-1}, y) \leftrightarrow \top, x_0, \dots, x_{n-1} \rhd y.$$

Leibniz deduction axioms: for all $m, n < \boldsymbol{\omega}$,

$$z_0 \leftrightarrow w_0, \ldots, z_{m-1} \leftrightarrow w_{m-1}$$
$$\rhd \gamma_{m+n}(z_0 \leftrightarrow w_0, \ldots, z_{m-1} \leftrightarrow w_{m-1}, x_0, \ldots, x_{n-1}, y) \leftrightarrow \gamma_n(x_0, \ldots, x_{n-1}, y).$$

Inference rules: for each $n < \boldsymbol{\omega}$,

$$(\mathrm{R}_{\Delta_n}) \quad \frac{x_0, \ldots, x_{n-1} \rhd y}{\rhd \gamma_n(x_0, \ldots, x_{n-1}, y) \leftrightarrow \top}.$$

Let $\boldsymbol{\mathcal{S}}$ be the unique deductive system whose rules are the derived sequents of $\boldsymbol{\mathcal{G}}$, that is, $\Gamma \vdash_{\boldsymbol{\mathcal{S}}} \psi$ iff $\vdash_{\boldsymbol{\mathcal{G}}} \varphi_0, \ldots, \varphi_{n-1} \rhd \psi$ for some $\varphi_0, \ldots, \varphi_{n-1} \in \Gamma$. Then $\boldsymbol{\mathcal{S}}$ is a finitely equivalential deductive system and $\boldsymbol{\mathcal{G}}$ is a fully adequate Gentzen system for it. Furthermore, if $\boldsymbol{\Delta} = \langle \Delta_n(x_0, \ldots, x_{n-1}, y) : n < \boldsymbol{\omega} \rangle$, where

$$\Delta_n(x_0, \ldots, x_{n-1}, y) = \{\gamma_n(x_0, \ldots, x_{n-1}, y) \leftrightarrow \top\}, \quad \text{for each } n < \boldsymbol{\omega},$$

then $\boldsymbol{\Delta}$ is a finite Leibniz-generating GDD system for $\boldsymbol{\mathcal{S}}$ over all Leibniz theories. However, $\boldsymbol{\mathcal{S}}$ does not have the multiterm DD theorem. These statements are verified in the following two theorems.

Theorem 5.11. *Let $\boldsymbol{\mathcal{G}}$ be the Gentzen system and $\boldsymbol{\mathcal{S}}$ the deductive system defined in the above example. $\boldsymbol{\mathcal{S}}$ is finitely equivalential and $\boldsymbol{\Delta}$ is a finite Leibniz-generating GDD system for $\boldsymbol{\mathcal{S}}$ over all Leibniz theories. $\boldsymbol{\mathcal{G}}$ is fully adequate for $\boldsymbol{\mathcal{S}}$.*

Proof. The singleton $E(x, y) = \{x \to y\}$ is a finite equivalence system for $\boldsymbol{\mathcal{S}}$ without parameters. This can be verified by showing that $E(x, y)$ defines the Leibniz congruences of $\boldsymbol{\mathcal{S}}$, which is not difficult, or it can be obtained directly from the original definition of equivalential logics (see [20] or [6]). Thus, by Cor. 3.11, T is a Leibniz theory of $\boldsymbol{\mathcal{S}}$ iff T is generated by a set of formulas of the form $\varphi \leftrightarrow \psi$. Hence $\boldsymbol{\Delta}$ is Leibniz-generating. We prove that $\boldsymbol{\Delta}$ is a GDD system over $\boldsymbol{\mathcal{S}}$ for every Leibniz theory, i.e., we verify that the equivalence

$$T, \varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi \quad \text{iff} \quad T \vdash_{\boldsymbol{\mathcal{S}}} \gamma_n(\varphi_0, \ldots, \varphi_{n-1}, \psi) \leftrightarrow \top \tag{14}$$

holds for every $T \in \mathrm{Th}^+ \boldsymbol{\mathcal{S}}$ and all $\varphi_0, \ldots, \varphi_{n-1}, \psi \in \mathrm{Fm}_\Lambda$.

Suppose T is a Leibniz theory of $\boldsymbol{\mathcal{S}}$ and

$$T, \varphi_0, \ldots, \varphi_{n-1} \vdash_{\boldsymbol{\mathcal{S}}} \psi.$$

Then, for some $\vartheta_0 \leftrightarrow \xi_0, \ldots, \vartheta_{m-1} \leftrightarrow \xi_{m-1} \in T$ we have

$$\vdash_{\mathcal{G}} \vartheta_0 \leftrightarrow \xi_0, \ldots, \vartheta_{m-1} \leftrightarrow \xi_{m-1}, \varphi_0, \ldots, \varphi_{n-1} \rhd \psi.$$

Thus, by (R_{Δ_n}),

$$\vdash_{\mathcal{G}} \rhd \gamma_{m+n}(\vartheta_0 \leftrightarrow \xi_0, \ldots, \vartheta_{m-1} \leftrightarrow \xi_{m-1}, \varphi_0, \ldots, \varphi_{n-1}, \psi) \leftrightarrow \top.$$

Then applying the Leibniz deduction axiom for m, n and using (cut) and the equivalence axioms several times we get

$$\vdash_{\mathcal{G}} \vartheta_0 \leftrightarrow \xi_0, \ldots, \vartheta_{m-1} \leftrightarrow \xi_{m-1} \rhd \gamma_n(\varphi_0, \ldots, \varphi_{n-1}, \psi) \leftrightarrow \top,$$

i.e.,

$$\vartheta_0 \leftrightarrow \xi_0, \ldots, \vartheta_{m-1} \leftrightarrow \xi_{m-1} \vdash_{\mathcal{S}} \gamma_n(\varphi_0, \ldots, \varphi_{n-1}, \psi).$$

Thus $T \vdash_{\mathcal{S}} \gamma_n(\varphi_0, \ldots, \varphi_{n-1}, \psi)$. This gives the implication from left to right in (14). The reverse implication is an immediate consequence of the graded detachment axiom for n.

Thus $\boldsymbol{\Delta}$ is a Leibniz-generating GDD system for $\mathcal{S}$ over every Leibniz theory. So by Thm. 4.8 $\mathcal{S}$ has a fully adequate Gentzen system; moreover, it has a presentation whose only proper inference rules (apart from (cut) and (structure)) are the g-sequents (R_{Δ_n}), for $n < \boldsymbol{\omega}$. Clearly any two Gentzen systems that are adequate for the same deductive system and have presentations with the same set of proper inference rules must coincide. Thus $\mathcal{G}$ is fully adequate for $\mathcal{S}$. ■

Theorem 5.12. *The deductive system $\mathcal{S}$ of Example* 5.10 *has the* GDD *theorem, but does not have the multiterm* DD *theorem.*

Proof. That $\mathcal{S}$ has the GDD theorem follows immediately from the preceding theorem by Cor. 5.5.

It is well known that, if $\mathcal{S}$ has the multiterm DD theorem, then, for every algebra $\boldsymbol{A}$, the lattice $\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ is distributive ([4, Corollary 4.4], [7, Corollary 2.6]). Thus it suffices to find an algebra $\boldsymbol{A}$ such that $\mathrm{Fi}_{\mathcal{S}}\,\boldsymbol{A}$ is not a distributive lattice, that is, a basic full g-model $\langle \boldsymbol{A}, \mathcal{C} \rangle$ of $\mathcal{S}$ such that $\mathcal{C}$ is not distributive. Since every full g-model reduces to a basic full g-model, and reduction preserves lattice structure up to isomorphism, it suffices to find a full g-model that is not distributive. Finally, since the Gentzen system $\boldsymbol{\mathcal{G}}$ of Exam. 5.10 is fully adequate for $\boldsymbol{\mathcal{S}}$, the problem reduces finally to finding a model $\langle \boldsymbol{A}, \mathcal{C} \rangle$ of $\boldsymbol{\mathcal{G}}$ such that $\mathcal{C}$ fails to be distributive.

Let A be any nonempty set, and let $0, 1$ be fixed but arbitrary elements of A. Let $\mathcal{C}$ be any algebraic closed-set system on A such that $1 \in \bigcap \mathcal{C}$ and $\mathrm{Clo}_{\mathcal{C}}(\{0\}) = A$. Then we shall show that there is a Λ-algebra

$$\boldsymbol{A} = \langle A, \leftrightarrow^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \gamma_n^{\boldsymbol{A}} \rangle_{n<\boldsymbol{\omega}}$$

that $\langle \boldsymbol{A}, \mathcal{C} \rangle \in \mathsf{Mod}\, \boldsymbol{\mathcal{G}}$.

Define $\top^{\boldsymbol{A}} = 1$,

$$a \leftrightarrow^{\boldsymbol{A}} b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise,} \end{cases}$$

for all $a, b \in A$, and, for every $n < \boldsymbol{\omega}$,

$$\gamma_n^{\boldsymbol{A}}(a_0, \ldots, a_{n-1}, b) = \begin{cases} 1 & \text{if } b \in \mathrm{Clo}_{\mathcal{C}}(\{a_0, \ldots, a_{n-1}\}) \\ 0 & \text{otherwise,} \end{cases}$$

for all $a_0, \ldots, a_{n-1}, b \in A$.

We show that $\langle \boldsymbol{A}, \mathcal{C} \rangle$ is a model of $\boldsymbol{\mathcal{G}}$ by verifying that each axiom and rule of inference is valid in $\langle \boldsymbol{A}, \mathcal{C} \rangle$. For each axiom, which is in the form of a sequent $\varphi_0, \ldots, \varphi_{n-1} \rhd \psi$, this means showing that, for every evaluation $h\colon \mathbf{Fm}_\Lambda \to \boldsymbol{A}$,

$$h(\psi) \in \mathrm{Clo}_{\mathcal{C}}(\{h(\varphi_0), \ldots, h(\varphi_{n-1})\}). \tag{15}$$

This is straightforward for the equivalence axioms. Consider for example the replacement axiom for the connective γ_n. If $h(x_i) \neq h(y_i)$, i.e., $h(x_i \leftrightarrow y_i) = h(x_i) \leftrightarrow^{\boldsymbol{A}} h(y_i) = 0$, for some $i < n$, or $h(z) \neq h(w)$, then $\mathrm{Clo}_{\mathcal{C}}\big(\{h(x_0 \leftrightarrow y_0), \ldots, h(x_{n-1} \leftrightarrow y_{n-1}), h(z \leftrightarrow w)\}\big) = A$, and (15) obviously holds. Otherwise, $h(x_i) = h(y_i)$ for all $i < n$ and $h(z) = h(w)$, and hence $h\big(\gamma_n(x_0, \ldots, x_{n-1}, z)\big) = \gamma_n^{\boldsymbol{A}}\big(h(x_0), \ldots, h(x_{n-1}), h(z)\big) = \gamma_n^{\boldsymbol{A}}\big(h(y_0), \ldots, h(y_{n-1}), h(w)\big) = h\big(\gamma_n(y_0, \ldots, y_{n-1}, w)\big)$. So

$$h\big(\gamma_n(x_0, \ldots, x_{n-1}, z) \leftrightarrow \gamma_n(y_0, \ldots, y_{n-1}, w)\big) = 1,$$

and obviously (15) again holds.

The graded detachment axiom for and $n < \boldsymbol{\omega}$:

Let $S = \{h(x_0), \ldots, h(x_{n-1})\}$ and $S' = S \cup \big\{h\big(\gamma_n(x_0, \ldots, x_{n-1}, y) \leftrightarrow \top\big)\big\}$. If $h(y) \in \mathrm{Clo}_{\mathcal{C}}(S)$, then obviously $h(y) \in \mathrm{Clo}_{\mathcal{C}}(S')$. If $h(y) \notin \mathrm{Clo}_{\mathcal{C}}(S)$, then $h\big(\gamma_n(x_0,\dots,x_{n-1},y)\big) = \gamma_n^{\boldsymbol{A}}(h(x_0),\dots,h(x_{n-1}),h(y)) = 0$. Hence $S' = S \cup \{0\}$, and again $h(y) \in \mathrm{Clo}_{\mathcal{C}}(S')$. So in all cases we have $h(y) \in \mathrm{Clo}_{\mathcal{C}}\big(\big\{h(\gamma_n(x_0,\dots,x_{n-1},y) \leftrightarrow \top), h(x_0),\dots,h(x_{n-1}),h(y)\big\}\big)$.

The Leibniz deduction axiom for $m, n < \boldsymbol{\omega}$:

Let $S = \{h(z_0 \leftrightarrow w_0),\dots,h(z_{m-1} \leftrightarrow w_{m-1})\}$ and $T = \{h(x_0),\dots,h(x_{n-1})\}$. Let

$$\varphi = \gamma_{m+n-1}(z_0 \leftrightarrow w_0,\dots,z_{m-1} \leftrightarrow w_{m-1},x_0,\dots,x_{n-1},y) \\ \leftrightarrow \gamma_n(x_0,\dots,x_{n-1},y).$$

If $h(z_i \leftrightarrow w_i) = 1$ for all $i < m$, then $S \cup T = T \cup \{1\}$, and hence $h(y) \in \mathrm{Clo}_{\mathcal{C}}(S \cup T)$ iff $h(y) \in \mathrm{Clo}_{\mathcal{C}}(T)$. So $h(\gamma_{m+n-1}(z_0 \leftrightarrow w_0,\dots,z_{m-1} \leftrightarrow w_{m-1},x_0,\dots,x_{n-1},y)) = h(\gamma_n(x_0,\dots,x_{n-1},y))$, and hence $h(\varphi) = 1$. Thus $h(\varphi) \in \mathrm{Clo}_{\mathcal{C}}(S)$. On the other hand, if $h(z_i \leftrightarrow w_i) = 0$ for some $i < m$, then $0 \in S$ and hence again we have $h(\varphi) \in \mathrm{Clo}_{\mathcal{C}}(S)$.

The inference rule (R_{Δ_n}) for $n < \boldsymbol{\omega}$:

If $h(y) \in \mathrm{Clo}_{\mathcal{C}}(\{h(x_0),\dots,h(x_{n-1})\})$, then $h\big(\gamma_n(x_0,\dots,x_{n-1},y)\big) = 1$ and hence $h\big(\gamma_n(x_0,\dots,x_{n-1},y) \leftrightarrow \top\big) = 1$.

This completes the verification of $\langle \boldsymbol{A}, \mathcal{C}\rangle$ as a model of $\boldsymbol{\mathcal{G}}$. To complete the proof of the theorem we only have to exhibit a model with the property that $\mathcal{C}$ is nondistributive. But this is easy. For example, let A be the 5-element set $\{1,a,b,c,0\}$ take $\mathcal{C} = \big\{\{1\},\{1,a\},\{1,b\},\{1,c\},A\big\}$. This satisfies the two conditions $1 \in \bigcap \mathcal{C}$ and $\mathrm{Clo}_{\mathcal{C}}\{0\} = A$, and $\mathcal{C}$ is isomorphic, as a lattice, to the 5-element nondistributive lattice M_3. ■

In the companion paper [16] we give an example of a finitely equivalential deductive system that is a "dual" of the one in Example 5.10 in the sense that it has the multiterm DD theorem but fails to have a fully adequate Gentzen system, thus confirming that among the five deduction-detachment-like properties considered previously, (V) does not imply (I) and hence also not (II). Like its dual this example is the result of an ad hoc construction designed specifically for the purpose at hand. The problem with finding examples in the extant literature of either of these two kinds is that there are very few examples of protoalgebraic deductive systems there that are not at least weakly algebraizable, and thus for which the existence of a fully adequate Gentzen system and of the multiterm DD theorem are equivalent. There is however a deductive system that is closely related to one that has appeared in the literature and that, like Exam. 5.10, has a fully adequate Gentzen system but not the multiterm DD theorem. See [16] for details.

It is interesting to compare the results in the present paper to those obtained in Chapter 4 of the monograph [15]. The two main results of [15] we are interested in present sufficient conditions for the existence of a fully adequate Gentzen system for selfextensional deductive systems, and are obtained by using a completely different technique, namely Rebagliato and Verdú's theory of algebraizability of Gentzen systems [21]. A system $\boldsymbol{\mathcal{S}}$ is *selfextensional* if the relation that holds between the formulas φ and ψ when they are interderivable (i.e., $\varphi \vdash_{\boldsymbol{\mathcal{S}}} \psi$ and $\psi \vdash_{\boldsymbol{\mathcal{S}}} \varphi$) is a congruence relation on the formula algebra. It is proved in [15, Theorem 4.45] that, if $\boldsymbol{\mathcal{S}}$ is selfextensional and satisfies the uniterm DD theorem, then it has a fully adequate Gentzen system. In Corollary 5.7 we have shown that, if $\boldsymbol{\mathcal{S}}$ is weakly algebraizable and has the multiterm DD theorem, then it has a fully adequate Gentzen system. These two cases seem to be orthogonal: there are selfextensional deductive systems with the uniterm DD theorem that are not weakly algebraizable, such as the quasi-normal modal logics, and there are weakly algebraizable systems with the multiterm DD theorem that are not selfextensional, such as the normal modal logics corresponding to S4 or S5, or Łukasiewicz's finitely valued logics $Ł_n$ for $n > 2$. However, the two cases have nonempty intersection, namely the weakly algebraizable and selfextensional deductive systems having the uniterm DD theorem, such as the classical or intuitionistic propositional calculi and all their fragments containing implication. All these systems have a fully adequate Gentzen system, and both Corollary 5.7 and the results in [15] provide presentations of this Gentzen system. It is interesting to observe that in this case the presentation in [15] is a redundant version of that given in the present paper, since it has the same initial sequents or axioms (the Hilbert-style rules of $\boldsymbol{\mathcal{S}}$), the rules (cut) and (structure), and the rules $(\mathrm{R}_{\Omega,n})$, as in our Corollary 5.7, plus the so-called *congruence rules*:

$$\frac{x_0 \rhd y_0 \ ; \ y_0 \rhd x_0 \ ; \ldots ; \ x_{k-1} \rhd y_{k-1} \ ; \ y_{k-1} \rhd x_{k-1}}{\lambda x_0 \cdots x_{k-1} \rhd \lambda y_0 \cdots y_{k-1}} \tag{16}$$

for each basic operation λ of Λ; k is the arity of the operation. By Theorem 4.8 and Corollary 5.7 we know these rules are no longer needed.

The other case treated in [15] concerns (not necessarily protoalgebraic) selfextensional deductive systems having a *conjunction*, that is, a term $\varepsilon(x,y)$ in two variables such that $\mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\{\varepsilon(x,y)\}\big) = \mathrm{Cn}_{\boldsymbol{\mathcal{S}}}\big(\{x,y\}\big)$. In [15, Theorem 4.27] it is proved that any selfextensional deductive system with conjunction has a fully adequate Gentzen system. This is used to show for instance that the conjunction-disjunction fragment of classical propositional calculus has a fully adequate Gentzen system, and to determine it. This deductive system fails to have the multiterm DD theorem, but is not a counterexample to the implication from (I) to (V) because it is not protoalgebraic. Merging these results of [15] with those of the present paper we unexpectedly find another class of deductive systems that falls simultaneously under both approaches:

Corollary 5.13. *Assume the deductive system $\boldsymbol{\mathcal{S}}$ is selfextensional and weakly algebraizable. If $\boldsymbol{\mathcal{S}}$ has conjunction then it has the uniterm DD theorem.*

Proof. From the assumption that $\boldsymbol{\mathcal{S}}$ is selfextensional and has conjunction we have by [15, Theorem 4.27] that $\boldsymbol{\mathcal{S}}$ has a fully adequate Gentzen system. Since $\boldsymbol{\mathcal{S}}$ is also weakly algebraizable, it has the multiterm DD theorem by Corollary 5.7. Finally, the conjunction operation transforms the finite set of formulas satisfying the DD theorem into a single formula, hence $\boldsymbol{\mathcal{S}}$ has the uniterm DD theorem. ■

By this result, a presentation of the fully adequate Gentzen system for these deductive systems can be obtained by two methods: In both cases one takes all Hilbert-style rules of $\boldsymbol{\mathcal{S}}$ as axioms (initial sequents); to these and to the rules (cut) and (structure) one has to add either the congruence rules (16), by [15, Definition 4.23], or the rules $(\mathrm{R}_{\Omega,n})$, by Corollary 5.7. Neither presentation is a subsystem of the other. Thus, a proof-theoretic consequence of this is that the sets of rules (16) and $(\mathrm{R}_{\Omega,n})$ are interderivable modulo the Hilbert-style rules of $\boldsymbol{\mathcal{S}}$ and the rules (cut) and (structure).

A more restricted and well behaved subcase of weak algebraizability is that of finite algebraizability. By [15, Proposition 4.29], a selfextensional and finitely algebraizable deductive system with conjunction is strongly finitely algebraizable (i.e., its equivalent quasivariety is actually a variety). There are selfextensional and finitely algebraizable systems that have the uniterm DD theorem but not a conjunction, such as the implication fragments of the classical or intuitionistic propositional calculi. Corollary 5.13

shows that the dual situation is not possible. It is also interesting to observe that under somewhat stronger assumptions the conclusion of this corollary is almost straightforward: In [11, Theorem 2.22] it is easily shown that a protoalgebraic and Fregean deductive system with at least one theorem and having conjunction has the uniterm DD theorem. A deductive system $\boldsymbol{\mathcal{S}}$ is *Fregean* if, for every theory T of $\boldsymbol{\mathcal{S}}$, the interderivability relation modulo T (i.e., the relation that holds between φ and ψ when $T, \varphi \vdash_{\boldsymbol{\mathcal{S}}} \psi$ and $T, \psi \vdash_{\boldsymbol{\mathcal{S}}} \varphi$) is a congruence on $\mathbf{Fm}_\Lambda$; every Fregean system is trivially self-extensional. The property of being protoalgebraic is a priori weaker than that of being weakly algebraizable, but for Fregean systems it is actually stronger than the assumptions of Corollary 5.7 because by [13, Theorem 2] a Fregean protoalgebraic logic with theorems is regularly finitely algebraizable. Fregean deductive systems are thoroughly investigated in [11, 12]. A more detailed analysis of examples concerning the existence and precise presentation of fully adequate Gentzen systems is contained in [15, Chapter 5] and in the last section of [16].

Finally, we pose an open problem that was suggested by the referee concerning the possibility of giving a Hilbert-style characterization of PGDD systems over all Leibniz theories; what we have in mind here is the way the standard deduction theorem of the classical and intuitionistic calculus is characterized in terms of modus ponens, Frege's syllogism $(x \to (y \to z)) \to ((x \to y) \to (x \to z))$, and the axiom $x \to (y \to x)$. If $\boldsymbol{\mathcal{S}}$ has a PGDD system $\langle\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{u}) : n \in \boldsymbol{\omega} \,\rangle$ over the Leibniz theories, then by the graded modus ponens property (Lem. 4.6)

$$\frac{\forall\bar{\vartheta}\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{\vartheta}),\ x_0, \dots, x_{n-1}}{y}$$

is an infinite rule of $\boldsymbol{\mathcal{S}}$. Hence, for each $n < \boldsymbol{\omega}$ there exists a finite subset $\Delta'_n(x_0, \dots, x_{n-1}, y, u_0, \dots, u_{m_n-1})$ of $\forall\bar{\vartheta}\, \Delta_n(x_0, \dots, x_{n-1}, y, \bar{\vartheta})$ such that

$$\frac{\Delta'_n(x_0, \dots, x_{n-1}, y, u_0, \dots, u_{m_n-1}),\ x_0, \dots, x_{n-1}}{y}$$

is a finite rule of $\boldsymbol{\mathcal{S}}$. It is an interesting open problem whether every deductive system $\boldsymbol{\mathcal{S}}$ with a PGDD system over Leibniz theories has a presentation in which the above rules are the only proper rules of inference.

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