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SOME MODEL THEORY FOR PROBABILITY STRUCTURES

A b s t r a c t. In this paper we study some model theory for Gaifman probability structures. A classical result of Horn-Tarski concerning the extension of probabilities on Boolean algebras will allow us to prove some preservation theorems for probability structures, the model-companion of logical probability, etc. extending some classical results in eastern model theory.

1. Introduction

Let L be a first-order language with a nonempty set C of constant symbols. If U is a set containing C, then $L(U)$ will be the language obtained by adjoining to L the constants of $U - C$. We shall write $\mathbf{c} \in C$ instead of $c_1, \ldots, c_n \in C$, $\varphi(\mathbf{x})$ instead of $\varphi(x_1, \ldots, x_n)$, $\varphi(\mathbf{c})$ instead of $\varphi(c_1, \ldots, c_n)$, etc. We shall denote by E the set of sentences of L and by $E(U)$ the set

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of sentences of $L(U)$. B (resp. $B(U)$) will denote the Lindebaum-Tarski algebra of L (resp. $L(U)$). Of course, B is isomorphic to a Boolean subalgebra of $B(U)$. The equivalence class of a sentence φ will be denoted by $[\varphi]$.
If A is an arbitrary Boolean algebra, then a *probability* on A is a function $m : A \to [0, 1]$ such that
(1.1) $m(a \vee b) = m(a) + m(b)$ for $a, b \in A$ such that $a \wedge b = 0$;
(1.2) $m(1) = 1$.

A *(logical) probability* on L is a probability μ on a subalgebra of B. Denote by $dom(\mu)$ the domain of μ.
A *probability structure* is a pair (U, u), where $C \subseteq U$ and u is a probability on the Boolean algebra $B(U)$ which satisfies the Gaifman condition:
(G) for any sentence $\exists \mathbf{x}\varphi(\mathbf{x})$ in $L(U)$ we have that
$u([\exists \mathbf{x}\varphi(\mathbf{x})]) = sup\{u([\varphi(a_1)] \vee \ldots \vee [\varphi(a_n)]) : a_1, \ldots, a_n \in U, n \in \omega\}$.
We shall also use the dual form of (G):
(G°) for any sentence $\forall \mathbf{x}\varphi(\mathbf{x})$ in $L(U)$ we have that
$u([\forall \mathbf{x}\varphi(\mathbf{x})]) = inf\{u([\varphi(a_1)] \wedge \ldots \wedge [\varphi(a_n)]) : a_1, \ldots, a_n \in U, n \in \omega\}$.
If μ is a probability on L and (U, u) is a probability structure, then (U, u) is a *model* of μ, denoted by $(U, u) \models \mu$, if μ is equal to the restriction of u to $dom(\mu)$. Recall the Gaifman completeness theorem [4]:

Theorem 1.1. Every probability μ on L has a model (U, u).

Let A be an arbitrary Boolean algebra, A' a subalgebra of A and m a probability on A'. Define the *interior measure* m_* and the *exterior measure* m^* of m by putting for any $a \in A$:
(1.3) $m_*(a) = sup\{m(x) : x \in A, x \leq a\}$;
(1.4) $m^*(a) = inf\{m(x) : x \in A, a \leq x\}$
We shall use the following Horn-Tarski result [6]:

Theorem 1.2. Let A be a Boolean algebra, A' a subalgebra of A, $a \in A - A'$ and $A'(a)$ the subalgebra of A generated by $A' \cup \{a\}$. If m is a probability on A' and $r \in [0, 1]$, then the following are equivalent:
(i) m can be extended to a probability m' on $A'(a)$ such that $m'(a) = r$;
(ii) $m_*(a) \leq r \leq m^*(a)$.

If (U, u) is a probability structure, φ is a sentence in $L(U)$ and $r \in [0, 1]$, then we shall say that (U, u) *satisfies* the pair (φ, r) ($(U, u) \models (\varphi, r)$) if

$u([\varphi]) = r$. A pair (φ, r), where $\varphi \in E, r \in [0,1]$, is *consistent* with a probability μ on L if there is a model of μ that satisfies (φ, r).

Lemma 1.3. Assume that μ is a probability on L, $\varphi \in E$ and $r \in [0,1]$. Then, the following are equivalent:
(i) (φ, r) is consistent with μ;
(ii) $\mu_*([\varphi]) \leq r \leq \mu^*([\varphi])$.

Proof. (i)$\Rightarrow$ (ii) Assume that there is a model (U, u) of μ that satisfies (φ, r). Then, for any $\psi \in E$,

$[\psi] \in dom(\mu), \vdash \psi \to \varphi \Rightarrow \mu([\psi]) = u([\psi]) \leq u([\varphi]) = r.$

Hence, $\mu_*([\varphi]) \leq r$ and, similarly, $r \leq \mu^*([\varphi])$.
(ii)$\Rightarrow$ (i) In accordance to Theorem 1.2 there exists a probability η that extends μ such that $[\varphi] \in dom(\eta)$ and $\eta([\varphi]) = r$. By Theorem 1.1, there is a model (U, u) of η, so $u([\varphi]) = \eta([\varphi]) = r$. Hence, $(U, u) \models (\varphi, r)$. ■

Let (U, u) and (V, v) be two probability structures such that $U \subseteq V$. (U, u) is a *substructure* of (V, v), denoted by $(U, u) \subseteq (V, v)$, if for any quantifier-free sentence φ in $L(U)$, $u([\varphi]) = v([\varphi])$. (U, u) is an *elementary substructure* of (V, v), denoted by $(U, u) \prec (V, v)$, if for any sentence φ in $L(U)$, $u([\varphi]) = v([\varphi])$.

Lemma 1.4. Let (U, u) be a substructure of (V, v). Then,
(i) for any existential sentence φ in $L(U)$, $u([\varphi]) \leq v([\varphi])$;
(ii) for any universal sentence φ in $\mathrm{L}(U)$, $v([\varphi]) \leq u([\varphi])$.

Proof. Using the conditions (G) and (G°). ■

Let $(U_\alpha, u_\alpha), \alpha < \lambda$ be a chain of probability structures. If $U = \cup_{\alpha<\lambda} U_\alpha$ then there exists a unique probability u on $B(U)$ such that $u|_{B(U_\alpha)} = u_\alpha$ for any $\alpha < \lambda$. The probability structure (U, u) will be called the *union* of the chain $(U_\alpha, u_\alpha), \alpha < \lambda$. This construction appears implicitly in the proof of Theorem 1 of [4].

Lemma 1.5. ([4]) If $(U_\alpha, u_\alpha), \alpha < \lambda$ is an elementary chain of probability structures and (U, u) is its union, then (U_α, u_α) is an elementary substructure of (U, u) for all $\alpha < \lambda$.

Proof. See the proof of Theorem 1 of [4] ■

For any U with $C \subseteq U$, let us denote by $D(U)$ the set of basic sentences in $L(U)$ and by $E_0(U)$ the set of quantifier-free sentences in $L(U)$. $B_0(U)$ will be the subalgebra of equivalence classes of quantifier-free sentences in $L(U)$.

2. Existentially closed probability structures

Let (U,u) be a substructure of (V,v). We say that (V,v) is an *existential extension* of (U,u), denoted by $(U,u) \prec_\forall (V,v)$, if $u([\varphi]) = v([\varphi])$ for any existential sentence φ in $L(U)$.
One remarks that (V,v) is an *existential extension* of (U,u) iff $u([\varphi]) = v([\varphi])$ for any universal sentence φ in $L(U)$.

Proposition 2.1. If $(U,u) \subseteq (V,v)$, then the following are equivalent:
(i) $(U,u) \prec_\forall (V,v)$;
(ii) there is an extension (W,w) of (V,v) such that $(U,u) \prec (W,w)$.

Proof. (i)⇒(ii) Suppose that $(U,u) \prec_\forall (V,v)$. We shall prove that there is a probability η on $L(V)$ such that:
(I) $B(U) \cup \{[\varphi] : \varphi \in D(V)\} \subseteq dom(\eta)$;
(II) η extends u;
(III) $\eta([\varphi]) = v([\varphi])$ for any $\varphi \in D(V)$.
Consider $\varphi \in E(U)$, $\mathbf{a} \in V - U$ and $\psi(\mathbf{a}) \in D(V)$ such that $\vdash \varphi \leftrightarrow \psi(\mathbf{a})$. It follows that $\vdash \varphi \leftrightarrow \forall \mathbf{x} \psi(\mathbf{x})$, therefore, since $\forall \mathbf{x}\psi(\mathbf{x})$ is an universal sentence in $L(U)$, $u([\varphi]) = u([\forall \mathbf{x}\psi(\mathbf{x})]) = v([\forall \mathbf{x}\psi(\mathbf{x})]) = u([\psi(\mathbf{a})])$.
Let now $\varphi \in D(V)$ such that $[\varphi] \notin B(U)$. We shall prove that
(2.1) $u_*([\varphi]) \leq v([\varphi]) \leq u^*([\varphi])$.
Let $\psi \in E(U)$ and $\mathbf{a} \in V - U$ such that $\vdash \psi \rightarrow \varphi(\mathbf{a})$. It follows that $\vdash \psi \rightarrow \forall \mathbf{x}\psi(\mathbf{x})$ and $\vdash \forall \mathbf{x}\varphi(\mathbf{x}) \leftrightarrow \varphi(\mathbf{a})$, hence $u([\psi]) \leq u([\forall \mathbf{x}\psi(\mathbf{x})]) = v([\forall \mathbf{x}\psi(\mathbf{x})]) = v([\varphi])$. By Theorem 1.2, there is a probability μ on the Boolean algebra $B(U)([\varphi])$ such that μ extends u and $\mu([\varphi]) = v([\varphi])$. Now, the construction of η will be done by a transfinite induction. Applying Theorem 1.1, there is a probability structure (W,w) fulfilling the required conditions.
(i)⇒(ii)Straightforward. ■

A probability μ on L is *model-complete* if $(U,u) \subseteq (V,v)$ implies $(U,u) \prec (V,v)$ for any models $(U,u), (V,v)$ of μ.

Proposition 2.2. For a probability μ on L the following are equivalent:
(i) μ is model-complete;
(ii) for any models $(U, u), (V, v)$ of μ, $(U, u) \subseteq (V, v)$ implies $(U, u) \prec_\forall (V, v)$.

Proof. Straightforward, using Proposition 2.1 and the proof of Robinson's model completeness test ([5],[7]). ■

A probability μ on L is called *inductive* if the class of its models is closed under unions of chains. By Lemma 1.5, every model-complete probability is inductive.
A probability structure (U, u) is *existentially closed* if $(U, u) \subseteq (V, v)$ implies that (V, v) is an existential extension of (U, u).

Proposition 2.3. If μ is an inductive probability on L, then any model of μ can be embedded into an existentially closed model of μ.

Proof. Let (U, u) be a model of μ and k be the cardinal number of the language $L(U)$. Consider an enumeration $\varphi_\alpha, \alpha < k$ of the existential sentences in $L(U)$. We shall define by induction a chain $(U_\alpha, u_\alpha), \alpha < k$ of models of μ.
(I) $(U_0, u_0) = (U, u)$;
(II) Assume that $\alpha = \beta + 1$ and that there was defined (U_γ, u_γ) for any $\gamma \leq \beta$. Since (U_β, u_β) is a model of μ, we have that there is a probability η on $L(U_\beta)$ such that η extends μ and the restriction of u_β to $B_0(U_\beta)$. By Lemma 1.3, there is a model (V, v) of η such that $v([\varphi_\beta]) = \eta^*([\varphi_\beta])$. We shall prove that for any model (W, w) of μ the following implication holds:
(2.2) $(U_\beta, u_\beta) \subseteq (W, w) \Rightarrow w([\varphi_\beta]) \leq v([\varphi_\beta])$.
Suppose that $(U_\beta, u_\beta) \subseteq (W, w)$. For any $\psi \in E_0(U_\beta)$ we get that $w([\psi]) = u_\beta([\psi]) = \eta([\psi])$, so (W, w) is a model of η. Since $(\varphi_\beta, w([\varphi_\beta]))$ is consistent with η, applying Lemma 1.3 we obtain that $w([\varphi_\beta]) \leq \eta^*([\varphi_\beta]) = v([\varphi_\beta])$. That is, (2.2) holds. We define $(U_\alpha, u_\alpha) = (V, v)$.
(III) If α is a limit ordinal, then (U_α, u_α) is the union of the chain (U_β, u_β), $\beta < \alpha$.
Hence, we have defined the chain $(U_\alpha, u_\alpha), \alpha < k$ of models of μ. Let (V_1, v_1) be the union of this chain. It follows that the following holds:
(2.3) If φ is an existential sentence in $L(U)$, $(W, w) \models \mu$ and $(V_1, v_1) \subseteq (W, w)$, then $v_1([\varphi]) = w([\varphi])$.
One can define by induction a chain $(V_n, v_n), n < \omega$ of models of μ such that for any $n < \omega$ a property similar to (2.3) holds. Let (V, v) be the

union of this chain. It follows that (V, v) is an existentially closed model of μ such that $(U, u) \subseteq (V, v)$. ■

If μ is a probability on L, then let us denote by $\mathbf{E}_\mu$ the class of existentially closed models of μ.

Proposition 2.4. Let μ be a probability on L. Then $\mathbf{E}_\mu$ is the unique class $\mathbf{C}$ of models of μ having the following properties:
(i) any model of μ can be embedded into a member of $\mathbf{C}$;
(ii) if $(U, u), (V, v)$ are members of $\mathbf{C}$, then $(U, u) \subseteq (V, v)$ implies $(U, u) \prec_\forall (V, v)$;
(iii) $\mathbf{C}$ is maximal with respect to the conditions (i) and (ii).

3. A preservation theorem

Given a probability structure $(U, u), \varphi \in E(U)$ and $r \in [0, 1]$, let us define
(3.1) $(U, u) \models^* (\varphi, r)$ iff $u([\varphi]) \geq r$;
(3.2) $\mu_\forall = \{(\varphi, \mu_*([\varphi]) : \varphi$ is an universal sentence in $L\}$.

Proposition 3.1. Let μ be a probability on L and (U, u) be a probability structure. Then (U, u) can be embedded in a model (V, v) of μ iff $(U, u) \models^* \mu_\forall$.

Proof. Suppose that $(U, u) \subseteq (V, v)$ and that $(V, v) \models \mu$. Let φ be an universal sentence in $L(U)$. It follows that for any $[\psi] \in dom(\mu)$ such that $\vdash \psi \to \varphi$, we have that $\mu([\psi]) = v([\psi]) \leq u([\psi]) \leq u([\varphi])$. Therefore, $\mu_*([\varphi]) \leq u([\varphi])$, so $(U, u) \models^* \mu_\forall$.
Conversely, suppose that $(U, u) \models^* \mu_\forall$. Let $[\varphi] \in dom(\mu)$ and $\psi \in D(U)$ such that $\vdash \psi \leftrightarrow \varphi$. We shall prove that $\mu([\varphi]) = u([\psi])$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be the constants from $U - C$ that appear in ψ. We get that $\vdash \varphi \leftrightarrow \psi(\mathbf{a})$, so $\vdash \neg\varphi \leftrightarrow \neg\psi(\mathbf{a})$, hence $\vdash \varphi \leftrightarrow \forall \mathbf{x} \psi(\mathbf{x})$ and $\vdash \neg\varphi \leftrightarrow \neg\forall \mathbf{x} \psi(\mathbf{x})$. Since $\forall \mathbf{x} \psi(\mathbf{x})$ is an universal sentence, it follows that $(U, u) \models^* (\forall \mathbf{x} \psi(\mathbf{x}), \mu_*([\forall \mathbf{x} \psi(\mathbf{x})]))$, that is $u([\forall \mathbf{x} \psi(\mathbf{x})]) \geq \mu_*([\forall \mathbf{x} \psi(\mathbf{x})])$. From the condition (G°) we obtain that $u([\psi(\mathbf{a})]) \geq u([\forall \mathbf{x} \psi(\mathbf{x})])$, hence $u([\psi(\mathbf{a})]) \geq u_*([\forall \mathbf{x} \psi(\mathbf{x})]) = u_*([\varphi])$. Similarly, from $(U, u) \models^* (\forall \mathbf{x} \neg\psi(\mathbf{x}), \mu_*([\forall \mathbf{x} \neg\psi(\mathbf{x}])),$ we get that $u([\neg\psi(\mathbf{a})]) \geq \mu_*([\neg\psi])$.
By [6], Corollary 1.18, p. 476 we get that $\mu_*([\neg\varphi]) + \mu^*([\varphi]) = 1$, hence $1 - u([\psi(\mathbf{a})]) \geq 1 - \mu^*([\varphi])$. We have got that $\mu_*([\varphi]) \leq u([\psi(\mathbf{a})]) \leq \mu^*([\varphi])$. Since $[\varphi] \in dom(\mu)$, we get that $\mu_*([\varphi]) = \mu([\varphi]) = \mu^*([\varphi])$. Hence,

$\mu([\varphi]) = u([\psi])$.
Let now $\varphi \in D(U)$ such that $[\varphi] \notin dom(\mu)$. We shall prove that μ can be extended to a probability η such that $[\varphi] \in dom(\eta)$ and $\eta([\varphi]) = u([\varphi])$. By Theorem 1.2, it suffices to prove that $\mu_*([\varphi]) \leq u([\varphi]) \leq \mu^*([\varphi])$. If $\varphi = \varphi(\mathbf{a})$, with $\mathbf{a} \in U - C$, then $\varphi(\mathbf{a}), \neg\varphi(\mathbf{a}) \in D(U)$ and $\vdash \varphi(\mathbf{a}) \leftrightarrow \forall\mathbf{x}\varphi(\mathbf{x}), \vdash \neg\varphi(\mathbf{a}) \leftrightarrow \forall\mathbf{x}\neg\varphi(\mathbf{x})$. Since $(U,u) \models^* (\forall\mathbf{x}\varphi(\mathbf{x}), \mu_*([\forall\mathbf{x}\varphi(\mathbf{x})]))$, we get that $u([\varphi]) = u([\forall\mathbf{x}\varphi(\mathbf{x})]) \geq \mu_*([\forall\mathbf{x}\varphi(\mathbf{x})]) = \mu_*([\varphi])$. Similarly, we have that $u([\neg\varphi]) \geq \mu_*([\neg\varphi])$. It follows that $u([\varphi]) \leq \mu^*([\varphi])$.
We can define by transfinite induction a probability ε on $L(U)$ such that
(3.3) $dom(\mu) \cup \{[\varphi] : \varphi \in D(U)\} \subseteq dom(\varepsilon)$;
(3.4) ε extends μ;
(3.5) $\varepsilon([\varphi]) = u([\varphi])$ for any $\varphi \in D(U)$.
Applying Theorem 1.2, there is a model (V,v) of ε such that $(V,v) \models \mu$ and $(U,u) \subseteq (V,v)$. ■

A probability μ on L is preserved under probability substructures if $(U,u) \subseteq (V,v)$ and $(V,v) \models \mu$ implies $(U,u) \models \mu$.

Corollary 3.2. Let μ be a probability on L. The following are equivalent:
(i) μ is preserved under probability substructures;
(ii) for any probability structure (U,u), $(U,u) \models^* \mu_\forall$ implies $(U,u) \models \mu$.

4. A characterization of inductive probabilities

A class Σ of probability structures is *inductive* if it is closed under unions of chains. Thus, a probability μ is inductive iff the class of its models is inductive. For a probability μ on L, let us denote
(4.1) $\Sigma_\mu = \{(U,u) : (U,u) \models^* \mu_\forall\}$
(4.2) $\mu_{\forall\exists} = \{(\varphi, \mu_*([\varphi])) : \varphi \text{ is an } \forall\exists - \text{sentence in } L\}$.

Proposition 4.1. Σ_μ is an inductive class.

Proof. Let (U_α, u_α), $\alpha < \lambda$ be a chain in Σ_μ and (U,u) be its union. Let $\forall\mathbf{x}\varphi(\mathbf{x})$ be an universal sentence in L. Then by the Gaifman condition (G):

$$\begin{aligned} u([\forall\mathbf{x}\varphi(\mathbf{x})]) &= inf\{u([\varphi(a_1)] \wedge \ldots \wedge [\varphi(a_n)]) : a_1, \ldots, a_n \in U\} \\ &= inf_{\alpha<\lambda} inf\{u([\varphi(a_1)] \wedge \ldots \wedge [\varphi(a_n)]) : a_1, \ldots, a_n \in U\} \\ &= inf_{\alpha<\lambda}\{u_\alpha([\forall\mathbf{x}\varphi(\mathbf{x})])\} \geq \mu_*([\forall\mathbf{x}\varphi(\mathbf{x})]) \end{aligned}$$

because $(U_\alpha, u_\alpha) \models^* (\forall \mathbf{x}\varphi(\mathbf{x}), \mu_*([\forall \mathbf{x}\varphi(\mathbf{x})]))$ for any $\alpha < \lambda$. It follows that $(U, u) \models^* (\forall \mathbf{x}\varphi(\mathbf{x}), \mu_*([\forall \mathbf{x}\varphi(\mathbf{x})]))$, so $(U, u) \models^* \mu_\forall$. ■

Lemma 4.2. If (V, v) is an existential extension of (U, u) and φ is an $\forall\exists$-sentence in $L(U)$ then $v([\varphi]) \leq u([\varphi])$.

Lemma 4.3. Let (U_α, u_α), $\alpha < \lambda$ be a chain of probability structures, (U, u) be its union, φ an $\forall\exists$-sentence in L and $r \in [0, 1]$. If $(U_\alpha, u_\alpha) \models^* (\varphi, r)$ for $\alpha < \lambda$ then $(U, u) \models^* (\varphi, r)$.

Proposition 4.4. For a probability μ on L the following are equivalent:
(i) μ is inductive;
(ii) For any probability structure (U, u), (U, u) is a model of μ iff $(U, u) \models^* \mu_{\forall\exists}$.

Proof. (i)⇒(ii) Assume that $(U, u) \models \mu$. Then for any $\varphi \in E$ and $\psi \in dom(\mu)$, $\vdash \psi \to \varphi$ implies $\mu([\psi]) = u([\psi]) \leq u([\varphi])$, so $\mu_*([\varphi]) \leq u([\varphi])$. Thus $(U, u) \models^* (\varphi, \mu_*([\varphi]))$. Now assume that $(U, u) \models^* \mu_{\forall\exists}$. We shall show that there is an existential extension (U_1, u_1) of (U, u) which is a model of μ. It suffices to prove that there exists a probability η on $L(U)$ such that the following hold:
(4.3) $\{[\varphi] : \varphi$ is an existential sentence in $L(U)\} \subseteq dom(\mu)$;
(4.4) η extends μ;
(4.5) If φ is an existential sentence in $L(U)$ then $\eta([\varphi]) = u([\varphi])$.
Consider $[\varphi] \in dom(\mu)$ and $\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})$ an existential sentence in $L(U)$ (with $\mathbf{a}$ in $U - C$) such that $\vdash \varphi \leftrightarrow \exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})$. We shall prove that $\mu([\varphi]) = u([\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})])$. We remark that $\vdash \varphi \leftrightarrow \forall \mathbf{y}\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y})$, $\vdash \neg\varphi \leftrightarrow \forall \mathbf{y}\neg\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y})$
for an appropriate choice of $\mathbf{y}$. Then one gets
$u([\varphi]) = u([\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})]) \geq u([\forall \mathbf{y}\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y})]) \geq \mu_*([\forall \mathbf{y}\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y})]) = \mu_*([\varphi])$
since $(U, u) \models^* (\forall \mathbf{y}\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y}), \mu_*([\forall \mathbf{y}\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{y})]))$. Similarly one gets
$u([\neg\varphi]) \geq \mu_*([\neg\varphi]) = 1 - \mu_*([\varphi])$, so $\mu_*([\varphi]) \leq u([\varphi]) \leq \mu^*([\varphi])$. But $[\varphi] \in dom(\mu)$ so $\mu_*([\varphi]) = \mu([\varphi]) = \mu^*([\varphi])$, hence $\mu([\varphi]) = u([\varphi]) = u([\exists \mathbf{x}\psi(\mathbf{x}, \mathbf{a})])$.
For any existential sentence φ in $L(U)$ with $[\varphi] \notin dom(\mu)$ we shall establish the inequality
(4.6) $\mu_*([\varphi]) \leq u([\varphi]) \leq \mu^*([\varphi])$.
If $\varphi = \varphi(\mathbf{a})$ with $\mathbf{a}$ in $U - C$ then $\forall \mathbf{x}\varphi(\mathbf{x})$ is an $\forall\exists$-sentence in L, so $(U, u) \models^* (\forall \mathbf{x}\varphi(\mathbf{x}), \mu_*([\forall \mathbf{x}\varphi(\mathbf{x})]))$. Thus

$u([\varphi]) = u([\forall \mathbf{x}\varphi(\mathbf{x})]) \geq \mu_*([\forall \mathbf{x}\varphi(\mathbf{x})]) = \mu_*([\varphi])$. Similarly from $(U, u) \models^*$ $(\forall \mathbf{x}\neg\varphi(\mathbf{x}), \mu_*([\forall \mathbf{x}\neg\varphi(\mathbf{x})]))$ one can infer that $u([\neg\varphi]) \geq \mu_*([\neg\varphi]) = 1 - \mu^*([\varphi])$, so (4.6) holds.
By Theorem 1.2, there is a probability ν defined on the Boolean subalgebra generated by $dom(\mu) \cup \{[\varphi]\}$ such that ν extends μ and $\nu([\varphi]) = u([\varphi])$. We can define by transfinite induction a probability η that satisfies the conditions (4.3)-(4.5).
Using the above argument and Proposition 2.1 one can define by induction a sequence $(U_n, u_n), n < \omega$ of probability structures such that the following conditions hold for all $n < \omega$:
(4.7) $(U_0, u_0) = (U, u)$;
(4.8) $(U_{2n}, u_{2n}) \prec (U_{2n+2}, u_{2n+2})$;
(4.9) $(U_{2n}, u_{2n}) \prec_\forall (U_{2n+1}, u_{2n+1})$;
(4.10) $(U_{2n+1}, u_{2n+1}) \models \mu$;
(4.11) $(U_{2n}, u_{2n}) \models^* \mu_{\forall\exists}$.
If we denote by (W, w) the union of this chain, then applying Lemma 1.5 we get that $(U, u) \prec (W, w)$. Since μ is inductive, (W, w) is a model of μ, hence $(U, u) \models \mu$.
(ii)⇒(i) By Lemma 4.3. ■

5. Model-companion of a probability

Let μ, η be two probabilities on L. We say that η is *model-consistent relative* to μ if any model of μ can be embedded into a model of η; μ and η are *mutually model-consistent* if each is model-consistent relative to the other. The probability η is called a *model-companion* of μ if η is model-complete and μ, η are mutually model-consistent.

Proposition 5.1. Let μ, η be two probabilities on L. If μ is inductive, then the following are equivalent:
(i) η is a model-companion of μ;
(ii) $\mathbf{E}_\mu$ is the class of models of the probability η.

Proof. (i)⇒(ii) We shall prove first that any model (U, u) of η is a model of μ. By Proposition 4.4, it suffices to show that $(U, u) \models^* \mu_{\forall\exists}$. Suppose that φ is an $\forall\exists$-sentence in L. By hypothesis and Proposition 2.3, (U, u) can be embedded into an existentially closed model (V, v) of μ. If ψ is a sentence in L such that $\vdash \psi \rightarrow \varphi$, then by Lemma 4.2,

$\mu([\psi]) = v([\psi]) \le v([\varphi]) \le u([\varphi])$, so $\mu_*([\varphi]) \le u([\varphi])$. This yields $(U,u) \models^* (\varphi, \mu_*([\varphi]))$, i.e. $(U,u) \models^* \mu_{\forall\exists}$.
Let **C** be the class of models of η. Assume (U,u) in **C** and $(U,u) \subseteq (V,v)$ with $(V,v) \models \mu$. Then $(V,v) \subseteq (W,w)$ for some model (W,w) of η, so $(U,u) \prec (W,w)$, η being model-complete. It follows that (V,v) is an existential extension of (U,u), so (U,u) is in E_μ.
In order to prove the converse inclusion $E_\mu \subseteq$ **C** assume (U,u) is an existentially closed model of μ. Thus there exists $(V,v) \models \eta$ such that $(U,u) \subseteq (V,v)$, so (V,v) is an existential extension of (U,u).
Let φ be an $\forall\exists$-sentence in L and $\psi \in E$ such that $\vdash \psi \to \varphi$. Thus $\eta([\psi]) = v([\psi]) \le v([\varphi]) \le u([\varphi])$, by Lemma 4.2. It follows that $\eta_*([\varphi]) \le u([\varphi])$, so $(U,u) \models^* (\varphi, \eta_*([\varphi]))$. Then $(U,u) \models^* \mu_{\forall\exists}$. But η is model-complete, so it is inductive. By Proposition 4.4 one can infer that $(U,u) \models \eta$, so $(U,u) \in$**C**.
(ii)$\Rightarrow$(i) If (U,u) is a model of η, then $(U,u) \subseteq (V,v)$ for some member (V,v) of E_μ, in accordance with Proposition 2.3. By hypothesis, $(V,v) \models \eta$. It is obvious that any model of η is a model of μ.
Assume now (U,u), (V,v) are two models of η such that $(U,u) \subseteq (V,v)$. Thus, they are in E_μ, so (V,v) is an existential extension of (U,u). By Proposition 2.2 η is model-complete. ■

Corollary 5.2. If η, ε are two model-companions of the probability μ on L then the models of η, ε are identical.

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References

[1] J. Barwise and A. Robinson, *Completing theories by forcing*, Annals of Math.Logic, **2** (1970), 119–142.

[2] C.C. Chang and H.J. Keisler, Model theory, North-Holland Publ.Comp., Amsterdam 1973.

[3] G. Georgescu and I. Voiculescu, *Eastern model theory for boolean-valued theories*, Zeit.f.math.Logik und Grund d.Math, **31** (1988), 79–88.

[4] H. Gaifman, *Concerning measures on first-order calculi*, Israel J.Math., **2** (1964), 1–18.

[5] J. Hirschfeld and W.H. Wheeler, *Forcing, arithmetic and division rings*, in: Lecture Notes in Mathematics, Springer 1975.

[6] A. Horn and A. Tarski, *Measures in Boolean algebras*, Trans.Am.Math.Soc., **64** (1948), 467–497.

[7] A. Robinson, Introduction to model theory and to metamathematics of algebra, North-Holland Publ.Comp., Amsterdam 1975.

[8] D. Scott and P. Krauss, *Assigning probabilities to logical formulas*, in: Aspects of inductive logic (Hintikka and Suppes, eds.), North-Holland Publ.Comp., Amsterdam 1966.

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