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A SHORT NOTE ON HOOPS AND CONTINUOUS t-NORMS

This short note contains no original results *per se*. It is meant as a modest contribution to the special issue of Reports on Mathematical Logic dedicated to the Workshop on Algebra & Substructural Logic held at the Japan Advanced Institute of Science and Technology (JAIST), November 10-17, 1999. The most relevant references for this note are a paper by Agliano, Ferreirim and Montagna [3] and a recent monograph by Hájek [15].

1. t-norms

According to the literature on Fuzzy Logic, t-norms are binary connectives which model “data fusion” [21]; they extend classical conjunction to the real interval [0,1] of truth values for uncertainty and approximate reasoning.

A *continuous t-norm* is a continuous map $* : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $\langle [0,1], * \rangle$ is a commutative po-monoid, i.e., for all $x, y, z \in [0,1]$, if $x \leq y$ then $x * z \leq y * z$.

There are three fundamental continuous t-norms: the Łukasiewicz t-norm defined by $x *_L y = \max(x + y - 1, 0)$; the Gödel (or lattice) norm

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defined by $x *_G y = x \wedge y$; and the product norm defined by $x *_P y = xy$. Indeed it is known (cf. [15], [20]) that every continuous t-norm behaves locally as one of the above. Every continuous t-norm induces a *residuation* (or *implication*) by the rule

$x \to y = \max\{z : z * x \leq y\}$.

The implications associated to the three fundamental norms are:

- $x \to_L y = \min(y - x + 1, 1)$;
- $x \to_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise;} \end{cases}$
- $x \to_P y = \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise.} \end{cases}$

The residual $\to$ of a continuous t-norm on $[0,1]$ satisfies

$$\begin{aligned} x \to x &= 1 \\ x \to 1 &= 1 \\ 1 \to x &= x \\ x \to y \approx y \to x &\approx 1 \quad \text{implies} \quad x \approx y. \end{aligned}$$

Hence the variety $\mathcal{V}_{\mathcal{K}}$ generated by any class $\mathcal{K}$ of algebras of the form $\langle [0,1], *, \to, 0 \rangle$, where $*$ is a continuous t-norm and $\to$ is its residual, is *ideal-determined* [14]. It is therefore the *equivalent algebraic semantics*, in the sense of [6], of its assertional logic. Historically, this connection was first considered for the variety generated by $\langle [0,1], *_L, \to_L, 0, 1 \rangle$. The algebras in this variety are term-equivalent to *Wajsberg algebras* [13] or *CN algebras* [17]. They are also term-equivalent to *MV-algebras* [10] and therefore the corresponding propositional calculus is Łukasiewicz infinite-valued logic.

In his recent monograph [15] Hájek presented a detailed study of the calculi associated to the three t-norms above and their equivalent algebraic semantics. Algebras in the variety $\mathcal{GA}$ generated by $\langle [0,1], *_G, \to_G, 0, 1 \rangle$ are *G-algebras* and *Gödel Logic* is its corresponding propositional calculus. Gödel Logic is an axiomatic extension of the Intuitionistic Propositional Calculus via the axiom $(p \to q) \vee (q \to p)$ (or $((p \to q) \to r) \to (((q \to p) \to r) \to r)$; see axiom (A6) below). Its equivalent algebraic semantics is therefore closely related to relative Stone algebras [16]. Similarly, algebras in the variety $\mathcal{PA}$ generated by $\langle [0,1], *_P, \rightarrow_P, 0, 1\rangle$ are *product algebras*; the corresponding propositional calculus is *Product Logic* [15], [1].

With the aim of introducing a deductive system which would encompass simultaneously the three calculi above, Hájek [15] defined *Basic Logic*, BL. If $\rightarrow$ and & are the logical connectives for implication and fusion, the axioms of BL are:

(A1) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$,
(A2) $(p \,\&\, q) \rightarrow p$,
(A3) $(p \,\&\, q) \rightarrow (q \rightarrow p)$,
(A4) $(p \,\&\, (p \rightarrow q)) \rightarrow (q \,\&\, (q \rightarrow p))$,
(A5) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \,\&\, q) \rightarrow r)$,
(A6) $((p \rightarrow q) \rightarrow r) \rightarrow (((q \rightarrow p) \rightarrow r) \rightarrow r)$,
(A7) $\overline{0} \rightarrow p$.

The only deduction rule of BL is *Modus Ponens*:

(MP) $p, p \rightarrow q \vdash q$.

The equivalent algebraic semantics of Basic Logic is the variety $\mathcal{BA}$ of *basic algebras*, which is a subvariety of residuated lattices. Indeed $\mathcal{BA}$ is generated by all linearly ordered residuated lattices that satisfy
$x \wedge y \approx x * (x \rightarrow y)$ [15]. It is therefore easy to see that the deductive system BL lacks the structural rule of contraction, but it has exchange and weakening. For recent developments in the study of residuated lattices and their relationship with substructural logics see [19] and [22].

Hájek conjectured that the variety $\mathcal{BA}$ of basic algebras is generated by all algebras of the form $\langle [0,1], *, \rightarrow, 0, 1\rangle$, where $*$ is a continuous t-norm on $[0,1]$. This is indeed the case, as it was proved by Cignoli, Esteva, Godo and Torrens in [11].

It seems that the most relevant algebraic aspect of any continuous t-norm on $[0,1]$ is the fact that the associated monoid is *residuated.* In addition, the usual order on [0,1] is indeed the order induced by the residual (by the rule $x \leq y$ iff $x \rightarrow y = 1$) and so if $*$ is a continuous t-norm then $\langle [0,1], *, \rightarrow, 0, 1\rangle$ is a *bounded hoop.*

2. Hoops and implicative subreducts

Hoops originated in a manuscript from the mid seventies by Büchi and Owens [9]. A thorough algebraic study of the class of hoops (a.k.a. naturally ordered pocrims) may be found in [5].

A *hoop* is an algebra $\mathbf{A} = \langle A, *, \rightarrow, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid and for all $x, y, z \in A$

(1) $x \rightarrow x \approx 1$,

(2) $x * (x \rightarrow y) \approx y * (y \rightarrow x)$,

(3) $x \rightarrow (y \rightarrow z) \approx x * y \rightarrow z$.

An algebra $\langle A, *, \rightarrow, 0, 1\rangle$ is a *bounded hoop* if $\langle A, *, \rightarrow, 1\rangle$ is a hoop and 0 is its least element. The variety of hoops is denoted by $\mathcal{HO}$ and it is the equivalent algebraic semantics of the deductive system $\mathsf{S}_{\mathcal{HO}}$. The axioms of $\mathsf{S}_{\mathcal{HO}}$ are (A1), (A5) and

(A8) $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$,
(A9) $p \rightarrow (q \rightarrow p)$,
(A10) $p \rightarrow (q \rightarrow (p \mathbin{\&} q))$,
(A11) $((p \rightarrow q) \mathbin{\&} p) \rightarrow ((q \rightarrow p) \mathbin{\&} q)$.

and inference rule *Modus Ponens* (MP).

BL proves all axioms of $\mathsf{S}_{\mathcal{HO}}$ (see [15, section 2.2], [24] and [5]); hence $\mathsf{S}_{\mathcal{HO}}$ is (strictly) weaker then BL.

An important subvariety of hoops, determined by the identity

$$(T) \qquad (x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$$

is the variety $\mathcal{WH}$ of *Wajsberg hoops*. Blok and Pigozzi [7] showed that bounded Wajsberg hoops are term-equivalent to *Wajsberg algebras*. Hence $\mathcal{WH}$ is the algebraic semantics of the positive fragment of Łukasiewicz's infinite-valued logic.

Finite linearly ordered Wajsberg hoops play a central role in the study of hoops in general. Consider the following

Example 2.1. Let $\mathbf{C}_n$ denote the Wajsberg hoop whose universe is $C_n = \{1 = a^0, a, a^2, ..., a^n\}$ and $a^k * a^m = a^{\min(k+m,n)}$, $a_k \to a_n = a^{\max(n-k,0)}$.

Similarly, let $\mathbf{Wa}_n$ denote the (basic) algebra $\langle C_n, \wedge, \vee, *, \to, 0, 1\rangle$, where $a^k \wedge a^m = a^{max\{k,m\}}$, $a^k \vee a^m = a^{min\{k,m\}}$ and $0 = a^n$.

We denote simply by **2** the two-element chain ($n = 1$), regardless of the fundamental operations considered.

An implicative subreduct of a hoop is always a BCK-algebra; moreover, if $\mathcal{V}$ is any variety of hoops, then the class $\boldsymbol{S}^{\to}(\mathcal{V})$, consisting of all implicative subreducts of algebras in $\mathcal{V}$, is a variety of BCK-algebras. This result appears in [8]; it may also be derived as a consequence of a more general theorem appearing in [2] and the fact that the identity

$$(J) \qquad (((x \to y) \to y) \to x) \to x \approx (((y \to x) \to x) \to y) \to y$$

holds in the variety of hoops.

The problem of determining an equational basis for the variety $\mathcal{HBCK}$ consisting of all implicative subreducts of hoops was posed by Wroński [25] and solved by the author [4]. Later Kowalski [18] found a syntactic proof.

Theorem 2.2. *The class $\mathcal{HBCK}$ is the variety of BCK-algebras satisfying identities (J) and*

$$(H) \qquad (x \to y) \to (x \to z) \approx (y \to x) \to (y \to z).$$

3. Basic hoops and basic algebras

The underlying order in a hoop is always a semilattice order, where $x \wedge y = x * (x \to y)$; however it is not in general a lattice order. Observe, nevertheless, that in any totally ordered hoop the join of any two elements always exists. Also, one can define in any hoop a *pseudo-join* via the binary term $x \dot{\vee} y := ((x \to y) \to y) \wedge ((y \to x) \to x$.

This binary operation is almost a join operation, in the sense that it is idempotent, commutative and $x \dot{\vee} y$ is an upper bound of x and y. We have

Proposition 3.1. *[9], [3] In a hoop* $\mathbf{A}$ *the following are equivalent:*

1. *the pseudo-join is associative,*
2. $x \leq y$ *implies* $x \dot{\vee} z \leq y \dot{\vee} z$,
3. $x \dot{\vee} (y \wedge z) \leq (x \dot{\vee} y) \wedge (x \dot{\vee} z)$,
4. *the pseudo-join is the join operation.*

Thus we may conclude that the class of hoops generated by totally ordered ones, for which the pseudo-join in indeed the join, is a variety whose subdirectly irreducible members are totally ordered. Proposition 3.1 provides an equational basis for this variety. This axiomatization involves both meet and implication and thus it is not suitable to characterize implicative subreducts. Here, the theory of basic algebras is very useful:

Theorem 3.2. *[3] The variety* $\mathcal{BA}$ *of basic algebras is term-equivalent to the variety of bounded hoops satisfying*

$$(B) \qquad (x \to y) \to z \leq ((y \to x) \to z) \to z.$$

Accordingly we say that a hoop satisfying (B) is a *basic hoop* and denote by $\mathcal{BH}$ the variety of basic hoops.

By using a technical lemma concerning basic hoops and the essential fact that hoops are 1-regular and their congruences are in 1-1 correspondence with filters, Agliano and Montagna have shown (see also [23]):

Theorem 3.3. *[3] The variety $\mathcal{BH}$ consists of hoops that are subdirect products of totally ordered hoops. Hence $\mathcal{BH}$ is the variety of hoops for which the pseudo-join is associative.*

Note that identity (B) corresponds to the logical axiom (A6). Hence we may conclude

Corollary 3.4. *Basic Logic is an axiomatic extension of $\mathsf{S}_{\mathcal{HO}}$ via the axiom (A6).*

Ordinal sums of hoops, as defined in [5], constitute an important tool in characterizing subdirectly irreducible hoops and will be used here to describe precisely the connection between basic hoops and basic algebras. As a matter of fact, one can derive from the structure of (finite) hoops the following

Corollary 3.5. *[3] $\mathbf{A}$ is a finite subdirectly irreducible basic hoop if and only if*

$$\mathbf{A} \cong \mathbf{C}_{n_1} \oplus \mathbf{C}_{n_2} \oplus \ldots \oplus \mathbf{C}_{n_k},$$

for some $k, n_1, \ldots, n_k \in \mathbb{N}$.

If $\mathbf{A}$ is a basic hoop, then $\mathbf{2} \oplus \mathbf{A}$ is a basic algebra of which $\mathbf{A}$ is a hoop subreduct. Since any hoop subreduct of a basic algebra is a basic hoop, the variety $\mathcal{BH}$ coincides with the class of hoop subreducts of basic algebras. Moreover, given a variety $\mathcal{V}$ of basic algebras, the class $\boldsymbol{S}^h(\mathcal{V})$ of hoop subreducts of $\mathcal{V}$ is always a variety of basic hoops [3]. Thus we obtain

Corollary 3.6. *[3] A finite basic algebra $\mathbf{A}$ is subdirectly irreducible if and only if there are $k, n_1, \ldots, n_k \in \mathbb{N}$ such that*

$$\mathbf{A} \cong \mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \ldots \oplus \mathbf{Wa}_{n_k}.$$

4. Generation by finite algebras

Given a class $\mathcal{K}$ of algebras, $\mathbf{A} \in \mathcal{K}$ has the *finite embedding property with respect to* $\mathcal{K}$ (FEP) if for any finite partial subalgebra $\mathbf{A}'$ of $\mathbf{A}$ there exists a finite algebra $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A}'$ is embeddable in $\mathbf{B}$. $\mathcal{K}$ has the FEP if each of its members has the FEP with respect to $\mathcal{K}$.

It is known that a variety $\mathcal{V}$ has the FEP if and only if it is generated as a quasivariety by its finite algebras, i.e. $\mathcal{V} = \boldsymbol{SPP}_u(\mathcal{V}_{fin})$. Moreover, if $\mathcal{V}$ is a finitely axiomatizable variety and has the FEP, then the quasi-equational theory of $\mathcal{V}$ is decidable. These results are due essentially to Evans [12], [5].

In [5] a method of constructing varieties of hoops with FEP was developed, having as a primary tool a Mal'cev product of varieties. With the variety $\mathcal{WH}$ of Wajsberg hoops as a starting point, this method generated a large family of varieties of hoops which have the FEP. In particular, the variety $\mathcal{HO}$ of all hoops has the FEP. Also, it follows from the structure of finite subdirectly irreducible basic hoops and algebras that

Proposition 4.1. *[3] The varieties $\mathcal{BH}$ and $\mathcal{BA}$ are generated as quasivarieties by their finite members. In particular*

$$\mathcal{BH} = \boldsymbol{SPP}_u(\mathbf{C}_{n_1} \oplus \ldots \oplus \mathbf{C}_{n_k} : k, n_1, \ldots, n_k \in \mathbb{N})$$
$$\mathcal{BA} = \boldsymbol{SPP}_u(\mathbf{Wa}_{n_1} \oplus \ldots \oplus \mathbf{Wa}_{n_k} : k, n_1, \ldots, n_k \in \mathbb{N})$$

Therefore the quasi-equational theories of $\mathcal{BH}$ and $\mathcal{BA}$ are decidable.

Now we can return to $\langle [0,1], *_L, \rightarrow_L, 1 \rangle$, where $*_L$ is the Łukasiewicz t-norm. It is not hard to see that

1. For every $a < b \in \mathbb{R}$, it is possible to define $*$ and $\rightarrow$ on $[a, b]$ in such a way that $\langle [a, b], *, \rightarrow, b \rangle \simeq \langle [0,1], *_L, \rightarrow_L, 1 \rangle$.
2. $\mathbf{C}_n$ is embeddable in $\langle [0,1], *_L, \rightarrow_L, 1 \rangle$, for all n.

These remarks as well as the structure of finite subdirectly irreducible basic hoops allow us to we reconfirm Hájek's conjecture that Basic Logic is the logic of continuous t-norms. Our result may be seen as a strengthening of [11, thm 4.2].

Theorem 4.2. *[3] The variety of basic hoops is generated as a quasivariety by all algebras of the form $\langle [0,1], *, \rightarrow, 1 \rangle$, where $*$ is a continuous t-norm on $[0,1]$ and $\rightarrow$ is its residual.*

Moreover the identity (B), which characterizes the variety $\mathcal{BH}$, also provides the equational basis for the variety of their implicative subreducts. Indeed if $\mathcal{BBCK}$ denotes the variety of BCK-algebras determined by (B), then

Theorem 4.3. *[3] The variety $\mathcal{BBCK}$ consists exactly of the implicative subreducts of basic hoops.*

5. Product hoops and product algebras

Recall from section 1 that Product Logic is the deductive system associated with the product norm $*_P$. Because of its novelty, Product Logic has drawn considerable attention in recent publications [15], [1]. If one defines negation by $\neg p = p \to \overline{0}$, Product Logic is an axiomatic extension [15] of Basic Logic via the axioms

(P1) $\neg\neg r \to ((p \,\&\, r \to q \,\&\, r) \to (p \to q))$,

(P2) $(p \wedge \neg p) \to \overline{0}$.

A close look at the definition and properties of the algebraic counterpart of this deductive system brings to mind strong connections with abelian ℓ-groups [15] and cancellative hoops [4]. Hence it is natural to investigate the class of hoop subreducts of product algebras. A major difficulty to overcome is to replace axioms (P1) and (P2) with negation-free axioms, because the deductive system $\mathsf{S}_{\mathcal{HO}}$ associated with hoops does not have negation. Since in bounded hoops on may define negation (by $\neg x := x \to 0$), a first approach consists in establishing a link between product algebras and bounded hoops. This topic has been discussed by Adillon and Verdú [1], who showed that product algebras are indeed the equivalent algebraic semantics of Product Logic and that they are term-equivalent to a variety of bounded hoops.

It is easy to observe that if $\mathbf{A}$ is a product algebra then $A \setminus \{0\}$ is the universe of a cancellative hoop, i.e., the underlying monoid is cancellative [1], [3]. It is known that cancellative hoops form a variety, given by the identity $y \to (x * y) \approx x$. Let $\mathcal{PH}$ denote the variety of hoop subreducts of product algebras.

Proposition 5.1. *[3] The subdirectly irreducible members of $\mathcal{PH}$ are the two-element chain* $\mathbf{2}$, *the subdirectly irreducible cancellative hoops and all hoops of the form* $\mathbf{2} \oplus \mathbf{C}$, *where* $\mathbf{C}$ *is any subdirectly irreducible cancellative hoop.*

One may sharpen the characterization of hoop subreducts of product algebras, henceforth called *product hoops*, by introducing an appropriate weakening of the cancellative law. We say that a hoop is *quasi-cancellative* if it satisfies

$$\forall\, xyz\, ((\exists\, w < z) \text{ implies } ((x * z) \to (y * z)) \to (x \to y) = 1).$$

We have

Proposition 5.2. *[3] A subdirectly irreducible hoop is quasi-cancellative if and only if it belongs to* $\mathcal{PH}$.

An implicative subreduct of a product hoop will be called a *product BCK-algebra*; we denote by $\mathcal{PBCK}$ the variety of product BCK-algebras.

Theorem 5.3. *[3] The variety $\mathcal{PH}$ (respectively $\mathcal{PBCK}$) consists of basic hoops (respectively BCK-algebras) satisfying the identity*

$$(PB) \qquad (x \to y) \to y \leq ((y \to z) \to ((y \to x) \to x)) \to ((y \to x) \to x).$$

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