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PROOF THEORY FOR FINITELY VALID SENTENCES

A b s t r a c t. We investigate infinitary sequent calculi which generate the finitely valid sentences of first-order logic, of simple type theory and of transitive closure logic, respectively.

1. Introduction

By a well-known theorem of Trakhtenbrot's [9] the set $Finval$ of finitely valid first-order sentences, i.e. those which are true in all finite models, is not an r.e. set. See [3] for a well-structured proof. Thus no *finitary* calculus can generate the set $Finval$. This result seems to have caused the opinion that there can be no (reasonable) proof theory dealing with $Finval$.

In this paper we give some pieces of proof theory for finitely valid (not only first-order) sentences. We think these pieces are interesting in themselves; they may also be of use in finite model theory. We introduce and investigate a very simple(-minded) *infinitary* inference rule (fin) which

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we call *the Finiteness Rule*. Added to various sequent calculi Σ this rule (fin) yields exactly the finitely valid sentences of the logic codified by Σ. In the present paper we treat three calculi: the calculus for first-order logic, the calculus for simple type theory, and the (ab ovo infinitary) calculus for transitive closure logic. In all three cases also the system enlarged by (fin) admits cut elimination. Finally, we show how to *approximate* finite validity in recursively enumerable ways by means of so-called finiteness schemes.

2. First Order Logic with the Finiteness Rule

Let LK_e be the classical sequent calculus for first-order logic with equality $(=)$. We adopt, however, the following rules for equality (attributed to Maehara) which allow us to eliminate *all* cuts from LK_e-proofs, including the so-called inessential cuts.

$$\frac{u = u, \Phi \implies \Psi}{\Phi \implies \Psi}; \quad \frac{u = v,\, \mathcal{F}[u], \Phi \implies \Psi}{u = v,\, \mathcal{F}[v],\, \Phi \implies \Psi}; \quad \frac{u = v,\, \Phi \implies \Psi,\, \mathcal{F}[u]}{u = v,\, \Phi \implies \Psi,\, \mathcal{F}[v]}$$

Here u, v are any terms.

For $n > 0$, $\exists^{\leq n}$ is short for $\exists\, x_1 \ldots x_n \forall y\; y = x_1 \vee \ldots \vee y = x_n$. Then *the Finiteness Rule* is the following infinitary rule:

$$(fin) \quad \frac{\exists^{\leq n},\ \Phi \implies \Psi \quad \text{for all } n \geq 1}{\Phi \implies \Psi}$$

In order to see how the rule (fin) works, one may prove in $LK_e + (fin) \setminus (cut)$ the following typical examples of first-order finite validities.

(1) $\forall x, y\, (fx = fy \to x = y) \implies \forall y\, \exists x\, fx = y$

(2) $\forall y\, \exists x\, fx = y \implies \forall x, y\, (fx = fy \to x = y)$

(3) $\forall x, y, z\, (R(x, y) \wedge R(y, z) \to R(x, z)), \forall x\, \exists y\, R(x, y) \implies \exists x\, R(x, x)$

(4) R is a linear ordering $\implies \exists x\, \forall y\, (x = y \vee R(x, y)) \wedge \exists x\, \forall y\, (x = y \vee R(y, x))$

We represent proofs in tree form. A proof(-tree) P in $LK_e + (fin)$ is in *fan form*, or P is a *fan proof*, if P has exactly one application of (fin) which occurs as P's last (i.e. lowermost) inference.

Theorem 1 (Fan Form Theorem). *If S is provable in $LK_e + (fin)$ [in $LK_e+(fin)\setminus(cut)$] then S has a fan proof in $LK_e+(fin)$ [in $LK_e+(fin)\setminus(cut)$].*

Proof. Suppose S is provable without an application of (fin). Then by left weakenings we can show $\exists^{\leq n}, S$ for each n. Then we apply (fin) to get S.

Now suppose that P contains an application (α) of (fin) not as its last inference. Then we show how to pull (α) one level down in the proof tree (preserving the endsequent of P) provided that the inference immediately below (α) is not another application of (fin). We illustrate this pulling down procedure for the case of $\wedge$: right. So let $\Phi \implies \Psi, \varphi$ be the conclusion of (α) which is also one premise of the $\wedge$: right in question whose other premise is $\Phi \implies \Psi, \psi$ yielding the conclusion $\Phi \implies \Psi, \varphi \wedge \psi$. To each of the premises $\exists^{\leq n}, \Phi \implies \Psi, \varphi$ of (α) together with $\Phi \implies \Psi, \psi$ we apply a $\wedge$: right obtaining the conclusions $\exists^{\leq n}, \Phi \implies \Psi, \varphi \wedge \psi$. Applying now (fin) we get the conclusion $\Phi \implies \Psi, \varphi \wedge \psi$ of the old $\wedge$: right. — Now we assume that the inference immediate following (α) is another application of (fin), say (β), with the conclusion $\Phi \implies \Psi$. We show how these two applications (α) and (β) of (fin) can be replaced by one application (γ) of (fin). The conclusion of (α) is of the form $\exists^{\leq n_0}, \Phi \implies \Psi$ for some positive integer n_0. Thus the premises of (α) are $\exists^{\leq n}, \exists^{\leq n_0}, \Phi \implies \Psi$ for each positive integer n. A left contraction yields the sequent $\exists^{\leq n_0}, \Phi \implies \Psi$. For each positive integer $n \neq n_0$ we find the sequent $\exists^{\leq n}, \Phi \implies \Psi$ among the premises of (β). Thus we have all premises for a new application (γ) of (fin) with conclusion $\Phi \implies \Psi$. ■

The best known infinitary rule is the ω-rule in arithmetic. There is, of course, no Fan Form Thm about the ω-rule. Indeed, the Fan Form Thm is typical for the Finiteness Rule.

Corollary 2 (to Thm 1). *If a sequent of the form $\exists^{\leq n}, \Phi \implies \Psi$ is provable in $LK_e + (fin)$ then $\exists^{\leq n}, \Phi \implies \Psi$ is already provable in LK_e.*

Theorem 3. *$LK_e+(fin)$ is correct and $LK_e+(fin)\setminus(cut)$ is complete w.r.t. the class of finite models.*

Proof. It is easily seen that $LK_e + (fin)$ is correct w.r.t. the class of finite models.

The completeness of $LK_e + (fin) \setminus (cut)$ w.r.t. finite models can be shown by the familar method of reduction trees for which one may consult [8]. The Fan Form Thm (Thm 1) allows us to start the construction of the reduction tree for a sequent S by inverting the rule (fin) just once. After that, we invert only the rules of $LK_e \setminus (cut)$. — However, instead of developing the completeness proof for $LK_e + (fin) \setminus (cut)$ w.r.t. to finite models from scratch it is also possible to use the completeness of $LK_e \setminus (cut)$ w.r.t. arbitrary models. Suppose then that S is not provable in $LK_e + (fin) \setminus (cut)$. Then there is a fortiori no fan proof P of S in $LK_e + (fin) \setminus (cut)$. This means that there is a positive integer n such that $\exists^{\leq n}, S$ has no proof in $LK_e \setminus (cut)$. Thus there is a model falsifying $\exists^{\leq n}, S$. Hence S is false in a model of size $\leq n$. ■

By Thm 3 we know for semantic reasons that $LK_e+(fin)$ admits cut elimination. But we want to give a *proof-theoretic (syntactic) proof* of the cut elimination theorem for $LK_e + (fin)$. There are two such proofs. — A first syntactic proof runs as follows: Let S be provable in $LK_e + (fin)$. Then by the syntactically proven Thm 1 there is a fan proof P of S. The premises $\exists^{\leq n}, S$ of the (fin) in P are provable in LK_e. For each n apply the cut elimination procedure for LK_e to get a cut-free proof of $\exists^{\leq n}, S$. Then apply (fin) again.

However, one can prove cut elimination for $LK_e+(fin)$ also in a self-contained way that does not rely either on the Fan Form Thm 1 or on the cut elimination theorem for LK_e. Such a proof is carried out in [2], following the pattern described in Schwichtenberg's article on cut elimination in [7].

3. Simple Type Theory with Finiteness Rules

By TK we understand a certain sequent calculus for the classical theory of all finite types, including an extensionality rule. We give a rough sketch of TK.

The *types* are defined as follows

(1) 0 is a type, the type of individuals

(2) If $\tau_1, \ldots, \tau_n$ for $n \geq 1$ are types then $[\tau_1 \ldots \tau_n]$ is a type, the type of n-ary relations with arguments of the respective types $\tau_1, \ldots, \tau_n$

In contradistinction to the type structure used in [6] we assume no types of propositions or of entities into whose construction propositions are entering. Besides that, the formation rules are as in [6]; in particular, we have abstracts $\lambda x_1^{\tau_1} \ldots x_n^{\tau_n} \mathcal{F}[x_1^{\tau_1}, \ldots, x_n^{\tau_n}]$ for relations of type $[\tau_1 \ldots \tau_n]$. The rules of the sequent calculus TK can be obtained from the corresponding positive-negative-part calculus in [6]. The abstraction and quantifier rules are such that full impredicative comprehension is derivable. Hence, TK is a formulation of simple (i.e. unramified) type theory. — We assume as known the notion of a Henkin model for the language of TK. A Henkin model is called *typewise finite* if (the domain of) every type is finite. We shall simply say *finite* for *typewise finite.* Two Henkin models are called *equivalent* if they make the same TK-sentences true.

TK is correct and $TK \setminus (cut)$ is complete w.r.t. Henkin models. This was first proved by Prawitz and Takahashi who thereby settled the socalled Conjecture of Takeuti, viz. that TK admits cut elimination. See [6] or [8]. — The finitely valid TK-sentences are those which are true in all finite Henkin models. Their set is not r.e. since it contains the set of finitely valid first-order sentences as an effectively detachable subset. This is prima facie somewhat surprising since one can express the finiteness of each type τ by a single sentence (without any nonlogical symbols), e.g. by $FIN[\tau] = \forall R^{[\tau\tau]} (R^{[\tau\tau]}$ is a wellorder $\rightarrow \lambda x^\tau y^\tau R^{[\tau\tau]}(y^\tau, x^\tau)$ is a wellorder). But finiteness as expressed by $FIN[\tau]$ is not real finiteness. Indeed, by an ultraproduct construction we can have a Henkin model $\mathcal{H}$ such that 1) $FIN[\tau]$ is true in $\mathcal{H}$ for all types τ and 2) $\mathcal{H}$ is not equivalent to a finite Henkin model.

However, there is again an escape via infinitary rules.

For each type τ we have the *Finiteness Rule of Type* τ:

$$(fin, \tau) \quad \frac{\exists^{\leq n,\tau},\ \Phi \implies \Psi \quad \text{for all } n \geq 1}{\Phi \implies \Psi}$$

Here $\exists^{\leq n,\tau}$ abbreviates $\exists x_1^\tau \ldots x_n^\tau \forall y^\tau\ y^\tau = x_1^\tau \vee \ldots \vee y^\tau = x_n^\tau$ where the equality $a^\tau = b^\tau$ is defined by $\forall x^{[\tau]} (x^{[\tau]}(a^\tau) \rightarrow x^{[\tau]}(b^\tau))$.

Theorem 4 (Up-and-Down-Finiteness). *Let σ and τ be any two types. Then the rules (fin, σ) and (fin, τ) are interdeducible in TK.*

Proof. We illustrate the proof in the special case $\sigma = 0$ and $\tau = [0]$. Prove for each $n \geq 1$ the sequent $\exists^{\leq n,0} \Longrightarrow \exists^{\leq 2^n,[0]}$; and prove for each $n \geq 1$ the sequent $\exists^{\leq n,[0]} \Longrightarrow \exists^{\leq \log n,0}$. Then use cuts. ■

Again we have *for each single* type τ a Fan Form Thm for $TK + (fin, \tau)$ and for $TK + (fin, \tau) \setminus (cut)$ — with the same proof as for Thm 1. — However, if $\mathbb{F}$ is a set of at least two typed finiteness rules then the methods in the proofs of Thms 1 and 4 yield a Fan Form Thm only for the system $TK + \mathbb{F}$ (with (cut)).

Theorem 5. *Let $\mathbb{F}$ be any nonempty subset of $\{(fin, \tau) : \tau$ is a type$\}$. Then $TK + \mathbb{F}$ is correct and $TK + \mathbb{F} \setminus (cut)$ is complete w.r.t. the class of finite Henkin models.*

Proof. The correctness proof is obvious. — That $TK + \mathbb{F} \setminus (cut)$ is complete w.r.t. the class of finite Henkin models can be proved much like the completeness of $TK \setminus (cut)$ w.r.t. to Henkin models. The only additional fact we have to use is that every Henkin model which satisfies a sentence $\exists^{\leq n,\tau}$ is equivalent to a finite Henkin model. ■

Corollary 6. *Let $\mathbb{F}$ be as in Thm 5. The calculus $TK + \mathbb{F}$ admits cut elimination.*

While the consistency of TK is provable in a fragment of primitive recursive arithmetic (just formalize the truth definition in the type structure over one individual) the cut elimination theorem for TK is excessively strong since it implies trivially the consistency of systems as strong as impredicative finite order arithmetic. See [8] for a proof.

How strong is the statement that $TK + \{(fin, \tau) : \tau$ is a type$\}$ (or any of the subsystems in Corollary 6) admits cut elimination? — The notions occurring in Thm 5 can be expressed in second-order arithmetic. Both infinitary proof trees and typewise finite Henkin models are certain sets of natural numbers. Then a closer analysis of the informal proof of Thm 5 reveals that it can be carried out in ACA (see Simpson's appendix to [8]). Therefore, the cut elimination property of $TK + \{(fin, \tau)\ :\ \tau$ is a type$\}$ is metamathematically rather weak. A more important point is the following: While there is, at present, no syntactic proof of the cut elimination theorem for TK in sight, cut elimination for $TK + \{(fin, \tau)\ :$ τ is a type$\}$ can be proved in a syntactic way. To give a rough outline: A *finite* proof figure in $TK + \{(fin, \tau)\ :\ \tau$ is a type$\}$ with an endsequent of the form $\exists^{\leq n,\tau},\ \Phi \implies \Psi$ is called *bounded*. It is sufficient to show that each bounded proof can be replaced by a cut-free proof with the same endsequent. The presence of $\exists^{\leq n,\tau}$ in the antecedens makes the impredicativity of the quantifier inferences (w.r.t. arbitrary abstraction terms) harmless and allows to move the cuts upwards in the proof tree, much like in Gentzen's original cut elimination proof for the first-order calculus.

4. Transitive Closure Logic with the Finiteness Rule

A logic with a prominent status in finite model theory is TC = transitive closure logic, as amply documented e.g. in [3]. It is well-known that the TC-sentences which are true in *all* (i.e. finite and infinite) models form a Π^1_1-complete set. Of course, the finitely valid TC-sentences form a Π^0_1-set. Below we give a proof-theoretic explanation for this jump from Π^1_1 down to Π^0_1.

We first sketch the classical sequent calculus TCK_e for transitive closure logic (with =). — In addition to the formation rules of a first-order language $\mathcal{L}$ we have the following formation rule for the TC-language (extending $\mathcal{L}$): If n is a positive integer and $\mathcal{F}[\overline{a}, \overline{b}]$ is a formula, then $TC^n(\overline{x}, \overline{y})\mathcal{F}[\overline{x}, \overline{y}]$ is an $(n+n)$-ary relation symbol. Here $\overline{a}, \overline{b}, \overline{x}$ and $\overline{y}$ are n-tuples of pairwise distinct variables.

Semantically, the relation symbol $TC^n(\overline{x}, \overline{y})\mathcal{F}[\overline{x}, \overline{y}]$ denotes, in a given model $\mathfrak{A} = \langle A, \dots \rangle$ for $\mathcal{L}$, the *standard* transitive closure of the relation $\{\langle \overline{x}, \overline{y} \rangle \in A^n \times A^n : \mathfrak{A} \models \mathcal{F}[\overline{x}, \overline{y}]\}$. We stressed the word *standard* because there are also nonstandard interpretations for the transitive closure construct. All models for TC referred to in this paper are standard.

$\mathcal{F}[\overline{u},\overline{v}]^{(1)}$ is short for $\mathcal{F}[\overline{u},\overline{v}]$; for $k \geq 2$ the expression $\mathcal{F}[\overline{u},\overline{v}]^{(k)}$ is defined by $\exists\overline{x}_1 \ldots \overline{x}_{k-1}(\mathcal{F}[\overline{u},\overline{x}_1] \wedge \mathcal{F}[\overline{x}_1,\overline{x}_2] \wedge \ldots \wedge \mathcal{F}[\overline{x}_{k-1},\overline{v}])$

The calculus TCK_e consists of the rules of LK_e together with the following special rules for TC.

$$(TC \Longrightarrow) \quad \frac{\mathcal{F}[\overline{u},\overline{v}]^{(k)},\ \Phi \implies \Psi \quad \text{for all } k \geq 1}{TC^n(\overline{x},\overline{y})\mathcal{F}[\overline{x},\overline{y}](\overline{u},\overline{v}),\ \Phi \implies \Psi}$$

$$(\Longrightarrow TC) \quad \frac{\Phi \implies \Psi,\ \mathcal{F}[\overline{u},\overline{v}]^{(k)}}{\Phi \implies \Psi, TC^n(\overline{x},\overline{y})\mathcal{F}[\overline{x},\overline{y}](\overline{u},\overline{v})}\ (k \geq 1)$$

Theorem 7. *TCK_e is correct and $TCK_e \setminus (cut)$ is complete w.r.t. to all models.*

The rule (fin) in the next theorem is the finiteness rule introduced at the beginning of section 2.

Theorem 8. *If S is provable in $TCK_e + (fin)$ [in $TCK_e + (fin) \setminus (cut)$] then there is a fan proof of S in $TCK_e + (fin)$ [in $TCK_e + (fin) \setminus (cut)$].*

Theorem 9. *$TCK_e + (fin)$ is correct and $TCK_e + (fin) \setminus (cut)$ is complete w.r.t. to all finite models.*

A detailed proof of Thm 7 is given in [2]. Thm 9 follows from Thm 7 by the methods illustrated in section 2 for the LK_e-case. As corollaries to Thms 7 and 9 we have cut elimination theorems; these can also be proved syntactically in both cases as done in [2], following the pattern in [7]. Since TCK_e can be considered as an impredicative extension of LK_e the existence of a syntactic cut elimination proof for TCK_e is important for foundational reasons. In [2] we give several proof-theoretic consequences of the cut elimination theorem for TCK_e, e.g. consistency proofs for axiom systems of arithmetic.

Corollary 10. *The set of sequents provable in $TCK_e + (fin)$ is Π^0_1.*

Proof. A fan proof (existing by Thm 8) of S has a (fin) as its last inference with premises $\exists^{\leq n}, S$ for each n. Above these sequents $\exists^{\leq n}, S$ there may also occur applications of the infinitary rule $(TC \implies)$; but in view of $\exists^{\leq n}$ as an antecedens formula and Thm 7 the set of sequents $\{\exists^{\leq n}, S \,:\, \exists^{\leq n}, S$ has a proof in $TCK_e\}$ is decidable. ■

4.1. An additional remark about $TCK_e + (fin)$

By Thm 8 provability of a sequent S in $TCK_e + (fin)$ reduces to provability in TCK_e of all sequents $\exists^{\leq m}, S$. TCK_e still contains an infinitary rule, viz. $(TC \implies)$. But in the case of sequents of the form $\exists^{\leq m}, S$ the infinitary rule $(TC \implies)$ can be mediated by infinitely many finitary rules as follows:

For each triple of positive integers m, n, l with $m^n < l$ we introduce the following (finitary) rule

$$(\mathit{blow\ up} : m^n \to l) \quad \frac{\exists^{\leq m}, \mathcal{F}[\overline{u}, \overline{v}]^{(k)}, \Phi \implies \Psi \quad \text{for all } k \text{ with } 1 \leq k \leq m^n}{\exists^{\leq m}, \mathcal{F}[\overline{u}, \overline{v}]^{(l)}, \Phi \implies \Psi}$$

Here $\overline{u}$ and $\overline{v}$ are n-tuples of terms.

How do we deduce the sequent $\exists^{\leq m}, TC^n(\overline{x}, \overline{y})\mathcal{F}[\overline{x}, \overline{y}](\overline{u}, \overline{v}), \Phi \implies \Psi$ as conclusion of a $(TC \implies)$? What we really must do is to prove the finitely many sequents $\exists^{\leq m}, \mathcal{F}[\overline{u}, \overline{v}]^{(k)}, \Phi \implies \Psi$, for k with $1 \leq k \leq m^n$. For then the rules $(\mathit{blow\ up} : m^n \to l)$ yield the remaining premises of $(TC \implies)$ automatically. The justification is found in the following Prop.

Proposition 11. *The rules* $(\mathit{blow\ up} : m^n \to l)$ *are deducible in* $TCK_e \setminus (TC \implies)$.

Proof. Let m, n, l satisfy $m^n < l$. Then one deduces the sequent

$$\exists^{\leq m}, \mathcal{F}[\overline{u}, \overline{v}]^{(l)} \implies \mathcal{F}[\overline{u}, \overline{v}]^{(1)} \vee \ldots \vee \mathcal{F}[\overline{u}, \overline{v}]^{(m^n)}$$

This sequent is logically valid since there are at most m^n distinct n-tuples in a model of size $\leq m$, hence every path longer than m^n must contain a loop which can be deleted to get a shorter path.

Then apply $\vee$: left to the premises of $(\mathit{blow\ up} : m^n \to l)$ and use a cut. ■

5. Approximations to Finite Validity

In set theory the notion of a finite set can be defined in many ways which are in general nonequivalent if the axiom of choice is absent. Let the set theoretic formula $\Phi(X)$ be a finiteness definition, i.e. the set X is finite (in a certain sense) if and only if $\Phi(X)$. We call $\Phi(X)$ an *elementary finiteness definition* if $\Phi(X)$ says that *for all* relations $R_1, \ldots, R_k$ (of various arities) on X the structure $\langle X, R_1, \ldots, R_k \rangle$ is a model of φ where $\varphi = \varphi[R_1, \ldots R_k]$ is a certain first-order sentence (with $=$) in the signature $\{R_1, \ldots R_k\}$. — Every such elementary finiteness definition $\Phi(X)$ gives rise to a *finiteness scheme* in the following way: Take the first-order sentence $\varphi[R_1, \ldots R_k]$ correlated with $\Phi(X)$. For $i = 1, \ldots, k$ the relation letter R_i of arity n_i may be replaced by a first-order formula $\mathcal{F}_i[x_1, \ldots, x_{n_i}]$ with the designated variables $x_1, \ldots, x_{n_i}$ shown. The formula $\mathcal{F}_i[x_1, \ldots, x_{n_i}]$ may contain other variables as parameters. Then each formula of the form $\varphi[\mathcal{F}_1, \ldots, \mathcal{F}_k]$ is an instance of *the finiteness scheme correlated with* the elementary finiteness definition $\Phi(X)$. — Let $\mathcal{L}$ be any first-order language (vocabulary). A finiteness scheme is called, more precisely, a *finiteness $\mathcal{L}$-scheme* if its instances are built from $\mathcal{L}$-formulae. We omit the prefix $\mathcal{L}$ if it is understood or irrelevant.

By assumption, each instance of a finiteness scheme is true in all finite models.

We give two examples.

1) A set X is *Russell-finite* if for all $R \subseteq X^2 : \langle X, R \rangle \models [R$ is transitive $\wedge \forall x \, \exists y \, R(x, y) \rightarrow \exists x \, R(x, x)]$. Then the instances of the Russell-finiteness scheme are (for each formula $\mathcal{F}[x, y]$):

$$\forall x, y, z \, (\mathcal{F}[x, y] \wedge \mathcal{F}[y, z] \rightarrow \mathcal{F}[x, z]) \wedge \forall x \, \exists y \, \mathcal{F}[x, y] \rightarrow \exists x \, \mathcal{F}[x, x]$$

2) A set X is *Dedekind-finite* if for all $R \subseteq X^2 : \langle X, R \rangle \models [R$ is an injective function $\rightarrow R$ is also surjective$]$. Then the instances of the Dedekind-finiteness scheme are:

$$\mathcal{F} \text{ is a function} \wedge \forall x, y, z (\mathcal{F}[x, z] \wedge \mathcal{F}[y, z] \rightarrow x = y) \rightarrow \forall y \, \exists x \, \mathcal{F}[x, y]$$

As a corollary to Thm 3 we have:

Proposition 12. *Let $Fin(\Phi)$ be the finiteness scheme correlated to the elementary finiteness definition $\Phi(X)$. Then every instance of $Fin(\Phi)$ is provable in $LK_e + (fin) \setminus (cut)$.*

Now we want to approximate the converse of Prop 12 in the following way: Let Θ be a theory in LK_e with language $\mathcal{L}$. If we add more and more finiteness $\mathcal{L}$-schemes to Θ we can deduce more and more finitely valid sentences, i.e. sentence true in all finite models of Θ. In some cases we may get even all finitely valid sentences by adding a finite (or r.e.) set of finiteness schemes. This is of course possible only if the set of finitely valid sentences in question is r.e. and hence recursive. Such a fortunate case is given by $\Theta = [R \text{ is a linear order}]$. In $LK_e + \Theta +$ [Russell-finiteness scheme] one can prove exactly the sentences which are valid in all finite linear orders. A proof of this can be extracted from [5], where instead of the Russell-finiteness scheme a related finiteness scheme is used, viz. the scheme $\forall x, y, z\,(\mathcal{F}[x, y] \wedge \mathcal{F}[y, z] \to \mathcal{F}[x, z]) \to \exists x\, \forall y\,(\mathcal{F}[x, y] \to \mathcal{F}[y, x])$ which may be called *the Whitehead-finiteness scheme.*

On the other hand, there is no r.e. set $\mathbb{M}$ of finiteness schemes such that LK_e + [*group axioms*] + $\mathbb{M}$ yields all sentences true in all finite groups.

Sometimes we need no special background theory Θ to delineate the intended models. A case in question is the language $L_k(f)$ of one unary function symbol together with $k \geq 0$ unary predicates. We pose the following

Problem 13. It is well-known that the set MU_k of $L_k(f)$-sentences which are valid in all finite k-colored monounaries $\langle A, f, C_1, \dots, C_k\rangle, C_i \subseteq A$, is recursive. Find a *finite* set $\mathbb{M}$ of finiteness $L_k(f)$-schemes such that $LK_e + \mathbb{M}$ yields the set MU_k.

Problem 14. The set AG_f of sentences true in all finite abelian group is decidable, as shown in [4]. We want to have a finite set $\mathbb{A}$ of finiteness schemes such that LK_e + [*abelian group axioms*] + $\mathbb{A}$ yields AG_f. By Thm 4. of [4] it is sufficient to deduce all of the statements $D(q, k, n)$ there mentioned. We suspect that $\mathbb{A}$ = {Dedekind-finiteness scheme, Whitehead-finiteness scheme} could do the job.

The *size* of a proof P is the number of sequents occurring in P.

Conjecture 15. 1) Let θ be a fixed first-order sentence. Suppose that for every sentence φ in the language of θ such that $LK_e + (fin) \vdash \theta \Longrightarrow \varphi$ there exists a natural number $k = k_\varphi$ such that for every n the sequent $\exists^{\leq n}, \theta \Longrightarrow \varphi$ possesses an LK_e-proof of size $\leq k$. Then the set $\{\varphi : LK_e + (fin) \vdash \theta \Longrightarrow \varphi\}$ is recursive. Here we assume that the logical signs can be introduced by so-called block inferences, i.e. more than one sign of the same kind in one inference.

2) Let $\mathcal{K}$ be the class of finite models of a single first-order sentence Φ such that validity in $\mathcal{K}$ is decidable. Then there exists a finite set $\mathbb{F}$ of finiteness schemes such that $LK_e + \Phi + \mathbb{F}$ yields exactly the $\mathcal{K}$-valid sentences. This generalizes the cases of linear orderings, of monounaries (in Problem 13) and of abelian groups (in Problem 14).

In some cases, a set of finitely valid *first-order* sentences may be approximated or even obtained via a detour through $TK +$ some finiteness axiom.

Another proof-theoretic topic related to *Approximations to Finite Validity* is the following: Given two finiteness $\mathcal{L}$-schemes we may ask whether one is LK_e-deducible from the other. This topic has some connections with set theory. For instance, in [1] it is shown that $\forall X(Dedekind\text{-}finite(X) \rightarrow Russell\text{-}finite(X))$ *is not* provable in ZF. It follows that the Russell-finiteness $\mathcal{L}$-scheme is not deducible from the Dedekind-finiteness $\mathcal{L}$-scheme, for $\mathcal{L}$ a sufficiently rich language.

On the other hand, it is also shown in [1] that $\forall X(Russell\text{-}finite(X) \rightarrow Dedekind\text{-}finite(X))$ *is* provable in ZF. But can one deduce the Dedekind-finiteness $\mathcal{L}$-scheme from the Russell-finiteness $\mathcal{L}$-scheme, for $\mathcal{L}$ being the full first-order language? — It is easily seen that for all $\mathcal{L}$ the Russell-finiteness $\mathcal{L}$-scheme can be deduced from the Whitehead-finiteness $\mathcal{L}$-scheme. But what about the converse?

It is also to be expected that there are two finiteness schemes neither of which is deducible from the other.

Admittedly, the considerations in this section are of a still tentative and conjectural nature. However, we think to have made clear the challenge to approximate and, whenever possible, to obtain the strength of the infinitary rule (fin) by finitarily expressible means of proof.

6. Remarks about the Intuitionistic Case

The intuitionistic system $LJ_e + (fin)$ arises from $LK_e + (fin)$ by allowing only sequents with at most one succedent formula. There are syntactic proofs of the fan form and cut elimination theorem for $LJ_e + (fin)$. Furthermore, the principle of excluded middle = *PEM* is not provable in $LJ_e + (fin)$. This shows that (contrary to some intuitionistic intuitions) *PEM* does not hold by apriori intuitionistic reasons in finite models. However, if we add for each n-ary predicate (including =) the axiom $\Longrightarrow P(u_1, \dots, u_n) \vee \neg P(u_1, \dots, u_n)$ to $LJ_e + (fin)$ we get all theorems of the classical calculus $LK_e + (fin)$. The proof is not completely trivial and uses the rules (fin) and (cut) in an essential way. Recall that, e.g., in Heyting Arithmetic we have *PEM* for atomic formulae without having *PEM* for all formulae.

What is a *finite intuitionistic model?* There are two natural options:

1) Finite Kripke models, i.e. frames consisting of finitely many worlds at each of which there is a finite classical model. 2) Finite Heyting-valued models, i.e. finite structures associated with a finite Heyting algebra.

We have not yet developed a semantic completeness proof for $LJ_e + (fin)$. Let TJ be the intuitionistic counterpart of TK. It could be interesting to develop a completeness proof for $TJ + \{(fin, \tau) : \tau \text{ is a type}\}$ w.r.t. a suitable model class.

7. A Concluding Remark on Infinite Validity

We say that a first-order sentence is *infinitely valid* iff it is true in all infinite models. $\exists^{\geq n}$ is short for $\exists x_1 \dots x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$.

For each positive integer n we introduce the following finitary rule:

$$(\infty, n) \quad \frac{\exists^{\geq n},\ \Phi \Longrightarrow \Psi}{\Phi \Longrightarrow \Psi}$$

By (∞) we understand the set $\{(\infty, n) : n \geq 1\}$ of all these rules. One can show that $LK_e + (\infty)$ is correct and $LK_e + (\infty) \setminus (cut)$ is complete w.r.t. the class of infinite models. In view of this, each single finitary rule (∞, n) could be called an *Infinity Rule*. It follows that the set of infinitely valid first-order sentences is recursively enumerable.

References

[1] J. W. Degen, *Several aspects and examples of infinity notions*, Math. Log. Quart. vol. **40** (1994), 111–124.

[2] J. W. Degen, *Proof theory of generalized closure logics*, manuscript.

[3] H.-D. Ebbinghaus and J. Flum, Finite model theory, Springer 1995.

[4] V. Huber Dyson, *On the decision problem for theories of finite models*, Israel J. Math. **2** (1964), 55–70.

[5] M. Mostowski, *The logic of finite models*, techn. report, Polish Acad. Sci. Warszawa 1988.

[6] K. Schütte, Proof theory, Springer, 1977.

[7] H. Schwichtenberg, *Proof theory: some applications of cut-elimination*, in: The Handbook of Mathematical Logic, ed. Barwise 1977 North-Holland.

[8] G. Takeuti, Proof theory, second edition, North-Holland, Amsterdam, 1987.

[9] B. A. Trakhtenbrot, *Impossibility of an algorithm for the decision problem in finite classes*, Dokl. Akad. Nauk SSSR **70** (1950), 569–572; also in: AMS Translations, ser. 3 vol. **23** (1963), 1–5.

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