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PROPERTY S

A b s t r a c t. A set-theoretic property called Property S is introduced as a generalization of the well-known Property B. Property S is named for Schrijver who first used it in a paper [13] in connection with the Boolean prime ideal theorem. It was independently introduced by Kolany [10] to give a uniform treatment for a variety of satisfiability problems using an abstract resolution method to determine satisfiability. We further investigate Property S and resolution.

1. Introduction

A collection \mathcal{E} of subsets of a set V is said to have **Property B** if there exists a partition, $\{X, V - X\}$, of V , such that both $X, V - X$ intersect every E in \mathcal{E} . A set which intersects every set in a family of sets is often called a **transversal** for that family; thus we can simply say: \mathcal{E} has Property B if there exists a subset X such that both $X, V - X$ are transversals for \mathcal{E} . Clearly $V - X$ is a transversal for \mathcal{E} if and only if X does not contain any E in \mathcal{E} . Hence, \mathcal{E} has Property B if and only if there is a transversal,

Received 8 January 2000.

X , for \mathcal{E} which does not contain any E in \mathcal{E} . We shall also say that the hypergraph $\langle V, \mathcal{E} \rangle$ has Property B if \mathcal{E} has the property.

The property is named for Felix Bernstein who proved, in 1908, that a countable system of infinite sets has Property B. Since then it has been studied extensively (see, for example, [5],[6], [7], [8], [11]). In graph theory Property B has been linked by Woodall [17] and Stein [15] with the 4-color problem. For example Woodall[17] showed that the 4-color conjecture was equivalent to: the odd circuits of a planar graph, when regarded as sets of edges, has Property B.

Property B is an NP-Complete property when restricted to finite families of finite sets; in fact, it remains NP-Complete even when all sets in the family have at most three elements (see [2]). In the next section we consider a generalization to two families of subsets and we name the generalization “Property S” for reasons explained below. Property B is then the special case when both families are the same.

2. Property S

Two families \mathcal{E}, \mathcal{F} of subsets of V will be said to have **Property S** if there exists a partition, $\{ X, V - X \}$, of V , such that X is a transversal for \mathcal{E} and $V - X$ is a transversal for \mathcal{F} ; the partition will be referred to as an **S-partition**. Equivalently, \mathcal{E}, \mathcal{F} have Property S if there is a transversal X for \mathcal{E} , which does not contain any of the subsets of \mathcal{F} .

If \mathcal{E}, \mathcal{F} has Property S, then $\mathcal{E} \cap \mathcal{F}$ has Property B, since every set in $\mathcal{E} \cap \mathcal{F}$ must then intersect both cells of the partition. If $\mathcal{E} = \mathcal{F}$, \mathcal{E}, \mathcal{F} has Property S if and only if \mathcal{E} has Property B.

From now on we shall often write $\langle V, \mathcal{E}, \mathcal{F} \rangle$ where \mathcal{E}, \mathcal{F} are collections of subsets of V and ask whether this “bihypergraph” has Property S. Instead of explicitly mentioned the partition of V , Schrijver[13] refers to disjoint transversals for the families \mathcal{E}, \mathcal{F} , but this is clearly equivalent. He proved the following “compactness” theorem for property S and showed that it is equivalent in ZF-set theory to BPI, the prime ideal theorem for Boolean algebras.

Theorem 2.1. *Let \mathcal{E}, \mathcal{F} be families of finite subsets of V . Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if $\langle V, \mathcal{E}_0, \mathcal{F}_0 \rangle$ has Property S for every finite $\mathcal{E}_0 \subset \mathcal{E}, \mathcal{F}_0 \subset \mathcal{F}$.*

Restrictions of Schrijver's Theorem are also equivalent to BPI; for example, the corresponding compactness result for Property B is equivalent to BPI even if all sets in the family have at most three elements (see [2]).

As mentioned in the Introduction, Bernstein showed that a countable system of infinite sets has Property B; this can be generalized as follows.

Theorem 2.2. *Let \mathcal{E}, \mathcal{F} be countable collections of infinite sets and let $V = (\cup \mathcal{E}) \cup (\cup \mathcal{F})$. Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S.*

Proof. Suppose $\mathcal{E} = \{A_n\}_{n=1}^\infty, \mathcal{F} = \{B_n\}_{n=1}^\infty$. For $n \geq 1$, choose elements x_n, y_n as follows:
 $x_1 \in A_1; y_1 \in B_1, y_1 \neq x_1; x_{n+1} \in A_{n+1}, x_{n+1} \neq y_i, 1 \leq i \leq n;$
 $y_{n+1} \in B_{n+1}, y_{n+1} \neq x_i, 1 \leq i \leq n+1$. Let $X = \cup_{n=1}^\infty \{x_n\}, Y = \cup_{n=1}^\infty \{y_n\}$. Then X, Y are disjoint transversals for \mathcal{E}, \mathcal{F} , respectively and the Theorem easily follows. ■

Much is known about Property B in the finite case, as well (see [5], [6],[7],[11],[17]). For example, it is easy to show that any family of subsets of a $(2n+1)$ element set V fails to have Property B if it contains all of the $(n+1)$ element subsets of V . We can generalize this as follows.

Theorem 2.3. *Suppose V is a set of $(2k+1)$ elements. Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ does not have Property S, if $\mathcal{E} \cap \mathcal{F}$ contains all the $(k+1)$ element subsets of V .*

Proof. Suppose $\mathcal{E} \cap \mathcal{F}$ contains all the $(k+1)$ element subsets of V . Any set in $\mathcal{E} \cap \mathcal{F}$ must intersect both cells of an S-partition for $\langle V, \mathcal{E}, \mathcal{F} \rangle$; however, any S-partition has one of its cells of cardinality at least $(k+1)$, and thus it must contain a set in $\mathcal{E} \cap \mathcal{F}$. ■

Next we generalize to Property S, a result of Woodall [17] for Property B. If \mathcal{A} is a family of subsets of V , let $\mathcal{I}(\mathcal{A}) = \{v \in V | v \supseteq a, \text{ for some } a \in \mathcal{A}\}$. Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if and only if there exists an X such that $X \notin \mathcal{I}(\mathcal{F})$ and $V - X \notin \mathcal{I}(\mathcal{E})$.

Theorem 2.4. *$\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if $|\mathcal{I}(\mathcal{E}) \cup \mathcal{I}(\mathcal{F})| < 2^{(|V|-1)}$.*

Proof. We claim that there is an X such that $X \notin \mathcal{I}(\mathcal{F})$ and $V - X \notin \mathcal{I}(\mathcal{E})$. If not, then for every $X \subseteq V$, either $X \in \mathcal{I}(\mathcal{F})$ or $V - X \in \mathcal{I}(\mathcal{E})$. Thus, either X or $V - X$ belongs to $\mathcal{I}(\mathcal{E}) \cup \mathcal{I}(\mathcal{F})$, for every $X \subseteq V$. However, this implies $|\mathcal{I}(\mathcal{E}) \cup \mathcal{I}(\mathcal{F})| \geq 2^{(|V|-1)}$. ■

Theorem 2.5. *Let V be a finite set. Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if $\sum_{A \in \mathcal{E} \cup \mathcal{F}} \left(\frac{1}{2^{|A|}}\right) < \frac{1}{2}$.*

Proof. Suppose $\sum_{A \in \mathcal{E} \cup \mathcal{F}} \left(\frac{1}{2^{|A|}}\right) < \frac{1}{2}$. The number of subsets of V which contain A is $2^{(|V|-|A|)}$. Thus $|\mathcal{I}(\mathcal{E}) \cup \mathcal{I}(\mathcal{F})| < \sum_{A \in \mathcal{E} \cup \mathcal{F}} 2^{(|V|-|A|)}$. But $\sum_{A \in \mathcal{E} \cup \mathcal{F}} 2^{(|V|-|A|)} = 2^{|V|} \sum_{A \in \mathcal{E} \cup \mathcal{F}} \frac{1}{2^{|A|}} < 2^{(|V|-1)}$. Hence $|\mathcal{I}(\mathcal{E}) \cup \mathcal{I}(\mathcal{F})| < 2^{(|V|-1)}$ and $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S, by the previous Theorem. ■

If, in the last result $\mathcal{E} = \mathcal{F}$, we get the proposition in [17].

Corollary 2.6. *If \mathcal{F} is a family of subsets of a finite set V and if $\sum_{A \in \mathcal{F}} \left(\frac{1}{2^{|A|}}\right) < \frac{1}{2}$, then \mathcal{F} has Property B.*

3. Property S and Satisfiability

If $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has property S, where $\{X, V - X\}$ is an S-partition of V , then X picks some elements from each of the subsets in \mathcal{E} but the totality of chosen elements does not contain one of the ‘forbidden subsets’ in \mathcal{F} . This problem of picking elements subject to constraints is very general; we give a few examples to indicate its wide applicability.

In propositional logic, a literal is either a statement variable or its negation, a clause is a finite collection of literals, and a conjunctive normal form (cnf) is a finite collection of clauses. The problem is to determine whether a given cnf is satisfiable; that is, does there exist an interpretation of the variables (as true or false) such that each clause contains at least one true literal. Let V be the set consisting of all the literals in the cnf, \mathcal{E} , the set of clauses of the cnf, and \mathcal{F} , the set of the pairs consisting of the statement letters and their negations. Then the satisfiability of the cnf is easily seen to be equivalent to $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has property S.

An n -coloring of a graph $G = \langle A, E \rangle$ is a function $f : A \rightarrow \{1, \dots, n\}$, such that $f(a_1) \neq f(a_2)$, if $a_1 E a_2$; G is n -colorable if such a coloring exists. Let V be the pairs, $\{a, j\}$, where $a \in A$, $1 \leq j \leq n$. Let \mathcal{E} be the sets $\{\{a, 1\}, \dots, \{a, n\}\}$, where $a \in A$. Let \mathcal{F} consist of all the pairs, $\{\{a_1, i\}, \{a_2, i\}\}$, with $a_1 E a_2$, $1 \leq i \leq n$. Then G is n -colorable if and only if $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has property S.

Suppose that for each vertex a of a graph $G = \langle A, E \rangle$, a list $L(a)$ of colors available for a is given. Then a *list coloring* from L is a proper coloring, f , such that $f(a) \in L(a)$; in case such a coloring exists we say that

G is L-list colorable. Let \mathcal{E} be the sets $\{\{a, i\} | i \in L(a)\}$, where $a \in A$. Let \mathcal{F} consist of all the pairs, $\{\{a_1, i\}, \{a_2, i\}\}$, with $a_1 E a_2$. Then G is L-list colorable if and only if $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has property S. A graph is said to be *k-choosable* if it has a list coloring for every assignment of k element lists to the vertices. (An introduction to list coloring and choosability can be found in West [16].)

(The “marriage problem”) Let $S = \{S_i\}_{i \in I}$ be an indexed family of finite sets. A system of distinct representatives (SDR) for S is a one-to-one function $f : I \rightarrow \cup S$ such that $f(i) \in S_i$, $i \in I$. Let V be the set of all pairs, $\{s, i\}$, where $s \in S_i$, $i \in I$. Let \mathcal{E} be all the sets $\{\{s, i\} | s \in S_i\}$, $i \in I$. Let \mathcal{F} consist of the sets, $\{\{s, i\}, \{s, j\}\}$, where $i \neq j$. Then S has an SDR if and only if $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has property S.

Since so many satisfiability problems can be treated as Property S problems, it makes sense to study methods that can determine whether Property S holds. In [3], we introduced a Tableau Method reminiscent of the Analytic Tableaux in logic of Raymond Smullyan [14]. The other main method, which also comes from logic, is resolution, which we turn to next.

4. Resolution

The Resolution proof procedure, used in logic to determine the satisfiability of conjunctive normal forms, has been generalized to provide a proof procedure for a variety of satisfiability problems by Cowen [1] and Kolany [10]. In Kolany [10] a notion of satisfiability on Hypergraphs was introduced which is equivalent to Property S. This enabled him to prove a far more useful resolution result than that of Cowen [1] whose definition of satisfiability was too restrictive.

Definition 4.1. Let $c_1, c_2, \dots, c_n, d, e$ be subsets of V . then e follows from c_1, c_2, \dots, c_n by **resolution** on d if $d = \{v_1, v_2, \dots, v_n\}$, where $v_i \in c_i$, $1 \leq i \leq n$, and $e = \cup_{i=1}^n c_i / v_i$; in this case, we write, $c_1, c_2, \dots, c_n \vdash_d e$.

If \mathcal{A}, \mathcal{D} are collections of subsets of V , $[\mathcal{A}]_{\mathcal{D}}$ will denote the closure of \mathcal{A} with respect to resolution on elements $d \in \mathcal{D}$. The following Theorem is elegantly proved by Kolany [10].

Theorem 4.2. $\langle V, \mathcal{E}, \mathcal{F} \rangle$ fails to have Property S if and only if $\emptyset \in [\mathcal{E}]_{\mathcal{F}}$.

It is interesting to note that Woodall [17] introduced for Property B a “reduction process” which is essentially the same as resolution and proved a result for Property B which is analagous to Kolany’s Theorem (see Proposition 2 of [17]).

Surely $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if and only if $\langle V, \mathcal{F}, \mathcal{E} \rangle$ has Property S. Thus, we have the following corollary to Kolany’s Theorem.

Corollary 4.3. $\langle V, \mathcal{E}, \mathcal{F} \rangle$ fails to have Property S if and only if $\emptyset \in [\mathcal{F}]_{\mathcal{E}}$

Therefore there are two distinct resolution methods which can be used in a particular case. (Of course, in the case of Property B, both methods are the same.) The next result shows how they can be combined for even greater flexibility.

Theorem 4.4. $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if and only if $\langle V, [\mathcal{E}]_{\mathcal{F}}, \mathcal{F} \rangle$ has Property S.

Proof. If $\langle V, [\mathcal{E}]_{\mathcal{F}}, \mathcal{F} \rangle$ has Property S, then so does $\langle V, \mathcal{E}, \mathcal{F} \rangle$, since $\mathcal{E} \subset [\mathcal{E}]_{\mathcal{F}}$. Suppose that $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S and X is a transversal for \mathcal{E} and $V - X$ is a transversal for \mathcal{F} . We claim that X is also a transversal for $[\mathcal{E}]_{\mathcal{F}}$. Suppose $c_i \cap X \neq \emptyset$, $1 \leq i \leq n$, and d follows from the c_i by resolution on f , where $f \in \mathcal{F}$, $f = \{v_1, \dots, v_n\}$, with $v_i \in c_i$, and $d = \cup_{i=1}^n c_i/v_i$. Then we must show that $d \cap X \neq \emptyset$, as well. Suppose, on the contrary that $d \cap X = \emptyset$. It follows that $c_i/v_i \cap X = \emptyset$, for $1 \leq i \leq n$. Since $V - X$ is a transversal for \mathcal{F} , $v_j \in V - X$, for some j , $1 \leq j \leq n$. Thus $v_j \notin X$ and this, together with $c_j/v_j \cap X = \emptyset$, implies $c_j \cap X = \emptyset$; however this contradicts $c_i \cap X \neq \emptyset$, $1 \leq i \leq n$. ■

It follows that $\langle V, \mathcal{E}, \mathcal{F} \rangle$ fails to have Property S if and only if $\emptyset \in [\mathcal{F}]_{[\mathcal{E}]_{\mathcal{F}}}$, or, if $\emptyset \in [\mathcal{E}]_{[\mathcal{F}]_{\mathcal{E}}}$, etc. Since this quickly becomes a typographical nightmare, we define, recursively, the following notation.

Definition 4.5. $[\mathcal{F}, \mathcal{E}, 0] = \mathcal{F}$; $[\mathcal{E}, \mathcal{F}, 0] = \mathcal{E}$; $[\mathcal{F}, \mathcal{E}, n + 1] = [\mathcal{F}]_{[\mathcal{E}, \mathcal{F}, n]}$; $[\mathcal{E}, \mathcal{F}, n + 1] = [\mathcal{E}]_{[\mathcal{F}, \mathcal{E}, n]}$.

We then have the following corollary to the previous Theorem.

Corollary 4.6. For $n > 0$, the following statements are equivalent.

1. $\langle V, \mathcal{E}, \mathcal{F} \rangle$ fails to have Property S

$$2. \emptyset \in [\mathcal{E}, \mathcal{F}, n]$$

$$3. \emptyset \in [\mathcal{F}, \mathcal{E}, n].$$

This allows resolution proofs which go back and forth; we illustrate some of these possibilities by giving three different proofs that a set \mathcal{E} of clauses in propositional logic is unsatisfiable. We number the clauses in \mathcal{E} for easy reference, as follows:

$$1)\{p, q\}, 2)\{p, \neg q, r\}, 3)\{p, \neg q, \neg r\}, 4)\{\neg p, q, r\}, 5)\{\neg p, q, \neg r\}, 6)\{\neg p, \neg q\}.$$

The clauses in \mathcal{F} consist of variables and their negations:

$$A)\{p, \neg p\}, B)\{q, \neg q\}, C)\{r, \neg r\}.$$

We annotate our proofs using the notation: $(x, y, z, \dots/w)$; this means that the clause was obtained from clauses labeled x, y, z, \dots by resolving on the clause labeled w . Our first proof is that $\emptyset \in [\mathcal{E}]_{\mathcal{F}}$.

$$\begin{array}{ll} 7) \{p, \neg q\} & (2,3/C) \\ 8) \{\neg p, q\} & (4,5/C) \\ 9) \{\neg q\} & (6,7/A) \\ 10) \{q\} & (1,8/A) \\ 11) \emptyset & (9,10/B) \end{array}$$

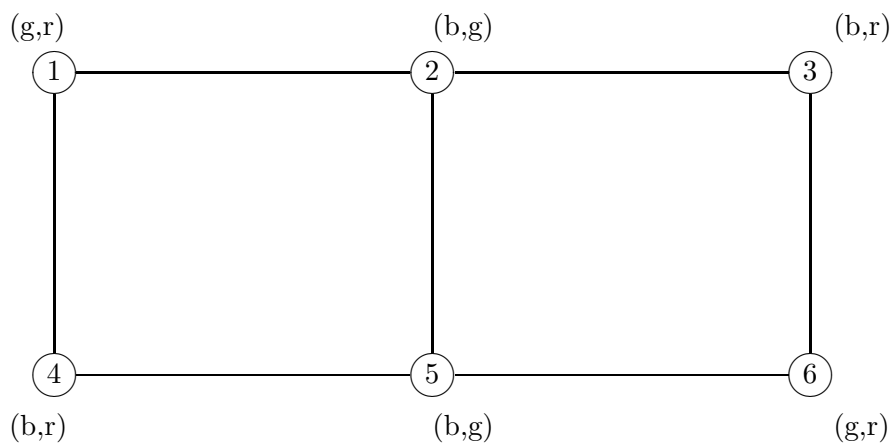
Our next proof shows that $\emptyset \in [\mathcal{F}]_{\mathcal{E}}$.

$$\begin{array}{ll} D) \{p, q\} & (A,B/6) \\ E) \{q, r\} & (D,B,C/3) \\ F) \{q\} & (D,B,E/2) \\ G) \{p, r\} & (A,D,C/5) \\ H) \{p\} & (A,D,G/4) \\ I) \emptyset & (H,F/1) \end{array}$$

Our last proof demonstrates that $\emptyset \in [\mathcal{F}]_{[\mathcal{E}]_{\mathcal{F}}}$. Lines 12 and 13 show membership in $[\mathcal{E}]_{\mathcal{F}}$; lines J,K show membership in $[\mathcal{F}]_{\mathcal{E}}$; L-N show membership in $[\mathcal{F}]_{[\mathcal{E}]_{\mathcal{F}}}$.

$$\begin{array}{ll} 12) \{p, \neg q\} & (2,3/C) \\ 13) \{\neg p, q\} & (4,5/C) \\ J) \{p, q\} & (A,B/6) \end{array}$$

K) $\{\neg p, \neg q\}$	(A,B/1)
L) $\{p\}$	(A,J/13)
M) $\{\neg q\}$	(B,K/13)
N) \emptyset	(L,M/12)



Next we prove, by resolution, that the graph depicted above is not L-list colorable, where L is the lists shown next to the vertices. Since each list has two elements, this will show that the graph is not 2-choosable (even though it is 2-colorable). Let \mathcal{E} be the collection of the following sets.

- 1) $\{g_1, r_1\}$, 2) $\{b_2, g_2\}$, 3) $\{b_3, r_3\}$, 4) $\{b_4, r_4\}$, 5) $\{b_5, g_5\}$, 6) $\{g_6, r_6\}$

Let \mathcal{F} be the following sets.

- A) $\{r_1, r_4\}$, B) $\{g_1, g_2\}$, C) $\{b_4, b_5\}$, D) $\{b_2, b_5\}$, E) $\{g_2, g_5\}$, F) $\{b_2, b_3\}$,
G) $\{r_3, r_6\}$, H) $\{g_5, g_6\}$

7) $\{r_1, b_2\}$	(1,2/B)
8) $\{b_3, g_6\}$	(3,6/G)
9) $\{b_2, b_5\}$	(2,5/E)
10) $\{b_2, b_4\}$	(4,7/A)
11) $\{b_3, b_5\}$	(5,8/H)
12) $\{b_2\}$	(9,10/C)
13) $\{b_5\}$	(11,12/F)
14) \emptyset	(12,13/D)

Thus $\emptyset \in [\mathcal{E}]_{\mathcal{F}}$, showing no list coloring exists.

We give next a dual proof, $\emptyset \in [\mathcal{F}]_{\mathcal{E}}$, which is even shorter.

I) $\{r_4, g_2\}$	(A,B/1)
J) $\{b_4, g_2\}$	(C,E/5)
K) $\{g_2\}$	(I,J/4)
L) $\{b_2, r_6\}$	(F,G/3)
M) $\{b_2, g_6\}$	(D,H/5)
N) $\{b_2\}$	(L,M/6)
O) \emptyset	(K,N/2)

5. Davis-Putnam Rules

Although Resolution is a complete proof procedure for propositional logic, it is often combined with other rules to increase its efficacy. The most widely used additions are due to Davis and Putnam [4] and use two rules, the pure literal rule and the unit clause rule to reduce the size of the problem. If these rules are not directly applicable, variables can be assigned truth values until the rules are applicable, but of course this leads to branching and may seem counter-productive; however, Davis-Putnam and its variations seem to yield the most effective algorithms for machine computations [12]. We give generalized versions of the D-P rules that are applicable to deciding Property S problems.

Definition 5.1. Suppose $\langle V, \mathcal{E}, \mathcal{F} \rangle$ is a bihypergraph and p belongs to some of the sets in \mathcal{E} . Then we say that p is free for \mathcal{F} in \mathcal{E} if each set of \mathcal{F} in which p occurs contains some element which does not occur in any set in \mathcal{E} .

Theorem 5.2 (Pure Literal Rule). *Let $\langle V, \mathcal{E}, \mathcal{F} \rangle$ be a bihypergraph. Suppose p is free for \mathcal{F} in \mathcal{E} and \mathcal{E}' consists of all sets in \mathcal{E} which do not contain p . Then $\langle V, \mathcal{E}', \mathcal{F} \rangle$ has Property S if and only if $\langle V, \mathcal{E}, \mathcal{F} \rangle$ does.*

Proof. Surely $\langle V, \mathcal{E}', \mathcal{F} \rangle$ has Property S if $\langle V, \mathcal{E}, \mathcal{F} \rangle$ does. Suppose then that p is free for \mathcal{F} in \mathcal{E} and $\langle X, V - X \rangle$ is an S-partition for $\langle V, \mathcal{E}', \mathcal{F} \rangle$. Let $Y = \bigcup \mathcal{F} / \bigcup \mathcal{E}$ and let $X' = (X \cup \{p\})/Y$. Then we claim that $\langle X', V - X' \rangle$ is an S-partition for $\langle V, \mathcal{E}, \mathcal{F} \rangle$. First, X' is a

transversal for \mathcal{E} , for it intersects all sets in \mathcal{E}' since X does and no elements of $\bigcup \mathcal{E}$ have been removed from X in forming X' ; while the elements of $\mathcal{E} - \mathcal{E}'$ all contain p , as does X' . Next $V - X'$ is a transversal for \mathcal{F} , since $V - X$ is a transversal for \mathcal{F} and p is the only element which is absent from $V - X'$ but belongs to $V - X$; however $\bigcup \mathcal{F} / \bigcup \mathcal{E} \subset V - X'$ and any set in \mathcal{F} which contains p , must also have an element in $\bigcup \mathcal{F} / \bigcup \mathcal{E}$, since p is free for \mathcal{F} in \mathcal{E} . ■

Theorem 5.3 (Unit Clause Rule). *Let $\langle V, \mathcal{E}, \mathcal{F} \rangle$ be a bihypergraph. Suppose that $\{p\} \in \mathcal{E}$ and $\{p, q\} \in \mathcal{F}$ and \mathcal{E}' results from \mathcal{E} by eliminating q from all clauses in \mathcal{E} . Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if and only if $\langle V, \mathcal{E}', \mathcal{F} \rangle$ does.*

Proof. Surely $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if $\langle V, \mathcal{E}', \mathcal{F} \rangle$ has Property S. Suppose then that $\langle X, V - X \rangle$ is an S-partition for $\langle V, \mathcal{E}, \mathcal{F} \rangle$. We claim that X is also transversal for \mathcal{E}' . Suppose false; then it must be the case that $q \in X$. Since X is a transversal for \mathcal{E} and $\{p\} \in \mathcal{E}$, $p \in X$ and so $\{p, q\} \subset X$. However this contradicts the fact that $V - X$ is a transversal for \mathcal{F} . Hence X is transversal for \mathcal{E}' and, of course, $V - X$ is a transversal for \mathcal{F} . ■

The unit clause rule cannot be applied if there are no pairs in \mathcal{F} ; however, for many applications, such as graph coloring, list coloring, etc., \mathcal{F} consists entirely of pairs.

Theorem 5.4 (Splitting Rule). *Let $\langle V, \mathcal{E}, \mathcal{F} \rangle$ be a bihypergraph and suppose that $p \in e_i \in \mathcal{E}$, $1 \leq i \leq n$. Let \mathcal{E}' be the result of removing p from each of the e_i in \mathcal{E} and \mathcal{E}'' , the result of replacing all the e_i by $\{p\}$ in \mathcal{E} . Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if and only if either $\langle V, \mathcal{E}', \mathcal{F} \rangle$ or $\langle V, \mathcal{E}'', \mathcal{F} \rangle$ has Property S.*

Proof. Surely if either $\langle V, \mathcal{E}', \mathcal{F} \rangle$ or $\langle V, \mathcal{E}'', \mathcal{F} \rangle$ has Property S, then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S. Suppose then that $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S and that $\langle X, V - X \rangle$ is an S-Partition. If $\langle V, \mathcal{E}'', \mathcal{F} \rangle$ doesn't have Property S then $p \notin X$. Since $X \cap e_i \neq \emptyset$, $1 \leq i \leq n$, and $p \notin X$, X is also a transversal for \mathcal{E}' . Also, $V - X$ is a transversal for \mathcal{F} . Hence $\langle V, \mathcal{E}', \mathcal{F} \rangle$ has Property S if $\langle V, \mathcal{E}'', \mathcal{F} \rangle$ does not. ■

Consider once again the list coloring problem solved above. Neither the pure literal rule or the unit clause rule is directly applicable. We therefore

turn to the splitting rule. There are many considerations when choosing a splitting variable; we choose b_2 for the splitting variable in this problem since it occurs twice in \mathcal{F} as does its mate in \mathcal{E} , g_2 . Then with $\{b_2\}$ replacing $\{b_2, g_2\}$, we can remove, by the unit clause rule, b_3 and b_5 from elements of \mathcal{E} , leading to further units: $\{g_5\}$, $\{r_3\}$. Since $\{r_3\}$ is a unit, we can eliminate r_6 from \mathcal{E} , getting $\{g_6\}$. Applying resolution to $\{g_5\}$, $\{g_6\}$, gives \emptyset . A similar argument can be given when $\{b_2, g_2\}$ is replaced by $\{g_2\}$. Thus we have shown $\langle V, \mathcal{E}, \mathcal{F} \rangle$ does not have Property S.

The Davis-Putnam rules can also lead us to conclude a bihypergraph does have Property S. Consider the problem of showing that the underlying graph in the list coloring example is 2-colorable—say with the colors, red and blue. Assume then that \mathcal{E} is the collection: $\{b_1, r_1\}, \dots, \{b_6, r_6\}$, and \mathcal{F} is the collection: $\{b_i, b_j\}, \{r_i, r_j\}$, for each $\{i, j\}$, an edge of the graph. If, in using the splitting rule, we first replace $\{b_1, r_1\}$ by $\{r_1\}$, the unit clause rule leads to the elimination of r_2, r_4 from \mathcal{E} , giving further units: $\{b_2\}, \{b_4\}$, and these lead to units: $\{r_5\}, \{r_3\}$ and then the unit, $\{b_6\}$. Then, if $T = \{r_1, b_2, r_3, b_4, r_5, b_6\}$, T is a transversal for \mathcal{E} which does not contain any set of \mathcal{F} ; thus, color the odd vertices red and the even vertices, blue.

6. Propositional Representation

We have seen in sections 3,4 that the satisfiability of a conjunctive normal form in propositional logic can be treated as a Property S problem. Conversely, we will show that any Property S problem can be represented as a conjunctive normal form problem. Let $\langle V, \mathcal{E}, \mathcal{F} \rangle$ be a bihypergraph. For each $v \in V$, take a propositional letter p_v . If $E \in \mathcal{E}$, $E = \{e_1, \dots, e_n\}$, let $c_E = (p_{e_1} \vee \dots \vee p_{e_n})$; if $F \in \mathcal{F}$, $F = \{f_1, \dots, f_k\}$, let $c_F = (\neg p_{f_1} \vee \dots \vee \neg p_{f_k})$.

Theorem 6.1. *Let $\langle V, \mathcal{E}, \mathcal{F} \rangle$ be a finite bihypergraph. Then $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S if and only if the cnf $C = \bigwedge_{E \in \mathcal{E}} c_E \wedge \bigwedge_{F \in \mathcal{F}} c_F$ is satisfiable.*

Proof. Suppose $\langle V, \mathcal{E}, \mathcal{F} \rangle$ has Property S and $\{X, V - X\}$ is an S-partition. Assign truth values to the p_v , $v \in V$ as follows: p_v is true if and only if $v \in X$. Since X is a transversal for \mathcal{E} , X intersects each $E \in \mathcal{E}$; thus, at least one $e_i \in E$ belongs to X and p_{e_i} is true. Therefore $\bigwedge_{E \in \mathcal{E}} c_E$ is true. If $F \in \mathcal{F}$, some $f_j \in F$ does not belong to X , since $F \not\subseteq X$. Therefore p_{f_j}

is false and $\neg p_{f_j}$ is true; hence c_F is true, $F \in \mathcal{F}$; that is, $\bigwedge_{F \in \mathcal{F}} c_F$ is true. Therefore C is true under the assignment.

Suppose C is satisfiable and let I be a satisfying assignment for the p_v , $v \in V$. Let X be the set of $v \in V$ such that p_v is true under I . Suppose $E \in \mathcal{E}$; since c_E must be true under I , at least one p_{e_i} , $e_i \in E$ must be true and hence $e_i \in X$. Thus X is a transversal for \mathcal{E} . Suppose $F \in \mathcal{F}$; since c_F is true under I , $\neg p_{f_j}$ must be true for at least one $f_j \in F$. Then p_{f_j} is false under I and so $f_j \notin X$. Therefore $V - X$ is a transversal for \mathcal{F} . Hence $\{X, V - X\}$ is an S-partition for $\langle V, \mathcal{E}, \mathcal{F} \rangle$. ■

Not only can any Property S problem be converted to a cnf satisfiability problem, but, as we shall see, any S-resolution proof of $\emptyset \in [\mathcal{E}]_{\mathcal{F}}$, can be directly converted to a resolution proof of cnf unsatisfiability. This is accomplished by replacing each line in the S-resolution proof by n lines in the cnf proof, where the resolution on that line is of order n . We only illustrate the translation process when $n=3$ —it should then be obvious how to proceed in general. Suppose $\{A, a\}, \{B, b\}, \{C, c\} \vdash_{\{a,b,c\}} \{A, B, C\}$, where A, B, C denote sequences of elements. Then this step can be replaced for the propositional cnf proof by the 3 steps,

- 1) $\{A, a\}, \{\neg a, \neg b, \neg c\} \vdash_{\{a, \neg a\}} \{A, \neg b, \neg c\}$
- 2) $\{A, \neg b, \neg c\}, \{B, b\} \vdash_{\{b, \neg b\}} \{A, B, \neg c\}$
- 3) $\{A, B, \neg c\}, \{C, c\} \vdash_{\{c, \neg c\}} \{A, B, C\}$

We note that in the case of graph colorings, where \mathcal{F} consists entirely of pairs, the translated proof of cnf unsatisfiability will be exactly twice as long as the proof that $\emptyset \in [\mathcal{E}]_{\mathcal{F}}$. It may be that even shorter proofs of cnf unsatisfiability can be found which do not arise from this translation process; however this seems unlikely to us.

7. Conclusion

The various resolution techniques outlined above have yet to be tried on large examples in logic or elsewhere. We feel that there may be advantages in treating the mathematical examples as Property S problems rather than translating them into cnf satisfiability problems since this preserves more of the structure of the original problem and should lead to much shorter

proofs. This would seem to be especially advantageous with the newer massively parallel random approaches to computation such as DNA computing. Finally, as we remarked earlier, much is known about the set-theoretic properties of Property B, in the cases of finite and infinite sets. How much of this carries over to Property S?

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