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SEQUENT CALCULI AND QUASIVARIETIES

A b s t r a c t. We discuss relatively point-regular quasivarieties related in some special sense to sequent calculi. We show that the free algebra in such a quasivariety is Fregean iff the sequent calculus has so-called *symmetric contraction* rules admissible. In the presence of the fusion connective this is equivalent to having contraction. With every sequent calculus $\mathcal{G}$ one can associate, in some way, a sequent calculus with fusion. If this calculus has a separability property then a quasivariety Q related to $\mathcal{G}$ is the class of fusion-less reducts of some quasivariety of algebras with fusion.

1. Introduction

We are interested in the nature of connection between Gentzen-style calculi and relatively point-regular quasivarieties of algebras. Our motivation is, among others, a connection between the sequent calculus LBCK and the quasivariety of BCK-algebras. An implication $t_1 \approx 1, \ldots, t_n \approx 1 \Rightarrow t \approx 1$ is a quasi-identity of BCK-algebras iff some sequent $s_1, \ldots, s_m \rightarrow t$,

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such that the set $\{s_1, \ldots, s_m\} \subseteq \{t_1, \ldots, t_n\}$, is derivable in LBCK. The proof-theoretical properties of LBCK are reflected in the algebraic properties of BCK-algebras. Among others, LBCK results from the implicational fragment of the system LJ by cancelling the contraction rule; this is reflected in the loss of the Fregean property in the quasivariety. We focus here on the quasivarieties that are in a relationship of a similar form with some sequent calculus.

Assume that for a quasivariety Q and a sequent calculus $\mathcal{G}$, an implication $t_1 \approx 1, \ldots, t_n \approx 1 \Rightarrow t \approx 1$ is a quasi-identity of Q iff some sequent $s_1, \ldots, s_m \to t$, such that the set $\{s_1, \ldots, s_m\} \subseteq \{t_1, \ldots, t_n\}$, is derivable in $\mathcal{G}$. We show (Theorem 3.8) that then the free algebra in Q is Fregean iff a *symmetric contraction* rule is admissible in Q. If there is a fusion in the language, then the free algebra in Q is Fregean iff $\mathcal{G}$ has the contraction rule (Proposition 4.4).

In Theorem 5.1 we also show, by adopting the argument used in [7], that if the sequent calculus obtained from $\mathcal{G}$ by adjoing fusion has the separability property, then every algebra in Q is embeddable into an algebra with fusion. We conclude that the quasivarieties for which a Gentzen system with the separability property exists, are rare.

The relationship between quasi-identities and derivable sequents discussed here is very special and there are many Gentzen system — quasivariety correspondences known in the literature which do not satisfy this condition. For example, it does not hold for modal algebras and the Gentzen system GK, for relevance logic and its Gentzen system and many others. The Gentzen systems for modal logics and for the relevance logic have contraction but the corresponding varieties are not Fregean.

In order to capture this general relationship, we propose to use the condition:

$$\text{if } \mathcal{G} \vdash t_1, \ldots, t_n \to t \text{ then } \mathrm{Q} \models \textstyle\bigwedge_{i=1}^{n} t_i \approx 1 \Rightarrow t \approx 1.$$

In the presence of the (CUT) rule, this is a weaker condition than the following:

$$\vdash_{\mathcal{G}} \frac{\{\to t_i : i = 1, \dots n\}}{\to t} \quad \text{iff} \quad \mathrm{Q} \models \bigwedge_{i=1}^{n} t_i \approx 1 \Rightarrow t \approx 1,$$

which we will mention here, but will consider in more details in another paper. An embedding result, analogous to the one obtained here, also holds if the relationship between algebras and sequent calculus is defined by this stronger condition.

2. Preliminaries

We assume the reader's familiarity with the universal algebraic notions of a variety, quasivariety and a relative congruence as well as with the idea of a Gentzen-system as a proof-theoretic presentation of a logic. We will refer to the Gentzen systems LK, LJ, LBCK, FL, $\mathrm{FL_w}$, the definitions of which can be found, for example, in [5]. The sequent calculus GK corresponding to the modal logic K results from LK by adding the rule $\frac{\Gamma \to x}{\Box\Gamma \to \Box x}$, where for a sequence $\Gamma = t_1, \dots, t_n$ of terms, $\Box\Gamma$ denotes $\Box t_1, \dots, \Box t_n$.

2.1. Point-regularity. Let an algebraic type Λ, containing a constant 1, be fixed throughout the paper. Most of the material of this subsection can be deduced from, or even found in, [8, Section 2], where the assertional logic for a pointed class of algebras was defined and properties of point-regular varieties, similar to the ones dicussed here, were obtained.

A quasivariety Q is *relatively point-regular* if there is a constant 1 in its type (i.e., Q is pointed) and for every algebra $\mathbf{A} \in \mathrm{Q}$, and every pair of relative Q-congruences θ, ψ on $\mathbf{A}$, if $1/\theta = 1/\psi$, then $\theta = \psi$. A variety V is called point-regular, if it is relatively point-regular as a quasivariety. For a set Γ of terms, the notation $\Gamma \approx 1$ is used instead of: $\bigwedge\{\gamma \approx 1 : \gamma \in \Gamma\}$. The following proposition in known.

Proposition 2.1. *Let* Q *be a pointed quasivariety. Then* Q *is relatively point-regular iff there exists a finite set* $\Delta(x, y) \subseteq \mathrm{Te}(x, y)^2$ *such that* $\mathrm{Q} \models \Delta(x, y) \approx 1 \Leftrightarrow x \approx y$.

Given a relatively point-regular quasivariety Q, the set Δ of binary terms such that $\mathrm{Q} \models \Delta(x, y) \approx 2 \Leftrightarrow x \approx 1$ will be called the system of *point-regularity terms* for Q. Notice that this condition is equivalent to the conjunction of: $\mathrm{Q} \models \Delta(x, x)$ and $\mathrm{Q} \models \Delta(x, y) \approx 1 \Rightarrow x \approx y$.

With every point-regular quasivariety one can associate its *assertional logic*, i.e., a logic AL_Q such that

$$\frac{\Gamma}{\gamma} \text{ is a rule of } \mathrm{AL}_\mathrm{Q} \text{ iff } \mathrm{Q} \models \Gamma \approx 1 \Rightarrow \gamma \approx 1.$$

If Q is relatively point-regular, then AL_Q is algebraizable and Q is an equivalent algebraic semantics for AL_Q. The set $\Delta(x, y)$ is a congruence system for AL_Q that satisfies the so-called *G*-rule ([2], [8]):

$$\frac{x, y}{\Delta(x, y)}.$$

Suppose that Q is a quasivariety that is not relatively point-regular, but there is a finite set Δ of binary terms that forms a congruence system for AL_Q. This means that Q satisfies the following conditions:

1. $\Delta(x, x) \approx 1$
2. $\Delta(x, y) \approx 1, x \approx 1 \Rightarrow y \approx 1$
3. $\Delta(x, y) \approx 1 \Rightarrow \Delta(\varphi(x), \varphi(y)) \approx 1$, for every unary polynomial φ on the term algebra.

Symmetry and transitivity:

$$\Delta(x, y) \approx 1 \Rightarrow \Delta(y, x) \approx 1,$$

$$\Delta(x, y) \approx 1,\ \Delta(y, z) \approx 1 \Rightarrow \Delta(x, z) \approx 1$$

follow from the conditions 1–3. For example, to show symmetry, suppose that $\mathbf{A} \in \mathrm{Q}$ and $a, b \in A$. Let $\delta(x, y) \in \Delta(x, y)$ and let $\varphi(x) = \delta(x, z)$.

Then φ is a unary polynomial on the term algebra and by 3, we obtain that $\mathbf{A} \models \Delta(x,y) \approx 1 \Rightarrow \Delta(\delta(x,z),\delta(y,z)) \approx 1$. Assume that $\Delta(a,b) = 1^{\mathbf{A}}$. Then $\Delta(\delta(a,a),\delta(b,a)) = 1^{\mathbf{A}}$, for all $\delta(x,y) \in \Delta(x,y)$. By 1 and 2, $\delta(b,a) = 1^{\mathbf{A}}$, for all $\delta(x,y) \in \Delta(x,y)$, finishing the proof of the symmetry condition.

We can also show that for an algebra $\mathbf{A} \in \mathrm{Q}$, if $\Delta'(x,y)$ is another set of binary terms that satisifies the conditions 1–3 on $\mathbf{A}$, then

$$\mathbf{A} \models \Delta(x,y) \approx 1 \Leftrightarrow \Delta'(x,y) \approx 1. \tag{1}$$

For let $\mathbf{A} \in \mathrm{Q}$. Suppose $\Delta(a,b) = 1$, for some elements $a,b \in A$. Let $\delta(x,y) \in \Delta'(x,y)$. It follows that $\Delta(\delta(a,a),\delta(a,b)) = 1$ by the condition 3 above. But $\delta(a,a) = 1$ in $\mathbf{A}$. So for each $\delta(x,y) \in \Delta'(x,y)$, we have that $\delta(a,b) = 1$, so $\Delta(a,b) = 1$.

Adding

$$\Delta(x,y) \approx 1 \Rightarrow x \approx y \tag{2}$$

to the axiomatization of Q we get an axiomatization of a relatively point-regular quasivariety $\mathrm{Q_r}$. The assertional logic $\mathrm{AL}_{\mathrm{Q_r}}$ equals the assertional logic AL_{Q}.

Proposition 2.2. *If* Q *is a variety, with a set* Δ *satisfying 1–3 in* Q, *then* $\mathrm{Q_r}$ *is the largest relatively point-regular quasivariety contained in* $\mathbf{V}$.

Proof. Suppose that $\mathrm{Q}' \subseteq \mathrm{Q}$ is a relatively point-regular subquasivariety of Q, and that $\Delta'(x,y)$ is a system of point-regularity terms for Q'. Let $\mathbf{A} \in \mathrm{Q}'$. Then $\mathbf{A} \models \Delta'(x,y) \approx 1 \Rightarrow x \approx y$. By (1), $\mathbf{A} \models \Delta'(x,y) \approx 1 \Rightarrow x \approx y$ and $\mathbf{A} \in \mathrm{Q_r}$.

For example, let $\mathcal{L}$ be some propositional logic axiomatized, Hilbert style, by a set of axioms Ax and rules $\mathcal{R}$. Assume that the connectives in the language of $\mathcal{L}$ are from $\Lambda \setminus \{1\}$, but that $\mathcal{L}$ satisfies the G-rule. Suppose that there exists a finite set Δ that is a congruence system for $\mathcal{L}$, i.e., the following three conditions hold.

1. $\vdash_{\mathcal{L}} \Delta(x,x)$

2. $\Delta(x,y) \vdash_{\mathcal{L}} \Delta(y,x)$
3. $\Delta(x,y) \vdash \Delta(\varphi(x),\varphi(y))$, for every unary polynomial φ on the formula algebra determined by the language of $\mathcal{L}$.

Let Q be the quasivariety of type Λ, axiomatized by $\{\alpha \approx 1 : \alpha \in \mathrm{Ax}\} \cup \{\bigwedge \Gamma \approx 1 \to \alpha \approx 1 : \frac{\Gamma}{\alpha} \in \mathcal{R}\}$. Then, if 1 is interpreted as the value of any tautology, Q is an algebraic semantics for $\mathcal{L}$. Now define the relatively point-regular quasivariety $\mathrm{Q_r}$ as above. Then $\mathrm{Q_r}$ is an equivalent algebraic semantics for the logic $\mathcal{L}$.

An example of a quasivariety obtained this way is the class of all BCK-algebras, the equivalent algebraic semantics for the BCK-logic. This logic is axiomatized by three axioms: (B) $((p \supset q) \supset ((q \supset r) \supset (p \supset r)))$, (C) $((p \supset (q \supset r)) \supset (q \supset (p \supset r)$ and (K) $(p \supset (q \supset p))$, and the modus ponens rule. The set $\Delta(x,y) = \{x \supset y, y \supset x\}$ is a congruence system that satisfies the G-rule. Let V be the variety axiomatized by setting the (B), (C) and (K) axioms equal to 1 and let Q be the quasivariety obtained by further adding the modus ponens: $x = 1,\ \Delta(x,y) = 1 \Rightarrow y = 1$. Neither V nor Q is (relatively) point-regular. To get a relatively point-regular quasivariety one adds the condition (2) to the axiomatization of Q. The resulting quasivariety is the quasivariety of all BCK-algebras. In the presence of (2), the modus ponens rule is derivable in $\mathrm{V_r}$, so $\mathrm{Q_r} = \mathrm{V_r}$. Thus the class of all BCK-algebras is the largest relatively point-regular quasivariety contained in V and the only proper quasi-identity in the axiomatization of this class is the quasi-identity (2) needed to ensure the point-regularity.

2.2. Sequent calculi. Below, the Greek capital letters will represent sequences or sets of terms, depending on the context. We use the following abbreviations. The concatenation of sequences Γ and Σ is denoted by juxtaposition $\Gamma\Sigma$. By $\{\Gamma\}$ we mean the set of all terms that are entries in Γ. If Γ is a sequence of terms, then $\widetilde{\Gamma}$ is used to denote some sequence such that $\{\widetilde{\Gamma}\} \subseteq \{\Gamma\}$. So $\widetilde{\Gamma}$ may have less terms than Γ and may differ from Γ in their order and numbers of repetition.

By a sequent we mean a pair $\langle \Gamma, t\rangle$, written as $\Gamma \to t$, such that Γ is a finite, possibly empty, sequence of terms and t is a term. A sequent calculus is a deductive system, usually defined in terms of rules, in which the role of the formulas is played by sequents. More technically, let Λ be a fixed type. Then $(\mathrm{Te}_\Lambda(X))^+$ is the set of Λ sequents, and $\langle t_1, \ldots, t_n\rangle$ for $n \geq 1$, is written as $\langle t_1, \ldots, t_{n-1}\rangle \to t_n$. A sequent calculus is then defined as a pair $\langle \Lambda, \mathrm{Cn}\rangle$, for some consequence operator $\mathrm{Cn} : \mathcal{P}((\mathrm{Te}_\Lambda(X))^+) \to \mathcal{P}((\mathrm{Te}_\Lambda(X))^+)$. As is true for deductive systems in general, the operator Cn is uniquely determined by the set of its rules. These rules are also being referred to as *derived rules* of the calculus, as opposed to the *admissible* rules, i.e., rules that do not lead out of the set of the theorems of the calculus. The theorems of the calculus will be called its *derived*, or *derivable sequents.*

Gentzen-style systems are a special case of sequent calculi in this sense. In a Gentzen system, the axiomatic rules are called *initial sequents* and are limited usually to the ones of the form $x \to x$, $x \to 1$, $\to 1$, or $0 \to x$. The rules take the form of schemata, with *second-order variables* ranging over sequences of formulas. More precisely, a generalized sequent is an expression of the form $\Gamma_0, t_1, \ldots, \Gamma_{n-1}, t_n, \Gamma_n \to t$, where each Γ_i is either a second-order variable or the empty sequence, t and t_i's are terms. A rule schema is a pair $\dfrac{S_1, \ldots, S_m}{S}$, where, for $i = 1, \ldots, m$, S_i and S are generalized sequents. A *second-order substitution* ([9]) is a pair of mappings: one assigns finite sequences of terms to the second-order variables and the other is a substitution in the usual sense.

A rule schema encodes the set of all rules that can be obtained from it by a second-order substitution. We say that a sequent calculus $\mathcal{G}$ *has a rule schema* (r) if all second-order substitutions of (r) are among the rules of $\mathcal{G}$. Gentzen systems known in the literature, are presented by means of only finitely many rule schemata which can be classified as either structural or else connective-introduction rules.

Among structural rules, the following rule schema, called (CUT), plays a special role: $\dfrac{\Gamma \to x, \quad \Sigma_1 x \Sigma_2 \to y}{\Sigma_1 \Gamma \Sigma_2 \to y}$. A sequent calculus $\mathcal{G}$ has (CUT) *elimination* if every sequent that can be derived in $\mathcal{G}$ can be derived in $\mathcal{G}$ without an application of (CUT). We will review in Section 4 other standard structural rules: weakening (W), contraction (C) and exchange (E). In substructural logics, some structural rules may be just missing or replaced by some nonstandard ones ([1]). We will consider later certain non-standard structural rules: a modification of (C) and a modification of (W).

The purpose of a Gentzen system is to model the derivation process in a logic in a constructive way. With properly chosen rules, the (CUT) elimination property leads to the separability and to a decision procedure for the logic. Also, algebraic results can be proved using this sequent calculi, like for example [11], [6], [7]).

If the assertional logic of a quasivariety has a Gentzen style presentation, then some model theoretic properties of the quasivariety are reflected particularly simply in the rules of the Gentzen system. For example, dropping the contraction rule from the Gentzen system LJ for the intuitionistic propositional calculus results directly in the loss of the so-called *Fregean* property in the related quasivariety.

A sequent calculus $\mathcal{G}$ will be called *congruential* if for every unary polynomial $\varphi(x)$ on the term algebra, it has a rule schema of the following form:

$$\frac{\Gamma_1 x \Sigma_1 \to y \qquad \Gamma_2 y \Sigma_2 \to x}{\Gamma \varphi(x) \Sigma \to \varphi(y)},$$

for some Γ, Σ such that $\{\Gamma\Sigma\} \subseteq \{\Gamma_1\Gamma_2\Sigma_1\Sigma_2\}$; Γ and Σ depend on Γ_1, Γ_2, Σ_1, Σ_2 in the sense that they can be obtained as concatenation terms depending on Γ_1, Γ_2, Σ_1, Σ_2; moreover, how this concatenation term is formed depends on the polynomial φ.

Finally, we say that a sequent calculus $\mathcal{G}$ *defines a polarity* if for every unary polynomial $\varphi(x)$ either

$$\frac{\Gamma x \Sigma \to y}{\Gamma \varphi(x) \Sigma \to \varphi(y)} \quad \text{or} \quad \frac{\Gamma x \Sigma \to y}{\Gamma \varphi(y) \Sigma \to \varphi(x)}$$

is a rule schema of $\mathcal{G}$. A polarity is a pair $M = \langle M^+, M^- \rangle$ and in the first case $\varphi \in M^+$, in the second: $\varphi \in M^-$.

2.3. Gentzen system and logic. The connection between a Gentzen-system $\mathcal{G}$ and a "*corresponding*" logic $\mathcal{L}$ involves, first of all, the condition that for every term (formula) t,

$$t \text{ is a tautology of } \mathcal{L} \quad \text{iff} \quad \text{the sequent } \to t \text{ is derivable in } \mathcal{G}. \tag{3}$$

If Q is a quasivariety and $\mathcal{L}$ its assertional logic, then the condition (3) translates to

$$\mathrm{Q} \models t \approx 1 \quad \text{iff} \quad \to t \ \text{ is derivable in } \mathcal{G}.$$

This is regarded as our *basic* adequacy condition. The condition does not, however, characterize the relationship between the derived sequents and the quasi-identities of the quasivariety, although, in the presence of (CUT), it implies that if a sequent $\Gamma \to t$ is derivable in $\mathcal{G}$ then $\Gamma \approx 1 \Rightarrow t \approx 1$ is an admissible quasi-identity of Q. More generally, (3) implies that if $\vdash_{\mathcal{G}} \dfrac{\{\to \gamma : \gamma \in \Gamma\}}{\to t}$ is a rule of $\mathcal{G}$ then $\Gamma \approx 1 \Rightarrow t \approx 1$ is an admissible quasi-identity of Q. Recall, that for a deductive system a rule is admissible if it does not lead outside the set of tautologies of the system; a quasi-identity is admissible for a quasivariety Q iff it is a quasi-identity of the free algebra in Q.

For one quasivariety Q there may be many sequent calculi satisfying our basic condition. For example, for the calculus defined as follows:

$$\vdash_{\mathcal{G}} \Gamma \to \gamma \quad \text{iff} \quad \frac{\{\Gamma\}}{\gamma} \text{ is a rule of } \mathrm{AL}_{\mathrm{Q}}. \tag{4}$$

with all admissible rules the property (3) obviously holds. The sequent calculus defined this way has all structural rules. In general, it need not have properties significant from the proof-theoretical point of view, like (CUT)-elimination, while for many nonclassical logics, sequent calculi with such properties exist. In fact, most nonclassical sequent calculi known in the literature, do not satisfy (4). In other words, if there is a general characterization of the quasi-identities of a quasivariety Q in terms of sequents derivable in a sequent calculus $\mathcal{G}$ adequate for Q, then it should be a weaker condition than the one used in (4).

In this paper, we will require that if a Gentzen system $\mathcal{G}$ is adequate for a quasivariety Q then:

$$\text{if } \mathcal{G} \vdash \Gamma \to t \text{ then } \mathrm{AL_Q} \vdash \frac{\{\Gamma\}}{t}. \tag{5}$$

We will also consider another condition:

$$\text{if} \quad \vdash_{\mathcal{G}} \frac{\{\to \gamma : \gamma \in \Gamma\}}{\to t} \quad \text{then} \quad \mathrm{AL_Q} \vdash \frac{\{\Gamma\}}{t}. \tag{6}$$

Note that in the presence of (CUT), (6) implies (5).

There are many examples that the converse of (5) does not hold. For example let $\mathcal{G}$ equal to LBCK and Q equal to the class of all BCK-algebras. Then

$$\vdash_{\mathrm{BCK}} \frac{X}{\gamma} \quad \text{iff} \quad \mathrm{LBCK} \vdash \Sigma \to \gamma \text{ for some } \Sigma \text{ such that } \{\Sigma\} = X. \tag{7}$$

Due to the weakening rule present in LBCK, the right-hand side of (7) can be replaced by a weaker condition: for the quasivariety Q of BCK-algebras and $\mathcal{G} = \mathrm{LBCK}$ we have

$$\vdash_{\mathrm{AL_Q}} \frac{X}{\gamma} \quad \text{iff} \quad \mathcal{G} \vdash \Sigma \to \gamma \text{ for some } \Sigma \text{ such that } \{\Sigma\} \subseteq \Gamma. \tag{8}$$

Clearly, the condition (8) implies (5) but it need not imply (6). Also, (8) characterizes the sequent calculus-quasivariety relationship only in some cases; in particular, it does not hold for the system GK corresponding to the modal logic K (and, in fact, for the Gentzen systems defined for other modal logics: $\mathrm{T}, \mathrm{S4}$ and S5):

For the logic K we have: $\vdash_{\mathrm{K}} \dfrac{\{t_1, \dots, t_n\}}{\gamma}$ iff there are $m_1, \dots, m_n, m$ such that $\mathrm{GK} \vdash \; [m_1]t_1, \dots, [m_n]t_n \to [m]\gamma$, where for m and γ, $[m]t := t \wedge \Box t \wedge \cdots \Box^m t$. This is directly related to the local deduction theorem for K (see [3]) and the fact that if there are the implication introduction rules and (CUT), then the derivability of a sequent $t_1, \dots, t_n \to t$ is equivalent to the derivability of the sequent $\to t_1 \supset \cdots \supset t_n \supset t$, association to the right, which by the basic adequacy condition is equivalent to this last formula being a tautology.

3. Sequent calculi and quasivarieties.

If Γ is a sequence of terms, and Σ is either a set or a sequence of terms, then the notation $\Gamma \to \Sigma$ abbreviates a collection of sequents $\Gamma_\sigma \to \sigma$ such that Γ is the concatenation of all Γ_σ. The notation $\widetilde{\Gamma} \to \sigma$ or $\widetilde{\Gamma} \to \Sigma$ abbreviates that there is a sequence Π such that $\{\Pi\} \subseteq \{\Gamma\}$ that can be put in the place of $\widetilde{\Gamma}$ and turn true the expression in which the original sequent occurs. For example, for terms t, s and a sequence Γ, by $\vdash_{\mathcal{G}} \widetilde{\Gamma} \to \Delta(t, s)$, we abbreviate the statement that there are some sequences Γ_δ, one for each $\delta \in \Delta(x, y)$, such that $\bigcup\{\{\Gamma_\delta\} : \delta \in \Delta(x, y)\} \subseteq \Gamma$ and $\vdash_{\mathcal{G}} \Gamma_\delta \to \delta(x, y)$, for each $\delta \in \Delta$. Finally, for a set $\Delta(x, y)$ of binary terms, we use $\widetilde{\Delta}(x, y)$ to denote certain sequence Γ such that $\{\Gamma\} \subseteq \Delta(x, y)$. Intuitively, but less precisely, $\widetilde{\Gamma} \to \sigma$ may be understood as: the sequent $\Gamma \to \sigma$ up to structural rules.

For the rest of the paper assume that $\mathcal{G}$ is a congruential sequent calculus that has the (CUT) rule and such that the sequents $x \to x$ are among the axioms of $\mathcal{G}$.

Define

$$\widetilde{\mathrm{Q}} := \mathrm{Mod}(\{\Gamma\} \approx 1 \Rightarrow t \approx 1 : \vdash_{\mathcal{G}} \Gamma \to t\}).$$

Then Q is the largest quasivariety such that (5) holds.

Suppose that $\Delta(x, y)$ is a finite set of binary terms and consider the following conditions:

$$\vdash_{\mathcal{G}} \frac{\Gamma \to x, \quad \Sigma_1 y \Sigma_2 \to z}{\Sigma_1 \Gamma \widetilde{\Delta}(x, y) \Sigma_2 \to z} \quad \text{and} \quad \vdash_{\mathcal{G}} \frac{\Gamma \to y, \quad \Sigma_1 x \Sigma_2 \to z}{\Sigma_1 \Gamma \widetilde{\Delta}(x, y) \Sigma_2 \to z}; \tag{9}$$

$$\vdash_{\mathcal{G}} \frac{\Gamma_1 x \Sigma_1 \to y, \quad \Gamma_2 y \Sigma_2 \to x}{\widetilde{\Pi} \to \Delta(x, y)}, \tag{10}$$

where $\Pi = \Gamma_1 \Sigma_1 \Gamma_2 \Sigma_2$.

If $\mathcal{G}$ has weakening then (10) implies

$$\vdash_{\mathcal{G}} \frac{\Gamma \to x, \quad \Sigma \to y}{\widetilde{(\Gamma\Sigma)} \to \Delta(x,y).} \tag{11}$$

Observe that (9) holds iff

$$\vdash_{\mathcal{G}} x\ \widetilde{\Delta}(x,y) \to y \quad \text{and} \quad \vdash_{\mathcal{G}} y\ \widetilde{\Delta}(x,y) \to x \tag{12}$$

and that (11) is equivalent to

$$\vdash_{\mathcal{G}} \widetilde{x,y} \to \Delta(x,y). \tag{13}$$

Also, (10) implies that

$$\vdash_{\mathcal{G}} \frac{x \to y \quad y \to x}{\to \Delta(x,y)} \tag{14}$$

and therefore (9) and (10) yield that the sequent $\to \Delta(x,y)$ is interderivable with the pair of sequents $x \to y;\ y \to x$.

Proposition 3.1. *Suppose $\mathcal{G}$ satisifies (9) and (10). The following are derivable in $\mathcal{G}$:*

$$\vdash_{\mathcal{G}} \to \Delta(x,x) \tag{15}$$

$$\vdash_{\mathcal{G}} \widetilde{\Delta}(x,y) \to \Delta(y,x) \tag{16}$$

$$\vdash_{\mathcal{G}} \widetilde{\Delta}(x,y)\widetilde{\Delta}(y,z) \to \Delta(x,z) \tag{17}$$

$$\vdash_{\mathcal{G}} \widetilde{\Delta}(x,y) \to \Delta(\lambda(x,\vec{z}), \lambda(y,\vec{z})), \tag{18}$$

for all operations λ from the type Λ.

Proof. Recall that (14) follows from (10). By (14) and the axiom $x \to x$ we get (15). By (10), the following is a derived rule

$$\frac{y\widetilde{\Delta}(x,y) \to x \quad x\widetilde{\Delta}(y,x) \to y}{\widetilde{\Delta}(x,y) \to \Delta(y,x)},$$

the premisses of which are derived sequents, by means of (12). This gives (16). To derive (18) we use again the same premisses and the following derivations

$$\frac{x\,\widetilde{\Delta}(x,y) \to y \quad y\,\widetilde{\Delta}(x,y) \to x}{\lambda(x,\vec{z})\,\Gamma_1(x,y) \to \lambda(y,\vec{z})} \quad \text{and} \quad \frac{x\,\widetilde{\Delta}(x,y) \to y \quad y\,\widetilde{\Delta}(x,y) \to x}{\lambda(x,\vec{z}),\Gamma_2(x,y) \to \lambda(y,\vec{z})},$$

for some $\{\Gamma_1\Gamma_2\} \subseteq \Delta(x,y)$, existing by the assumption that $\mathcal{G}$ is congruential. Then (18) follows from (10). Next, using (18) we get $\vdash_{\mathcal{G}} \Delta(x,y) \to \Delta(\delta(y,z),\delta(x,z))$ for all $\delta \in \Delta(x,y)$. So $\vdash_{\mathcal{G}} \widetilde{\Delta}(x,y)\widetilde{\Delta}(y,z) \to \Delta(x,z)$, (17).

Corollary 3.2. *Under the assumptions of Proposition 3.1, $\Delta(x,y)$ is a congruence system for $\widetilde{\mathrm{Q}}$.*

Proof. The properties: $\widetilde{\mathrm{Q}} \models \Delta(x,x) \approx 1$ and $\widetilde{\mathrm{Q}} \models \Delta(x,y) \approx 1 \Rightarrow \Delta(\lambda(x,\vec{z}),\lambda(y,\vec{z}))$ follow from the above proposition. Also, $\vdash_{\mathcal{G}} x,\Delta(x,y) \to y$, so the modus ponens property: $\widetilde{\mathrm{Q}} \models \Delta(x,y) \approx 1, x \approx 1 \Rightarrow y \approx 1$ holds.

Definition 3.3. *Let* Q *be a relatively point-regular quasivariety with a point-regularity system $\Delta(x,y)$. Let $\mathcal{G}$ be a congruential sequent calculus with (*CUT*) and such that the sequents $x \to x$ are initial in $\mathcal{G}$. Then we say that $\mathcal{G}$ is* adequate *for* Q *if the conditions (9)–(11) hold, and for all sequences Γ of terms and for every term t:*

$$\mathrm{Q} \models t \approx 1 \text{ iff } \vdash_{\mathcal{G}} \to t$$

and (5), i.e., if $\vdash_{\mathcal{G}} \Gamma \to t$ then $\mathrm{Q} \models \Gamma \approx 1 \Rightarrow t \approx 1$.

Recall the definition of Q_r from subsection 2.1. For a sequent calculus $\mathcal{G}$ satisfying (9)–(11), define $\mathrm{Q}(\mathcal{G}) := (\widetilde{\mathrm{Q}})_r$. Then

$$\mathrm{Q}(\mathcal{G}) := \mathrm{Mod}\Big(\{\Gamma\} \approx 1 \Rightarrow t \approx 1 : \vdash_{\mathcal{G}} \Gamma \to t\} \cup \{\Delta(x,y) \approx 1 \Rightarrow x \approx y\}\Big).$$

Proposition 3.4. *A Gentzen system $\mathcal{G}$ satisfying (9)–(11) is adequate for $\mathrm{Q}(\mathcal{G})$. The quasivariety $\mathrm{Q}(\mathcal{G})$ is the largest relatively point-regular quasivariety for which $\mathcal{G}$ is adequate.*

Proof. The first statement is clear. Suppose that Q' is a relatively point-regular quasivariety, such that $\mathcal{G}$ is adequate for Q'. Then $\mathrm{Q}' \subseteq \widetilde{\mathrm{Q}}$.

Since $\Delta(x,y)$ is a system of point-regularity terms for $\widetilde{\mathrm{Q}}$, it follows from Proposition 2.2 that $\mathrm{Q}' \subseteq (\widetilde{\mathrm{Q}})_r$.

For the next lemma we require that the sequent calculus $\mathcal{G}$ has weakening and axiom $\to 1$. So $\Gamma \to 1$ and $\dfrac{\Gamma_1\Gamma_2 \to x}{\Gamma_1 1 \Gamma_2 \to x}$ are a derivable sequent and a rule of $\mathcal{G}$. This yields that the sequents $\Gamma_1\Gamma_2 \to x$ and $\Gamma_1 1 \Gamma_2 \to x$ are interderivable.

Lemma 3.5. *Assume that $\mathcal{G}$ satisfies (9)–(11), has weakening and axiom: $\to 1$.*

1. *Suppose that $\mathrm{Q}(\mathcal{G}) \models \bigwedge_{i=1}^n t_i \approx s_i \Rightarrow t \approx s$. Then $\vdash_{\mathcal{G}} t\Gamma \to s$ and $\vdash_{\mathcal{G}} s\Sigma \to t$ for some sequences Γ and Σ such that $\{\Gamma\Sigma\} \subseteq \bigcup_{i=1}^n \Delta(t_i, s_i)$.*
2. *If $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is a rule of AL_Q, then a sequent $\beta_1 \ldots \beta_n \to \alpha$ is derivable in $\mathcal{G}$, for some terms $\beta_1, \ldots, \beta_m \in \{\alpha_1, \ldots, \alpha_n\}$.*

Proof. Let us first observe that sequents $\widetilde{\Delta}(t,1) \to t$ and $t \to \Delta(t,1)$ are derivable in $\mathcal{G}$ for every term t, because $1, \widetilde{\Delta}(t,1) \to t$ and $t, 1 \to \Delta(t,1)$ are derivable, by (12) and (13). By this observation, the second part of the lemma follows from the first.

To prove the first part of the lemma, let S be the set of all quasi-equations of the form $\bigwedge_{i=1}^n t_i \approx s_i \Rightarrow t \approx s$ for which there is $\{\Gamma\} \subseteq \bigcup_{i=1}^n \Delta(t_i, s_i)$ such that $\vdash_{\mathcal{G}} \Gamma, t \to s$. We prove by induction that every quasi-identity of $\mathrm{Q}(\mathcal{G})$ is in S. Suppose that $\bigwedge_{i=1}^n t_i \approx 1 \Rightarrow t \approx 1$ is in the basis of $\widetilde{\mathrm{Q}}$. Then there exists a derivable sequent $\Gamma \to t$ such that $\{\Gamma\} \subseteq \{t_1, \ldots, t_n\}$. By our assumption on weakening and $\to 1$, also the sequents $1, \Gamma \to t$ and $t, \Gamma \to 1$ are derivable. In view of our observation, this suffices to show that $\bigwedge_{i=1}^n t_i \approx 1 \Rightarrow t \approx 1 \in S$. Also, the quasi-identity $\Delta(x,y) \Rightarrow x \approx y$ belongs to S, as well as the following axiom and quasi-identities: $x \approx x$, $x \approx y, y \approx z \Rightarrow x \approx z$, $x \approx y \Rightarrow \lambda(x, \vec{z}) \approx \lambda(y, \vec{z})$, for every $\lambda \in \Lambda$.

Clearly, S is closed under substitution. The quasi-identity $x \approx y \Rightarrow x \approx y$ is in S, because $\vdash_{\mathcal{G}} x, \Delta(x,y) \to y$. For a subset $\{t_i \approx s_i : i = 1, \ldots, n\}$ of $\{t_i \approx s_i : i = 1, \ldots, m\}$, if $\bigwedge_{i=1}^n t_i \approx s_i \Rightarrow t \approx s$ is in S, then so is $\bigwedge_{i=1}^{m} t_i \approx s_i \Rightarrow t \approx s$. Finally, suppose that for each $j = 1, \ldots, m$ a quasi-identity $\bigwedge_{i=1}^{n_j} t_1^j \approx s_i^j \Rightarrow t^j \approx s^j$ is in S and that $\bigwedge_{j=1}^{m} t^j \approx s^j \Rightarrow t \approx s$ is in S as well. Then for each j, the sequents $t^j, \Gamma^j \to s^j$ and $s^j, \Gamma^j \to t^j$are derivable for some $\{\Gamma^j \Sigma^j\} \subseteq \bigcup_{i=1}^{n_j} \Delta(t_i^j, s_i^j)$. Then $\Pi^j \to \Delta(t^j, s^j)$ is derivable as well, for a sequence Π^j such that $\widetilde{\Pi}^j \subseteq \bigcup_{i=1}^{n_j} \Delta(t_i^j, s_i^j)$. Using the sequent the derivability of which is implied by the assumption that $\bigwedge_{j=1}^{m} t^j \approx s^j \Rightarrow t \approx s$ belongs to S, and the (CUT) rule, we obtain that

$$\bigwedge_{j=1}^{m} \bigwedge_{i=1}^{n_j} t_i^j \approx s_i^j \Rightarrow t \approx s$$

is in S as well.

Corollary 3.6. *Let $\mathcal{G}$ be as above. Then $t \approx s$ is an identity of $\mathrm{Q}(\mathcal{G})$ iff the sequents $t \to s$ and $s \to t$ are derivable in $\mathcal{G}$.*

Recall that the filters of the logic $\mathrm{AL_Q}$ on an algebra $\mathbf{A} \in \mathrm{Q}$ coincide with the equivalence classes of the element $1^{\mathbf{A}}$ modulo the Q-congruences on $\mathbf{A}$. The filters of the logic $\mathrm{AL_Q}$ will be called Q-filters. The symbol $\mathrm{Fg}_{\mathrm{Q}}^{\mathbf{A}}(a)$ denotes the Q-filter on $\mathbf{A}$ generated by a. Our next lemma and theorem refer to $\mathrm{Q} = \mathrm{Q}(\mathcal{K})$. Let $\mathbf{F}$ be the free algebra in Q and let $\equiv_{\mathrm{Q}}$ be the congruence such that $\mathbf{F} = \mathrm{Te}/{\equiv_{\mathrm{Q}}}$. Then $t \equiv_{\mathrm{Q}} s$ iff $\mathrm{Q} \models t \approx s$. By Corollary 3.6, $t \equiv_{\mathrm{Q}} s$ iff the sequents $t \to s$ and $s \to t$ are derivable in $\mathcal{G}$.

Lemma 3.7. *Suppose that $\mathcal{G}$ is a sequent calculus with weakening and axiom $\to 1$, such that (9)–(11) hold. Let $\mathbf{F} \in \mathrm{Q}(\mathcal{G})$ be the free algebra in $\mathrm{Q}(\mathcal{G})$ and let t and s be terms. Then*

$$t/{\equiv_{\mathrm{Q}}} \in \mathrm{Fg}_{\mathrm{Q}}^{\mathbf{A}}(s/{\equiv_{\mathrm{Q}}}) \quad \textit{iff} \quad \textit{a sequent } s, \ldots, s \to t \textit{ is derivable in } \mathcal{G},$$

for some number, possibly 0, of repetitions of the term s.

Proof.

The implication from right to left is clear. In the other direction, let $\mathrm{Q} = \mathrm{Q}(\mathcal{G})$ and assume that $t/{\equiv_{\mathrm{Q}}} \in \mathrm{Fg}_{\mathrm{Q}}^{\mathbf{A}}(s/{\equiv_{\mathrm{Q}}})$. If $t/{\equiv_{\mathrm{Q}}} = s/{\equiv_{\mathrm{Q}}}$ then in view of discussion preceeding the lemma, the sequent $s \to t$ is derivable.

For the inductive step, suppose that there is a rule $\dfrac{\alpha_1,\dots,\alpha_n}{\alpha}$ of AL_{Q} and a homomorphism $h : \mathrm{Te} \longrightarrow \mathbf{F}$ such that for each $i = 1,\dots,n$, $h(\alpha_i) \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(s/{\equiv_{\mathrm{Q}}})$ and $h(\alpha) = t$. For every generator x of the term algebra, let s_x be a term such that $h(x) = s_x/{\equiv_{\mathrm{Q}}}$. Define a function f on the generators, by putting $f(x) = s_x$. Then f extends to a homomorphism such that for every term γ, $h(\gamma) = f(\gamma)/{\equiv_{\mathrm{Q}}}$. In particular, for each $i = 1,\dots,n$, $h(\alpha_i) = f(\alpha_i)/{\equiv_{\mathrm{Q}}}$. By the induction hypothesis, for each $i = 1,\dots,n$, a sequent $s,\dots,s \to f(\alpha_i)$ is derivable in $\mathcal{G}$. By lemma 3.5, a sequent $\beta_1,\dots,\beta_m \to \alpha$ is derivable for some $\{\beta_1,\dots,\beta_m\} \subseteq \{\alpha_1,\dots,\alpha_n\}$. So also $\vdash_{\mathcal{G}} f(\beta_1),\dots,f(\beta_m) \to f(\alpha)$, and by (CUT), a sequent $s\dots,s \to f(\alpha)$ is derivable as well. But $\vdash_{\mathcal{G}} f(\alpha) \to t$ because $h(\alpha) = t$. Applying (CUT) again yields $\vdash_{\mathcal{G}} s,\dots,s \to t$.

An algebra $\mathbf{A}$ in a relatively point-regular quasivariety Q is *Fregean* relatively to Q if for all $a, b \in A$, $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(a) = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(b)$ iff $a = b$. A quasivariety Q is Fregean iff every algebra in it is Fregean relatively to Q.

A *symmetric contraction* rule is any rule of the form

$$(\mathrm{sC}_{nm}) \qquad \frac{x\dots x \to y \quad y\dots y \to x}{x \to y},$$

where $m \geq 1$ is the number of the occurrences of x in the first premiss and $n \geq 1$ the number of the occurrences of y in the second premiss of the rule.

Theorem 3.8. *Assume that the quasivariety* Q *is Fregean and that there is a sequent calculus* $\mathcal{G}$ *with weakening and axiom* $\to 1$*, such that (9)–(11) hold and such that* $\mathrm{Q} = \mathrm{Q}(\mathcal{G})$*. Then all the symmetric contraction rules are admissible in* $\mathcal{G}$*. If all the symmetric contraction rules are admissible in* $\mathcal{G}$ *then the free algebra in* Q *is Fregean.*

Proof. First suppose that Q is Fregean and that, for some terms t and s and numbers m and n, sequents $s \cdots s \to t$ and $t \cdots t \to s$ are derivable in $\mathcal{G}$. Then $t/{\equiv_{\mathrm{Q}}} \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(s/{\equiv_{\mathrm{Q}}})$ and $s/{\equiv_{\mathrm{Q}}} \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(t/{\equiv_{\mathrm{Q}}})$, so the two filters are equal. Hence $t/{\equiv_{\mathrm{Q}}} = s/{\equiv_{\mathrm{Q}}}$ and therefore, $s \to t$ is derivable. This shows that the symmetric contraction rules are admissible.

For the other direction, consider the free algebra $\mathbf{F}$ in Q. Let $t/{\equiv_{\mathrm{Q}}}$ and $s/{\equiv_{\mathrm{Q}}} \in F$ b e such that $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(t/{\equiv_{\mathrm{Q}}}) = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{Q}}(s/{\equiv_{\mathrm{Q}}})$. By the preceeding lemma, $s\ldots, s \to t$ and $t\ldots, t \to s$ are derivable sequents. If the number of repetitions of s in the first of these sequents is 0 then $\to t$ is derivable and $t \approx 1$ is an identity of Q. Then also $s \approx 1$ is an identity and the sequents $s \to t$ and $t \to s$ are derivable. Now suppose that the number of repetitions in neither of the two derivable sequents equals 0. Then by the admissibility of the symmetric contraction, $t \to s$ and $s \to t$ are derivable.

Let us finish this section by a remark that in general, a quasivariety corresponding to a sequent calculus is not of the form $\mathrm{Q}(\mathcal{G})$. A stronger condition, namely (6) is more appropriate in these cases. One can define a quasivariety $\mathrm{K}(\mathcal{G})$ as the class of models of all quasi-identities $\bigwedge_{i=1}^{n} t_i \approx 1 \Rightarrow t \approx 1$ such that $\vdash_{\mathcal{G}} \dfrac{\to t_i, \ldots, \to t_n}{\to t}$. Then $\mathrm{K}(\mathcal{G}) \subseteq \widetilde{\mathrm{Q}}(\mathcal{G})$. The quasivarieties of this kind will be considered in another paper. Even for substructural logics, the two classes need not be the same. For example, consider the system $\mathrm{FL_w}$. The rule $\dfrac{q}{p \supset p \,\&\, q}$ is a rule of the assertional logic of $\mathrm{K}(\mathrm{FL_w})$, while not of $\mathrm{Q}(\mathrm{FL_w})$, for no sequent $q \ldots q \to p \supset p \,\&\, q$ can be derived in $\mathrm{FL_w}$.

4. The fusion connective

For a logic that has a Gentzen-style presentation, a connective & is called a *fusion* if it repaces the comma in the sequences of terms occurring in the antecedents of the sequents. The algebraic models of a Gentzen system with fusion are monoids with some additional operations. In this section, we consider three notions of fusion: algebraic fusion, fusion in ordered algebras and fusion defined by a Gentzen system or a sequent calculus. The first two notions differ, while the fusion in sequent calculi can be regarded as a fusion in ordered algebras.

We say that a binary connective & is an *a-fusion* (algebraic fusion) on an algebra $\mathbf{A}$ if & is associative and, moreover, $\mathbf{A}$ satisfies the following conditions:

$$1 \mathbin{\&} 1 = 1 \tag{19}$$

$$x \mathbin{\&} y = 1 \Rightarrow x = 1 \tag{20}$$

$$x \mathbin{\&} y = 1 \Rightarrow y = 1 \tag{21}$$

Our definition implies that $x \mathbin{\&} 1 = 1$ iff $x = 1$, but does not imply that $x \mathbin{\&} 1 = x$. If also implies that the relative-congruence classes of 1, i.e., filters of the assertional logic of $\mathbf{A}$ are closed under the a-fusion. It follows from (19)–(21) that on $\mathbf{A}$, the finitely generated filters, are principal: for all $a, b \in A$, $\mathrm{Fg}(a, b) = \mathrm{Fg}(a \mathbin{\&} b)$. In particular, $\mathrm{Fg}(a) = \mathrm{Fg}(a \mathbin{\&} a)$.

A deductive system $\mathcal{S}$ is Fregean, if for every matrix model $\mathfrak{A}$ of $\mathcal{S}$ and for all $a, b \in A$, we have: $\mathrm{Fg}_{\mathcal{S}}(a) = \mathrm{Fg}_{\mathcal{S}}(b)$ implies that $a = b$. A relatively point-regular quasivariety Q is Fregean, if the logic AL_{Q} is Fregean. The next proposition follows.

Proposition 4.1. *If a relatively point-regular quasivariety* Q *has an a-fusion and is Fregean, then the fusion is idempotent and commutative in* Q.

An ordered algebra is a structure $\mathfrak{A} = \langle \mathbf{A}, \leq \rangle$, where $\mathfrak{A}$ is an algebra and $\leq$ is a partial order relation. We will also assume, in addition, that the constant 1 is a maximal element of $\mathfrak{A}$. Suppose that all the unary polynomials on the term algebra of the given type are classified into two subsets: M^+ and M^-. The pair $M = \langle M^+, M^- \rangle$ is called a *polarity.* We say that $\mathfrak{A}$ is M-ordered if every polynomial from M^+ defines a monotone operation on $\mathfrak{A}$, and every polynomial from M^- defines an antimonotone operation on $\mathfrak{A}$.

We say that an operation & on an ordered algebra $\mathfrak{A} = \langle \mathbf{A}, \leq \rangle$ is a $\leq$*-fusion* on $\mathfrak{A}$ if it is an a-fusion on $\mathbf{A}$ and $\mathfrak{A}$ satisfies the following condition:

$$\frac{x \leq y, z \leq u}{x \mathbin{\&} z \leq y \mathbin{\&} u}. \tag{22}$$

One can consider order-filters on an ordered algebra with an $\leq$-fusion. Without any special conditions connecting orders with congruences, the order filters are not the same as the filters of the assertional logic. By a $(\leq, \&)$-filter we mean an upward closed subset of A that is closed under & and contains the 1. It follows that an $(\leq, \&)$-filter $\mathrm{Fg}^{(\leq)}(a, b)$ generated by a, b is a sub-$(\leq, \&)$-filter of the $(\leq, \&)$-filter $\mathrm{Fg}^{(\leq)}(a \& b)$ generated by $a \& b$: $\mathrm{Fg}^{(\leq)}(a, b) \subseteq \mathrm{Fg}^{(\leq)}(a \& b)$ The inclusion can be reversed if the $\leq$-fusion satisfies also the condition

$$x \,\&\, y \leq x; \quad x \,\&\, y \leq y. \tag{w}$$

The meet operation, if it exists, will be called conjunction on an ordered algebra $\mathfrak{A}$. Notice that if & is an $\leq$-fusion on $\mathfrak{A}$, then & is a conjunction on $\mathfrak{A}$ iff it is a commutative and idempotent operation that satisfies (w). The commutativity of & is expressed as

$$x \,\&\, y \leq y \,\&\, x. \tag{e}$$

and the idempotency as the conjunction of the following two conditions:

$$x \leq x \,\&\, x \tag{c}$$

and

$$x \,\&\, x \leq x. \tag{ww}$$

Notice that although the idempotency and commutativity of fusion can be expressed both algebraically and in terms of the orders, the conditions (w) and (ww) do not have purely algebraic versions. For a fixed polarity M, an $\leq$-fusion & is called an *M-fusion* if for every $\varphi \in M^+$ and for every $\psi \in M^-$

$$\begin{array}{c} \dfrac{x \,\&\, y \,\&\, z \leq u}{x \,\&\, \varphi(y) \,\&\, z \leq \varphi(u)} \quad \text{and} \\[2ex] \dfrac{x \,\&\, y \,\&\, z \leq u}{x \,\&\, \psi(u) \,\&\, z \leq \psi(y)} \end{array} \tag{23}$$

Finally, suppose that $\mathcal{G}$ is a Gentzen system. A connective & is called a g-fusion for $\mathcal{G}$ if the following two rules are admissible for $\mathcal{G}$:

$$\frac{\Gamma, x, y, \Delta \to z}{\gamma, x \,\&\, y, \Delta \to z} \tag{$\& \to$}$$

(→ &) $$\frac{\Gamma \to x \quad \Delta \to y}{\Gamma, \Delta \to x \,\&\, y}$$

If the Gentzen system $\mathcal{G}$ with a g-fusion has the (CUT) rule, then it is easy to show, that a sequent $t_1, \ldots, t_n \to \gamma$ is derivable in $\mathcal{G}_\&$ iff $t_1 \,\&\, \cdots \,\&\, t_n \to \gamma$ is derivable in $\mathcal{G}$. A connective $\wedge$ is a conjunction in a Gentzen system $\mathcal{G}$ iff $\mathcal{G}$ has the following two rules:

(∧ →) $$\frac{\Gamma, x, \Delta \to z}{\Gamma, x \wedge y, \Delta \to z}, \quad \frac{\Gamma, y, \Delta \to z}{\Gamma, x \wedge y, \Delta \to z}$$

(→ ∧) $$\frac{\Gamma \to x \quad \Gamma \to y}{\Gamma \to x \wedge y}$$

Suppose that $\mathcal{G}$ has (CUT). If & is a g-fusion for $\mathcal{G}$ and $\mathcal{G}$ the following rules (C) and (W):

(C) $$\frac{\Gamma, x, x, \Delta \to y}{\Gamma, x, \Delta \to y}$$

(W) $$\frac{\Gamma, \Delta \to z}{\Gamma, x, \Delta \to z}$$

then the connective & is a conjunction in $\mathcal{G}$. Conversely, in the presence of (C) and (W), a connective $\wedge$ such that the two rules for conjunction are admissible in $\mathcal{G}$, is also a fusion in $\mathcal{G}$. In fact,

Proposition 4.2. *Suppose that & is a g-fusion in $\mathcal{G}$. Then the following are equivalent:*

(i) *& is a conjunction in $\mathcal{G}$,*

(ii) *$\mathcal{G}$ has (W) and (C),*

(iii) *$\mathcal{G}$ has (W), (C) and (E).*

Proposition 4.3. *Suppose that $\wedge$ is a conjunction for $\mathcal{G}$. Then the following are equivalent:*

(i) *$\wedge$ is a g-fusion for $\mathcal{G}$,*

(ii) *$\mathcal{G}$ has (W) and (C),*

(iii) *$\mathcal{G}$ has (W), (C) and (E).*

The Gentzen systems LJ and GK, for example, have conjunction and all structural rules; so the conjunction is fusion for these systems. In the proofs from left to the right of the above propositions, it is essential that there is a connective that satisfies both the rules for g-fusion and the rules for conjunction. In the deductions, the (CUT) rule must be used. It was remarked in [5, page 225] that although in $\mathrm{FL_{cw}}$ (which is the sequent calculus defined as FL with (C) and (W) added) the rule (E) can be derived from the other rules, the (CUT) rule is necessary for this derivation and moreover, (CUT) cannot be eliminated from $\mathrm{FL_{cw}}$, while it can be eliminated from $\mathrm{FL_{cew}}$ (the calculus obtained from $\mathrm{FL_{cw}}$ by adding (E), i.e., LJ). Recall that (CUT) elimination means that if a sequent is a theorem of the system, then there is a proof of this sequent that does not use the (CUT) rule. So there are sequents derivable by means of the rules of FL, (C), (W) and (E); and if we want to replace the use (E) by its derivation from FL with (C) and (W), we have to use (CUT) that cannot be eliminated. If we do not want to use (CUT) we do not have to, but we need to use (E) instead.

The following proposition can also be easily observed.

Proposition 4.4. *Suppose that $\mathcal{G}$ is congruential, has (W) and the following axiom $\to 1$. Let $\mathrm{Q} = \mathrm{Q}(\mathcal{G})$ be a relatively point-regular quasivariety with a-fusion. Then the free algebra in Q is Fregean iff $\mathcal{G}$ has (C)*

Proof. First observe that if & is a g-fusion then its interpretation in Q is an a-fusion. As observed earlier, $\mathrm{Fg}(x) = \mathrm{Fg}(x \,\&\, x)$ in the free algebra of Q. So if it is Fregean, $x = x \,\&\, x$ is an identity of Q and $x \longrightarrow x \,\&\, x$, which yields (C). Conversely, (C) implies all the rules of symmetric contraction, so the free algebra in Q is Fregean.

Let us also remark that in the presence of a g-fusion & and the following weakened version of (W):

$$\text{(WW)} \qquad \frac{\Gamma x \Sigma \to y}{\Gamma x x \Sigma \to y},$$

contraction follows directly from the admissibility of symmetric contraction. Indeed, $x, x \to x \,\&\, x$ and $x \,\&\, x \to x$ are derived sequents, whenever we have a g-fusion and (WW). By symmetric contraction, $x \to x \,\&\, x$ is also a derived sequent. This yields (C). If the nature of the sequent calculus – quasivariety relationship is different than the one considered in the assumption of Proposition 4.4, the conclusion of the proposition need not hold. For example, the Gentzen system GK, strongly adequate to the class of modal algebras, has conjunction. In the presence of all the structural rules, this conjunction is also a fusion. GK has contraction but the class of modal algebras is not Fregean.

Defining the class of ordered algebras as the class of models of the theory of the axioms of partial order and the following implications:

$$t_1 \le s_1, \ldots, t_n \le s_n \Rightarrow t \le s,$$

for every

$$\vdash_{\mathcal{G}} \frac{t_1 \to s_1, \ldots, t_n \to s_n}{t \to s}$$

of $\mathcal{G}$, we obtain a so-called ordered quasivariety $\langle \widetilde{\mathcal{K}}(\mathcal{G}), \le \rangle$ such that $\mathcal{G}$ is strongly adequate for $\widetilde{\mathrm{K}}(G)$. Suppose that $\mathcal{G}$ has a g-fusion. Then $\langle \widetilde{\mathcal{K}}(\mathcal{G}), \le \rangle$ has a $\le$-fusion and $\mathcal{G}$ has rules: (W), (WW), (E) and (C) respectively, iff $\langle \widetilde{\mathcal{K}}(\mathcal{G}), \le \rangle$ satisfies (w), (ww), (e) and (c) respectively. Also, if $\mathcal{G}$ defines a polarity M then the fusion in $\langle \widetilde{\mathcal{K}}(\mathcal{G}), \le \rangle$ is an M-fusion. A connective $\wedge$ is a conjunction in $\mathcal{G}$ iff its interpretation in $\langle \widetilde{\mathcal{K}}(\mathcal{G}), \le \rangle$ is a meet operation on the members of $\langle \mathcal{K}(\mathcal{G}), \le \rangle$. Conversely, for a quasivariety K of ordered algebras, one can introduce a sequent calculus that is strongly adequate for K. This idea will also be discussed elsewhere.

5. Embedding

5.1. The quasivariety $\mathrm{Q}_{\&}$. For a quasivariety Q let $\mathrm{Q}_{\&}$ be the class $\{\langle \mathbf{A}, \& \rangle : \mathbf{A} \in \mathrm{Q}, \& \text{ is an a-fusion}\}$. Thus, $\mathrm{Q}_{\&}$ is a quasivariety axiomatized by the quasi-identities of Q and (26)–(28). Observe that if Q is relatively point-regular, then so is $\mathrm{Q}_{\&}$. Let $\langle \mathbf{A}, \& \rangle \in \mathrm{Q}_{\&}$ and let $F \subseteq A$. Then F is a $\mathrm{Q}_{\&}$-filter on $\langle \mathbf{A}, \& \rangle$ iff F is a Q-filter on $\mathbf{A}$ and is closed under the & operation. For a sequent calculus $\mathcal{G} = \langle \Lambda, \mathrm{Cn} \rangle$ let $\mathcal{G}_{\&}$ be the sequent calculus $\langle \Lambda_{\&}, \mathrm{Cn}_{\&} \rangle$, where $\Lambda_{\&}$ results from Λ by adjoining the binary connective symbol & and $\mathrm{Cn}_{\&}$ results from Cn by adding the two rules for fusion: $(\& \to)$ and $(\to \&)$.

Observe that always $\mathrm{Q}(\mathcal{G}_{\&}) \subseteq (\mathrm{Q}(\mathcal{G}))_{\&}$ and $\boldsymbol{K}(\mathcal{G}_{\&}) \subseteq (\boldsymbol{K}(\mathcal{G}))_{\&}$.

5.2. Embedding theorems. We discuss two related embedding results. The proofs are based on [7]. We say that a sequent calculus $\mathcal{G}_{\&}$ has the *separability property* over & if every sequent S derivable in $\mathcal{G}$ and such that & does not occur in S, has a derivation in the &-less fragment of $\mathcal{G}$. This property is usually a consequence of the (CUT)-elimination property for $\mathcal{G}_{\&}$, which in turn, when the introduction rules are standard, often is a consequence of the (CUT)-elimination in $\mathcal{G}$.

For the second theorem in this section we consider the strong separability property. A sequent calculus $\mathcal{G}$ has this property if every rule of the form $\dfrac{\to t_1, \ldots, \to t_n}{\Gamma \to t}$ in $\mathcal{G}_{\&}$ in which & does not occur, can be derived by means of the rules of $\mathcal{G}$. If $\mathbf{B}$ is an algebra the type of which has fusion, then by $\mathbf{B}^{\mathrm{r}}$ we denote the fusion-less reduct of $\mathbf{B}$.

Theorem 5.1.

(i) *Let* Q *be a point-regular quasivariety with the point-regularity system* $\Delta(x, y)$ *and let* $\mathcal{G}$ *be a congruential sequent calculus satisfying (9)–(11) and such that* $\mathcal{G}_{\&}$ *has separability property.*

(ii) *Assume that* $\mathcal{G}$ *is adequate for* Q*. Then for every algebra* $\mathbf{A} \in$ Q *there is an algebra* $\mathbf{B} \in \mathrm{Q}_{\&}$ *such that* $\mathbf{A}$ *is embeddable into* $\mathbf{B}^{\mathrm{r}}$*. In fact, the algebra* $\mathbf{B} \in \mathrm{Q}(\mathcal{G}_{\&})$.

Proof. Let $\mathbf{A} \in$ Q and define $X_A := \{x_a : a \in A\}$ to be a set of distict variables. Define also $v_0 : \mathrm{Te}(X_A) \to A$ by $v_0(x_a) = a$. For a term $t \in \mathrm{Te}(X_A)$, let $t^{\mathbf{A}} := v_0(t)$, and if $\Gamma = t_1, \dots, t_n$ is a sequence of terms, then let $\Gamma^{\mathbf{A}} = t_1^{\mathbf{A}}, \dots, t_n^{\mathbf{A}}$. We write $\Gamma^{\mathbf{A}} = 1^A$ to abbreviate the statement that $t_1^{\mathbf{A}} = 1^A, \dots, t_n^{\mathbf{A}} = 1^A$. Let $\mathrm{Te}_{\&}(X_A)$ be the algebra of terms in the type of $\mathbf{A}$ extended with the binary operation symbol of &. Define a congruence relation $\equiv$ on $\mathrm{Te}_{\&}(X_A)$ as follows: $s \equiv t$ iff there exist sequences of terms Γ, Σ from $\mathrm{Te}(X_A)$ such that

$$\vdash_{\mathcal{G}_{\&}} \Gamma, s \to t \quad \text{and} \quad \vdash_{\mathcal{G}_{\&}} \Sigma, t \to s$$

and such that $\Gamma^{\mathbf{A}} = 1^{\mathbf{A}}, \Sigma^{\mathbf{A}} = 1^{\mathbf{A}}$. Then $\equiv$ is a congruence relation on $\mathrm{Te}_{\&}(X_A)$: it is reflexive, because $x \to x$ is an initial sequent; it is transitive, because (CUT) is a rule of $\mathcal{G}$ and hence also of $\mathcal{G}_{\&}$; it is symmetric by definition and since $\mathcal{G}$, and therefore $\mathcal{G}_{\&}$, are congruential, $\equiv$ is a congruence.

Suppose that $t, s \in \mathrm{Te}(X_A)$. If $s \equiv t$, then for some sequences Γ, Σ from $\mathrm{Te}(X_A)$ such that $\Gamma^{\mathbf{A}} = 1^{\mathbf{A}}, \Sigma^{\mathbf{A}} = 1^{\mathbf{A}}$, we have: $\mathcal{G}_{\&} \vdash \Gamma, s \to t$ and $\Sigma, t \to s$. By the separation property of $\mathcal{G}_{\&}$, the sequents $\Gamma, s \to t$ and $\Sigma, t \to s$ also are derivable in $\mathcal{G}$. By (12),$\mathcal{G} \vdash \Gamma, \Sigma \to \Delta(s,t)$, so $\Delta^{\mathbf{A}}(s,t) = 1^{\mathbf{A}}$. It follows that $t^{\mathbf{A}} = s^{\mathbf{A}}$. Conversely, suppose that $t, s \in \mathrm{Te}(X_A)$ and $t^{\mathbf{A}} = s^{\mathbf{A}}$. Then $\Delta(t,s)^{\mathbf{A}} = 1^{\mathbf{A}}$ and by (15), the sequents $t, \Delta(t,s) \to s$ and $s, \Delta(s,t) \to t$ are derivable. So $s \equiv t$. Hence for $t, s \in \mathrm{Te}(X_A)$, we have: $t \equiv s$ iff $t^{\mathbf{A}} = s^{\mathbf{A}}$.

The mapping $f : A \to \mathrm{Te}_{\&}(X_A)/\!\equiv$ defined by $f(a) = x_a/\!\equiv$ is therefore $1-1$. It also is a homomorphism. For let λ be an n-ary operation symbol and let $a_1, \dots, a_n$ be a sequence of elements of A. We want to show that $x_{\lambda(a_1,\dots,a_n} \equiv \lambda(x_{a_1}, \dots, x_{a_n})$. But $x^{\mathbf{A}}_{\lambda(a_1,\dots,a_n)} = \lambda(x_{a_1}, \dots, x_{a_n})^{\mathbf{A}}$ and both terms are members of $\mathrm{Te}(X_A)$. So the equivalence follows from our previous claim.

We claim that $B \in \mathrm{Q}(\mathcal{G}_{\&})$. For let $\Gamma \to t$ be a sequent derivable in $\mathcal{G}_{\&}$ and assume that for some homomorphism h, $\{h(\Gamma)\} = 1^{\mathbf{B}}$. Then for each

$\gamma \in \{\Gamma\}$ there is a sequence of terms Σ_γ from $\mathrm{Te}(X_A)$ such that $\Sigma_\gamma \to h(\gamma)$ is derivable in $\mathcal{G}_\&$ and $\Sigma_\gamma^{\mathbf{A}} = 1^{\mathbf{A}}$. Applying the substitution h to $\Gamma \to t$ and then the (CUT), we obtain the derivable sequent $\Sigma \to h(t)$, where Σ is the concatenation of all the Σ_i's and as such has the property that $\{\Sigma\}^{\mathbf{A}} = 1^{\mathbf{A}}$. It follows that $h(t) = 1^{\mathbf{B}}$, which finishes the proof that $\mathbf{B} \in \mathrm{Q}(\mathcal{G}_\&)$.

Corollary 5.2. *Suppose that a sequent calculus $\mathcal{G}$ is congruential and satisfies the conditions (12)–(13) with a set $\Delta(x, y)$. Suppose that $\mathcal{G}_\&$ has separability property. Then*

$$\mathrm{Q}(\mathcal{G}) = \{\mathbf{B}^{\mathrm{r}} : \mathbf{B} \in \mathrm{Q}(\mathcal{G}_\&)\}.$$

In a similar manner, one can prove that if $\mathcal{G}$ is a sequent calculus strongly adequate to a relatively point-regular quasivariety K then every algebra $\mathbf{A}$ from K is embeddable into some algebra from $\mathrm{K}(\mathcal{G}_\&)$, under assumption that $\mathcal{G}_\&$ has a strong version of the separability property, i.e., every rule of the form $\dfrac{\to t_1, \ldots, \to t_n}{\Gamma \to t}$ in $\mathcal{G}_\&$ in which & does not occur, can be derived by means of the rules of $\mathcal{G}$.

The separability property of $\mathcal{G}_\&$ plays an essential role in the proof of Theorem 5.1. Usually, this property follows from the (CUT) elimination for $\mathcal{G}$. We can conclude from the theorem that we can't define a notion, applying to any quasivariety Q, of a Gentzen system "corresponding" to a Q and expect that such a system would have "nice" proof-theoretic properties (leading to the separability) and be adequate for Q. For if such a Gentzen system exists, then Q must be embeddable in $\mathrm{Q}_\&$, which is a rare property.

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