REPORTS ON MATHEMATICAL LOGIC
35 (2001), 75–85

Joanna GRYGIEL

BOOLEAN CONSTRUCTIONS OF INDEPENDENT SETS OF GENERATORS FOR FILTERS

A b s t r a c t. Let F be a filter in a Boolean algebra. We consider the problem if it possible to construct an independent set of generators for F from any its set of generators. It turns out that the answer to this question depends on the minimal cardinality of the set of generators of F.

Let $\mathcal{B} = \langle B, \wedge, \vee, -, \mathbf{0}, \mathbf{1} \rangle$ be a non-degenerated Boolean algebra. A subset A of B is said to be *independent* (or *free*) iff for any choice of different elements $a_1, \ldots, a_n$ from A we have

$$\varepsilon a_1 \wedge ... \wedge \varepsilon a_n \neq \mathbf{0},$$

where $\varepsilon a \in \{a, -a\}$.

Independent sets were thoroughly investigated not only in algebra (see [6]) but in topology as well (see [1]). It is well known that every independent subset of $\mathcal{B}$ generates some free subalgebra of $\mathcal{B}$.

Let F be a filter of the Boolean algebra $\mathcal{B}$. Let us recall that the set G is a set of generators of the filter F if for every $x \in F$ there are

Received 8 January 2000.

$g_1, ..., g_m \in G$ such that

$$g_1 \wedge ... \wedge g_m \leq x.$$

Then we shall write $F = [G)$.

If there is an independent set G such that $F = [G)$ then we shall say that F is *freely generated*.

Here we are going to consider the boolean constructibility of an independent set of generators of a freely generated filter from any its set of generators.

One should stress that the defined above notion of generators of filters (which has logical roots) does not coincide with the (algebraical) notion of generators of Boolean algebras. Thus the set of generators of a proper filter (even ultrafilter) of a Boolean algebra $\mathcal{B}$ need not generate the algebra $\mathcal{B}$. Therefore, theorems concerning construction of independent sets of generators of Boolean algebras have nothing to do with theorems about the construction of independent sets of generators of filters in Boolean algebras.

The main problem is if the filter $[G)$ generated by a set G has got an independent set of generators in the subalgebra $\mathcal{B}(\mathcal{G})$ of $\mathcal{B}$ generated by the set G. In other words, we are going to say, when it is possible, for a freely generated filter in $\mathcal{B}$ to built up an independent set of its generators consisting of Boolean combinations of any generating it set.

The possibility of constructing such the set of generators from the given one depends, of course, on the set from which we start out our construction and on the filter itself. In particular, it depends on the minimal cardinality of its set of generators.

The similar parameters appear if one considers the problem of the existence of the independent set of generators of filters in Boolean algebras (see [3]).

Let F be a filter in $\mathcal{B}$ and let $m(F)$ denotes the minimal cardinality of the set of its generators.

If $m(F) = 1$ then F is a principal filter.

If $m(F) = \omega$ we will call F countably generated. It was proved in [2] that every countably generated filter containing only finitely many coatoms of the algebra is freely generated. Hence, for instance, every countably generated filter of a free Boolean algebra is freely generated.

If $m(F)$ is a regular and uncountable cardinal we shall say that F is regularly generated. We proved in [3] that F is freely generated if $m(F)$ is either a regular cardinal or a singular cardinal with uncountable cofinality.

In this paper we are going to show that for every countably generated filter there is a set of generators from which it is impossible to construct by means of boolean operators an independent set of generators. In contrary, if F is a regularly and freely generated filter then we can construct its independent set of generators from any generating it set. The same situation happens if $m(F)$ is a singular cardinal with uncountable cofinality.

First, let us observe that:

Theorem 1. *If L is an infinite descending chain of elements of $\mathcal{B}$ then there is no independent set of generators of the filter $[L)$ built from boolean combinations of elements of L.*

Proof. Let L be an infinite descending chain of elements of $\mathcal{B}$ and let $\mathcal{B}'$ be the subalgebra of $\mathcal{B}$ generated by L. No independent set of generators for the filter $[L)$ of $\mathcal{B}$ is included in $\mathcal{B}'$.

Indeed, let F be the filter generated by L in $\mathcal{B}'$. Since $\mathcal{B}'$ is superatomic (see [4]) then the filter F is not freely generated (see [3]). If the filter $[L)$ in $\mathcal{B}$ were freely generated by a subset of B', the same subset would be the set of generators of F in $\mathcal{B}'$, whence it is not the case. ■

Clearly, each countably generated filter is generated by an infinite and descending chain. Hence

Corrolary 1. *For every countably generated filter F there is a set of generators G of F such that no independent set of generators for F can be built up by use of Boolean combinations of the elements of G.*

The situation is quite different if regularly generated filters are concerned.

Theorem 2. *Let F be a freely and regularly generated filter of a Boolean algebra $\mathcal{B}$. Then for every set Y of generators of F we have:*

(i) *The set of all finite meets of elements of Y contains an independent subset of the same cardinality as Y.*

(ii) *An independent set of generators of F is included in the subalgebra of $\mathcal{B}$ generated by the set Y.*

The proof of this theorem is based on the following set-theoretic result (see [4]):

Lemma 1 (Δ - system Lemma). *If λ is a regular uncountable cardinal and $\mathcal{A} = \{A_\alpha : \ \alpha < \lambda\}$ is a system of finite sets then there is a subsystem $\mathcal{B}$ of $\mathcal{A}$ and a finite set Δ such that $card\,\mathcal{A} = card\,\mathcal{B}$ and $A_i \cap A_j = \Delta$ for any different sets $A_i, A_j \in \mathcal{B}$.*

Proof of Theorem 2. Let X be an independent set of generators of the filter F. As X and Y generate the same filter then for any $a \in X$ there is a finite set $Y(a) \subseteq Y$ such that $a \geq \bigwedge Y(a)$ and at the same time for every $b \in Y$ there is a finite set $X(b) \subseteq X$ such that $b \geq \bigwedge X(b)$. Using the axiom of choice one can uniquely define $X(b)$ and $Y(a)$.

Let $Z(a) = \bigcup\{X(b);\ b \in Y(a)\}$.
Then for every $a \in X$ the set $Z(a)$ is a finite subset of X and

$$(1) \quad a \geq \bigwedge Y(a) \geq \bigwedge Z(a).$$

As X is an independent, we have

$$(2) \quad a \in Z(a) \text{ for any } a \in X.$$

Let us consider the family $\mathcal{Z} = \{Z(a) : \ a \in X\}$. Because X is an independent set then (see [3])

$$(3) \quad \operatorname{card} X = m(F).$$

Since every element of the family $\mathcal{Z}$ is a finite set and $m(F)$ is a regular cardinal then the cardinality of the family equals $m(F)$. By Lemma 1, there is a set $Z \subseteq X$ and a finite set $S \subseteq X$ such that:

$$(4) \quad \operatorname{card} Z = \operatorname{card} X,$$

$$(5) \quad Z(a_1) \cap Z(a_2) = S \text{ for any different } a_1, a_2 \in Z.$$

Let $Y_1 = \{\bigwedge Y(a) :\ a \in Z \setminus S\}$.
We are going to show that

$$(6) \quad \operatorname{card} Y_1 = m(F).$$

Let a_1, a_2 be different elements of $Z \setminus S$.
If $\bigwedge Y(a_1) = \bigwedge Y(a_2)$ then, by (1), $a_2 \geq \bigwedge Y(a_1) \geq \bigwedge Z(a_1)$.

Thus, as $a_1 a_2 \in X$, which is an independent set, we get $a_2 \in Z(a_1)$. Hence, by (2) $a_2 \in Z(a_1) \cap Z(a_2)$, and then from (5) $a_2 \in S$, which contradicts our assumptions.
Therefore $\bigwedge Y(a_1) \neq \bigwedge Y(a_2)$ for any different $a_1, a_2 \in Z \setminus S$. It means that $\operatorname{card} Y_1 = \operatorname{card}(Z \setminus S)$. Hence, by (3) and (4) we get (6).

Now, we shall show that Y_1 is an independent set.
Let us suppose that

$$\bigwedge Y(a_1) \wedge ... \wedge \bigwedge Y(a_n) \leq \bigwedge Y(a_{n+1}) \vee ... \vee \bigwedge Y(a_m)$$

for some different $a_1, ..., a_m \in Z \setminus S$.
Then, by (1)

$$\bigwedge \{x : \ x \in \bigcup_{i=1}^{n} Z(a_i)\} \leq \bigvee_{j=n+1}^{m} a_j,$$

which, according to the independence of X, means that for some j, where $n+1 \leq j \leq m$, we have

$$a_j \in \bigcup_{j=1}^{n} Z(a_i),$$

but, from (2) $a_j \in Z(a_j)$, hence, by (5),

$$a_j \in \bigcup_{i=1}^{n} Z(a_i) \cap Z(a_j) = S,$$

which is impossible.

We have shown that Y_1 is an independent subset of Y of cardinality $m(F)$, what proves (a).

Since $\operatorname{card}(Z \setminus S) = \operatorname{card} X$ then there is a bijection $f : Z \setminus S \longrightarrow X$. In addition, as by (5), the sets $Z(c) \setminus S$ for $c \in Z$ are pairwise disjoint, hence the set

$$\{c \in Z \setminus S : \ (Z(c) \setminus S) \cap Z(a) \neq \emptyset\}$$

is finite for every $a \in X$.
Let us denote for $a \in Z \setminus S$:

$$(7) \ \ G(a) = \{c \in Z \setminus S : \ (Z(c) \setminus S) \cap Z(f(a)) \neq \emptyset\}.$$

$$(8) \ \ T(a) = \bigcup \{Y(c) : \ c \in G(a)\}.$$

$$(9) \ \ W = \{ \bigwedge Y(a) \wedge (- \bigwedge T(a) \vee \bigwedge Y(f(a))) : \ a \in Z \setminus S\}.$$

We are going to show that W is an independent set of generators of the filter F.
First, let us observe that $W \subseteq F$. Indeed, $f(a) \in X$ for every $a \in Z \setminus S$. Moreover, $Y(a) \subseteq Y \subseteq F$ for every $a \in X$, so $\bigwedge Y(a) \in F$. Therefore

$$\bigwedge Y(a) \wedge (-\bigwedge T(a) \vee \bigwedge Y(f(a))) \in F$$

for every $a \in Z \setminus S$.

It is easy to show that $X \subseteq [W)$. Indeed, for every $c \in X$ there is an $a \in Z \setminus S$ such that $c = f(a)$. So, by (1)

$$c \geq \bigwedge Y(f(a)) \geq \bigwedge Y(a) \wedge (-\bigwedge T(a) \vee \bigwedge Y(f(a))) \wedge \bigwedge T(a).$$

However, $\bigwedge T(a) \in [W)$ because $\bigwedge Y(b) \in [W)$ for every $b \in Z \setminus S$. Thus $c \in [W)$ for every $c \in X$, which means that the set W generates the filter F.

In the end, we shall prove that the set W is independent.

Let us suppose that

$$\bigwedge Y(b_1) \wedge ... \wedge \bigwedge Y(b_n) \wedge (-\bigwedge T(b_1) \vee \bigwedge Y(f(b_1))) \wedge ... \wedge$$

$$\wedge (-\bigwedge T(b_n) \vee \bigwedge Y(f(b_n))) \leq \bigwedge Y(c_1) \vee ... \vee \bigwedge Y(c_m)$$

for some different elements $b_1, ..., b_n, c_1, ..., c_m \in Z \setminus S$. ¿From (1) we get

$$\bigwedge Y(c_1) \vee ... \vee \bigwedge Y(c_m) \leq \bigvee_{i=1}^{m} c_i,$$

hence

$$\bigwedge Y(b_1) \wedge ... \wedge \bigwedge Y(b_n) \wedge (-\bigwedge T(b_1) \vee \bigwedge Y(f(b_1))) \wedge \tag{10}$$

$$\wedge ... \wedge (-\bigwedge T(b_n) \vee \bigwedge Y(f(b_n))) \leq \bigvee_{i=1}^{m} c_i.$$

Let us suppose that $c_i \in G(b_n)$ for some $1 \leq i \leq m$. Then, by (1) and (8) we have $c_i \geq \bigwedge Y(c_i) \geq \bigwedge T(b_n)$, so

$$-c_i \leq -\bigwedge T(b_n) \vee \bigwedge Y(f(b_n)),$$

and hence, by (10) we get

$$\bigwedge Y(b_1) \wedge ... \wedge \bigwedge Y(b_n) \wedge (-\bigwedge T(b_1) \vee \bigwedge Y(f(b_1))) \wedge$$

$$\wedge ... \wedge (-\bigwedge T(b_{n-1}) \vee \bigwedge Y(f(b_{n-1}))) \leq \bigvee_{i=1}^{m} c_i.$$

Thus, we can assume that there is an r such that $0 \leq r \leq n$ and

$$\bigwedge Y(b_1) \wedge ... \wedge \bigwedge Y(b_n) \wedge Y(f(b_1)) \wedge ... \wedge \bigwedge Y(f(b_r)) \leq \bigvee_{i=1}^{m} c_i,$$

and additionaly,

$$(11) \;\; c_i \notin G(b_j) \text{ for } 1 \leq i \leq m, \; 1 \leq j \leq r.$$

Therefore, by (1)

$$\bigwedge Z(b_1) \wedge ... \wedge \bigwedge Z(b_n) \wedge \bigwedge Z(f(b_1)) \wedge ... \wedge \bigwedge Z(f(b_r)) \leq \bigvee_{i=1}^{m} c_i.$$

As the set X is independent, then there is an $i \leq m$ such that

$$c_i \in Z(b_1) \cup ... \cup Z(b_n) \cup Z(f(b_n)) \cup ... \cup Z(f(b_r)).$$

Then, by (2), one of the following two possibilities must hold:

1. $c_i \in Z(b_j) \cap Z(c_i)$ for some $1 \leq j \leq n$,
2. $c_i \in Z(f(b_j)) \cap Z(c_i)$ for some $1 \leq j \leq r$.

However, in the first case, we have by the assumption $b_j \neq c_i$ and then from (5) $c_i \in S$, which contradicts our assumptions.
In the second case, by (7) and (11) we get

$$(Z(c_i) \setminus S) \cap Z(f(b_j)) = \emptyset,$$

and again we have $c_i \in S$, which is impossible.

It means that W is an independent set of generators of the filter F. Moreover, we conclude by (8) and (9) that elements of W are built from the elements of the set Y by use of boolean operators only, which completes the proof. ■

The similar result holds if $m(F)$ is a singular cardinal with uncountable cofinality:

Theorem 3. *Let F be a freely generated filter of a Boolean algebra $\mathcal{B}$ such that $m(F) = \kappa$, where κ is a singular cardinal and $cf\,\kappa > \omega$. Then from any set of generators of F we can construct by use boolean operators only the independent set of generators of F.*

The proof is analogous to the proof of Theorem 3 but it is based on the following double Δ-system lemma (see [5]):

Lemma 2. *Let κ be a singular cardinal such that $cf\,\kappa > \omega$ and let $< \lambda_\alpha; \alpha < cf\,\kappa >$ be a strictly increasing sequence of successor cardinals with supremum α and such that $\lambda > cf\,\kappa$ for $\alpha < cf\,\kappa$. Suppose $\mathcal{A} =< A_\xi;\ \xi < \kappa >$ is a system of finite sets. Then there exist sets $\Gamma \subseteq cf\,\kappa$ and Δ and sequences $< \Sigma_\alpha;\ \alpha \in \Gamma >$, $< \Delta_\alpha;\ \alpha \in \Gamma >$ satisfying the following conditions:*

(a) *$< \Sigma_\alpha;\ \alpha \in \Gamma >$ is a system of pairwise disjoint subsets of κ with $card\,\Sigma_\alpha = \lambda_\alpha$ for all $\alpha \in \Gamma$,*

(b) *$card\,\Gamma = cf\,\kappa$,*

(c) *$< A_\xi;\ \xi \in \Sigma_\alpha >$ is a Δ-system with kernel Δ_α for every $\alpha \in \Gamma$, i.e. $A_\zeta \cap A_\eta = \Delta_\alpha$ for any different $\zeta, \eta \in \Sigma_\alpha$,*

(d) *for different $\alpha, \beta \in \Gamma$ we have $\Delta_\alpha \cap \Delta_\beta = \Delta$. Further, if $\xi \in \Sigma_\alpha$ and $\eta \in \Sigma_\beta$ then $A_\xi \cap A_\eta = \Delta$.*

Proof of Theorem 3. Let X be an independent set of generators of the filter F and let Y be an arbitrary set of generators of F. We can define the sets $Y(a), X(b)$ and $Z(a)$ in the same way as in the proof of Theorem 2.
Then the cardinality of the family $\{Z(a) :\ a \in X\}$ equals $\operatorname{card} X = m(F) = \kappa$. Hence, by Lemma 2 there are $\Gamma \subseteq cf\,\kappa$ and sequences $< Z_\alpha :\ \alpha \in \Gamma >$ and $< S_\alpha :\ \alpha \in \Gamma >$ fulfilling the conditions (a) – (d) and such that $Z_\alpha \subseteq X$ for every $\alpha \in \Gamma$.
Let $Z = \bigcup_{\alpha\in\Gamma} Z_\alpha$.
Then $Z \subseteq X$ and $\operatorname{card} Z = \operatorname{card} X$, by (b). Now, we can take

$$Y_1 = \{\bigwedge Y(a) :\ a \in Z \setminus \bigcup_{\alpha\in\Gamma} S_\alpha\}.$$

Similarly, as in the case of Theorem 2 we can prove that Y_1 is an independent set of cardinality κ.
Since there is a bijection $f : (Z \setminus \bigcup_{\alpha\in\Gamma} S_\alpha) \to X$ and the sets $Z(c) \setminus \bigcup_{\alpha\in\Gamma} S_\alpha$ are pairwise disjoint for $c \in Z$ we can define the set W by the equation (9) taking $S = \bigcup_{\alpha\in\Gamma} S_\alpha$.
The further proof is analogical as for Theorem 2. ■

There arises the question if it is possible to generalize the costruction in results above to cover the case of any filter F containing any independent set of cardinality $m(F)$. The answer is negative. The mere existence of an independent set of cardinality $m(F)$ in a filter F does not guarantee that the filter is freely generated. For example, in every complete Boolean algebra there is an independent set of cardinality equal to the cardinality of the whole algebra, hence in every ultrafilter F of the algebra there is an independent subset of cardinality not less than $m(F)$. However, no ultrafilter of a complete Boolean algebra is freely generated (see [3]).

By use of Theorems 2 and 3 we can prove that:

Theorem 4. *If a Boolean algebra $\mathcal{B}$ can be homomorphically embedded into a free Boolean algebra then*

1. *every regularly generated filter of $\mathcal{B}$ is freely generated,*
2. *if F is a filter of $\mathcal{B}$ such that $m(F)$ is a singular cardinal with uncountable cofinality then F is freely generated.*

Proof. Let $\mathcal{B}'$ be a free Boolean algebra and let h be an injective homomorphism from $\mathcal{B}$ into $\mathcal{B}'$.

1. Suppose that F is a regularly generated filter of the algebra $\mathcal{B}$. Let Y be a set of generators of F such that $\operatorname{card} Y = m(F)$. Then $h(Y)$ is the set of generators of the filter $F' = [h(F))$ of the algebra $\mathcal{B}'$. Since $m(F') = m(F)$ then F' is a regularly generated filter of $\mathcal{B}'$. Therefore (see [3]) F' is freely generated and then, according to Theorem 2, we can construct from elements of $h(Y)$, by use of boolean operators only, an independent set G of generators of F'. It means that $G \subseteq h(F)$, so $h^{-1}(G) \subseteq F$.

We are going to show that $h^{-1}(G)$ is an independent set of generators of the filter F.
Let us suppose

$$x_1 \wedge ... \wedge x_n \leq y_1 \vee ... \vee y_t,$$

for some different $x_1, ..., x_n, y_1, ..., y_t \in h^{-1}(G)$. Then

$$h(x_1) \wedge ... \wedge h(x_n) \leq h(y_1) \vee ... \vee h(y_t),$$

which is impossible, as $h(x_1), ..., h(x_n), h(y_1), ..., h(y_t)$ are different elements of an independent set G. It means that the set $h^{-1}(G)$ is independent.
Furthermore, for any $y \in F$ there are $x_1, ..., x_k \in G$ such that

$$h(y) \geq x_1 \wedge ... \wedge x_k$$

and then

$$y \geq h^{-1}(x_1) \wedge ... \wedge h^{-1}(x_k),$$

which completes the proof of 1.

2. The proof is similar. ■

Corrolary 2. *If $L = \{a_i\}_{i<\kappa}$ is a descending chain of elements of $\mathcal{B}$ and $cf\kappa$ is an uncountable cardinal then the filter $[L)$ is not freely generated.*

Proof. Let F be a filter generated by the descending chain L. Then $m(F) = cf\kappa$, hence F is regularly generated. If F is freely generated then, according to Theorem 2, we can construct an independent set of generators of F from elements of the chain L by use of boolean operators, which contradicts Theorem 1. ■

Let us observe that we cannot omit here the assumption that the cofinality of κ is an uncountable cardinal. Indeed, if $cf\kappa = \aleph_0$ then $[L)$ is countably generated and it may have an independent set of generators which cannot be constructed from elements of the chain L.

References

[1] B. Balcar, F. Franek, *Independent families in complete Boolean algebras*, TAMS **274**: 607–618, 1982.

[2] J. Grygiel, *Absolutely independent sets of generators of filters in Boolean algebras*, Reports on Mathematical Logic **24**: 25–35, 1990.

[3] J. Grygiel, *Freely generated filters in free Boolean algebras*, Studia Logica **54**: 139–147, 1995.

[4] S. Koppelberg, *Free Constructions*, in Handbook of Boolean algebras, edited by J.D. Monk , R. Bonnet, North Holland, 1989.

[5] J. D. Monk, *Independence in Boolean algebras*, Periodica Mathematica Hungarica **14**: 269–308, 1983.

[6] R. Sikorski, Boolean algebras, Berlin, Gottingen, Heidelberg, 1960.

Institute of Mathematics
Pedagogical College of Częstochowa
Al. Armii Krajowej 13/15
42-200 Częstochowa
Poland