REPORTS ON MATHEMATICAL LOGIC
34 (2000), 79–106

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KOMORI IDENTITIES IN ALGEBRAIC LOGIC

A b s t r a c t. A variety generated by a class $\mathbb{K}$ of BCK-algebras consists of BCK-algebras if and only if it satisfies a certain kind of identity, first discovered by Komori. A similar phenomenon is shown to hold more generally in a certain class of quasivarieties of logic that includes not only the class of BCK-algebras but also such classes as the quasivariety of biresiduation algebras and quasivarieties of algebras with an equivalence operation. We describe a set of identities (which we call *Komori identities*), and show that the variety generated by a class $\mathbb{K}$ of algebras in one of the quasivarieties considered is contained in the quasivariety if and only if it satisfies a Komori identity. We use the result to establish (i) that the subvarieties of any of the quasivarieties studied are congruence 3-permutable and (ii) that the varietal join of two subvarieties of any of the quasivarieties studied is contained in the quasivariety.

Mathematics Subject Classification: 08C15, 08C10, 03B47, 03G25
Key words and phrases: quasivariety, algebraizable deductive system, equivalence, congruence permutable, lattice of subvarieties
Received 18 April 2000

1. Introduction

The "equivalent algebraic semantics" of an algebraizable deductive system (in the sense of [2]) can always be assumed to be a quasivariety. If the quasivariety is a variety, the deductive system is said to be *strongly algebraizable*. Most familiar algebraizable deductive systems, such as the classical propositional calculus, the intuitionistic propositional calculus, the standard modal logics and many-valued logics, are strongly algebraizable: their equivalent algebraic semantics are the varieties of Boolean algebras, Heyting algebras, various varieties of modal algebras and varieties of Łukasiewicz algebras respectively. We refer to the equivalent algebraic semantics of algebraizable deductive systems as *"(quasi)varieties of logic"*. The most familiar example of a quasivariety of logic that fails to be a variety is the class of BCK-algebras. It is the equivalent algebraic semantics of the implicational system of BCK-logic, first formulated by Meredith (see [14]); this logic is therefore algebraizable but not strongly algebraizable. Since the class does not form a variety the question arises whether the varieties of BCK-algebras can be characterized. In [8] a precise and relatively simple syntactic description is given of a set of identities that serve this purpose: a class of BCK-algebras is a variety of BCK-algebras if and only if it satisfies one of the identities in the set; the discovery of the set of identities, and a crucial lemma pertaining to them are attributed to Komori. In [3] a proof is given of a slightly stronger result. Using the methods of this last paper, [12] obtained syntactic characterizations of the subvarieties of a large quasivariety of logic consisting of algebras with an "equivalence connective".

In the present paper these results are generalized to a much larger class of quasivarieties of logic. In order to guarantee that a quasivariety of algebras is the equivalent algebraic semantics of some (algebraizable) deductive system it suffices to assume that the quasivariety is relatively point-regular, or, equivalently, that there is a finite set of binary terms

$\Delta_i(x,y)$, $i \in I$ and a (definable) constant 0, such that the following are satisfied:

$$\forall x(\Delta_i(x,x) \approx 0), \tag{1}$$

and

$$\forall x,y(\bigwedge_{i\in I} \Delta_i(x,y) \approx 0 \Longrightarrow x \approx y). \tag{2}$$

Although these conditions are not necessary to ensure a quasivariety is a quasivariety of logic, they are satisfied in most familiar cases. In all examples mentioned above except BCK-algebras, the formulas $\Delta_0(x,y) = x \rightarrow y$, $\Delta_1(x,y) = y \rightarrow x$, and the constant 1 serve the purpose; while in the example of BCK-algebras one uses the formulas $\Delta_0(x,y) = x \dot{-} y$, $\Delta_1(x,y) = y \dot{-} x$, and the constant 0. We will consider quasivarieties of algebras the fundamental operations of which consist of a finite collection of binary operations Δ_i, $i \in I$ and a nullary operation 0, that satisfy the identitiy (1) and quasi-identity (2) displayed above; such quasivarieties have then enough structure to ensure they are quasivarieties of logic. Furthermore we assume the free algebra on one generator in the quasivariety possesses only two elements. Note that this assumption is satisfied in the class of BCK-algebras and also in the class of equivalential algebras studied in [12]. Another interesting class in which these assumptions are satisfied is the quasivariety of *biresiduation algebras*, i.e., the residuation subreducts of the quasivariety of all left and right residuated integral partially ordered monoids (see section 6.1.1).

Let $\mathbb{Q}$ be a quasivariety as described in the previous paragraph, and $\mathbb{V}$ the variety defined by the identities $\forall x(\Delta_i(x,x) \approx 0)$ and the identities of the form $\Delta_i(x,0) \approx 0$, $\Delta_i(x,0) \approx x$, $\Delta_i(0,x) \approx 0$, and $\Delta_i(0,x) \approx x$ that hold in $\mathbb{Q}$. The quasivariety $\mathbb{Q}$ is defined, relative to $\mathbb{V}$, by the quasi-identity (2). In Section 2 we show that $\mathbb{Q}$ is a *splitting quasivariety* in $\mathbb{V}$ with respect to a certain three-element algebra. This allows us to prove that a subvariety of $\mathbb{V}$ is 0-regular (and hence congruence modular) if and only if it is contained in $\mathbb{Q}$ (Corollary 3.5). The next section contains the main result of the paper, viz. a syntactic characterization of the subvarieties of $\mathbb{Q}$. We introduce a set of identities with the following property: given a class $\mathbb{K}$ of algebras in $\mathbb{Q}$, the variety generated by $\mathbb{K}$ is contained in $\mathbb{Q}$ if and only if $\mathbb{K}$ satisfies one of the identities in the set. We call these identities *Komori identities for* $\mathbb{Q}$ *relative to* $\mathbb{V}$. In Section 4 we use the result to establish two facts concerning subvarieties of $\mathbb{Q}$. It is easy to find examples that show that a subvariety of $\mathbb{Q}$ need not be congruence-permutable. It turns out, however, that any subvariety of $\mathbb{Q}$ is congruence 3-permutable (Theorem 5.2). This generalizes a similar result on varieties of BCK-algebras obtained in [8]. The second application concerns the question whether the varietal join of two subvarieties of $\mathbb{Q}$ is contained in $\mathbb{Q}$. Theorem 5.3 answers the question affirmatively, generalizing the analogous result for varieties of BCK-algebras obtained in [3]. In the last section we show what form the results of the paper take in the quasivariety of biresiduation algebras, and we show how the results specialize to familiar ones in the quasivariety of BCK-algebras and the algebras studied in [12].

2. Preliminaries

We denote algebras by boldface capitals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ and their respective underlying sets by $A, B, C, \dots$. We indicate the set of non-negative integers by ω.

Let $\mathbb{K}$ be a class of algebras of a fixed similarity type and $\mathbf{A} \in \mathbb{K}$. We will make standard use of the operators I, S, H, and P, denoting respectively the operations of forming isomorphic copies, subalgebras, homomorphic images, and direct products of the algebras in $\mathbb{K}$. $\mathsf{V}(\mathbb{K})$ and $\mathsf{Q}(\mathbb{K})$ will denote the variety and quasivariety generated by the class $\mathbb{K}$.

Let $\mathbb{K}$ be a quasivariety and let $\mathbf{A} \in \mathbb{K}$. A congruence relation θ of $\mathbf{A}$ is a $\mathbb{K}$-*congruence* (or a relative congruence, if the class $\mathbb{K}$ is clear from the context) if $\mathbf{A}/\theta \in \mathbb{K}$. The principal relative congruence of $\mathbf{A}$ generated by a set $X \subseteq A^2$ is the smallest relative congruence of $\mathbf{A}$ containing X and will be denoted by $\theta^{\mathbf{A}}_{\mathbb{K}}(X)$. If $X = \{(a, b)\}$ we will abbreviate $\theta^{\mathbf{A}}_{\mathbb{K}}(X)$ by $\theta^{\mathbf{A}}_{\mathbb{K}}(a, b)$. The set of all $\mathbb{K}$-congruences of an algebra $\mathbf{A} \in \mathbb{K}$, ordered by inclusion, forms an algebraic lattice denoted by $\mathbf{Con}_{\mathbb{K}}(\mathbf{A})$. We say that $\mathbf{A}$ is *relatively congruence modular* (resp. *distributive*) if the lattice $\mathbf{Con}_{\mathbb{K}}(\mathbf{A})$ is modular (resp. distributive). $\mathbf{A}$ is *(relatively) congruence n-permutable* if for any (relative) congruences θ_0, θ_1 on $\mathbf{A}$, $\overbrace{\theta_0 \circ \theta_1 \circ \theta_0 \cdots}^{n} = \overbrace{\theta_1 \circ \theta_0 \circ \theta_1 \cdots}^{n}$. A (relatively) congruence 2-permutable algebra is called *(relatively) congruence permutable.* The class $\mathbb{K}$ is *congruence n-permutable* if all algebras in $\mathbb{K}$ are. Similarly, we say that $\mathbb{K}$ is *congruence permutable* if all algebras in $\mathbb{K}$ are.

We say that $\mathbf{A}$ is *relatively* 0*-regular* if $0 \in A$ and for all relative congruences θ, θ' of $\mathbf{A}$, $\theta = \theta'$ iff $0^{\mathbf{A}}/\theta = 0^{\mathbf{A}}/\theta'$. The quasivariety $\mathbb{K}$ is said to be *relatively* 0*-regular* if all algebras in $\mathbb{K}$ are. Any 0-regular variety is congruence modular (see [6]). This property, however, does not hold for quasivarieties. For an example, see [4].

A *tolerance* on $\mathbf{A}$ is a reflexive, symmetric binary relation τ on A which is also compatible with the fundamental operations of $\mathbf{A}$, i.e. it is also a subuniverse of $\mathbf{A} \times \mathbf{A}$. A congruence is thus a tolerance relation that is transitive. We use $\tau^{\mathbf{A}}(X)$ to denote the *tolerance of* $\mathbf{A}$ *generated by the set* $X \subseteq A^2$, that is, the smallest tolerance of $\mathbf{A}$ containing X. The set of all tolerances on an algebra $\mathbf{A}$ forms an algebraic lattice, which will be denoted by $\mathbf{Tol}(\mathbf{A})$. For $\tau, \eta \subseteq A \times A$, we denote the relational product $\tau \circ \eta$ by $\tau\eta$, and we define τ^n recursively by: $\tau^0 := \Delta_{\mathbf{A}} = \{(a, a) : a \in A\}$, and $\tau^{n+1} := \tau^n\tau$ $(n \in \omega)$. Notice that for any tolerance relation τ of an algebra $\mathbf{A}$, the congruence generated by τ is nothing but the transitive closure of τ. If it exists, the least positive $n \in \omega$ such that τ^n is a congruence for every $\tau \in \mathrm{Tol}(\mathbf{A})$, is called the *tolerance number* of $\mathbf{A}$, and is denoted by $\mathrm{tn}(\mathbf{A})$. If $\mathrm{tn}(\mathbf{A}) \leq \mathrm{n}$, then $\mathbf{A}$ is congruence $(n+1)$-permutable [15].

3. Basic properties

Let $\mathcal{L}$ be the language $\{\Delta_i : i \in I\} \cup \{0\}$, where I is finite, Δ_i is a binary operation symbol for each $i \in I$, and 0 is a constant symbol. We will center our attention on varieties over the language $\mathcal{L}$ for which the free algebra on one generator has exactly two elements, and on their subquasivarieties defined by the quasi-identity

$$\bigwedge_{i\in I} \Delta_i(x,y) \approx 0 \Longrightarrow x \approx y. \tag{3}$$

Given any such variety $\mathbb{W}$, we see that for each $i \in I$, the variety $\mathbb{W}$ must satisfy exactly one of the following identities:

$$\Delta_i(x,0) \approx 0$$
$$\Delta_i(x,0) \approx x.$$

Similarly, $\mathbb{W}$ must satisfy exactly one of the following identities for each $i \in I$:

$$\Delta_i(0,x) \approx 0$$
$$\Delta_i(0,x) \approx x.$$

These conditions allow us to distinguish, for each such variety $\mathbb{W}$, the following two subsets J and K of I:

$$J = \{i \in I :\ \mathbb{W} \models \Delta_i(x,0) \approx x\}, \qquad K = \{i \in I :\ \mathbb{W} \models \Delta_i(0,x) \approx x\}.$$

Notice that if $J = K = \emptyset$ then the quasivariety $\mathbb{Q}$ consists of the trivial algebra $\langle \{0\}, \langle \Delta_i \rangle_{i\in I}, 0\rangle$. For if $\mathbf{A}$ is any algebra in $\mathbb{Q}$ and $a \in A$, then $\Delta_i^{\mathbf{A}}(a,0) = 0$ $(i \in I)$ implies $a = 0$, by (3). Similarly, $\Delta_i^{\mathbf{A}}(0,a) = 0$ $(i \in I)$ implies $0 = a$. Since we are only interested in the case in which the subquasivariety $\mathbb{Q}$ of $\mathbb{W}$ is nontrivial, we will from now on assume that J and K are fixed subsets of the set I with $J \cup K \neq \emptyset$.

Let $\mathbb{V}_{J,K}$ be the variety over the language $\mathcal{L}$ defined by the following conditions:

$$\begin{aligned} &\Delta_i(x,x) \approx 0 && \text{for all } i \in I,\\ &\Delta_i(x,0) \approx x && \text{if } i \in J,\\ &\Delta_i(x,0) \approx 0 && \text{otherwise},\\ &\Delta_i(0,x) \approx x && \text{if } i \in K,\\ &\Delta_i(0,x) \approx 0 && \text{otherwise.} \end{aligned} \tag{4}$$

In particular, the free algebra on one generator in the variety $\mathbb{V}_{J,K}$ has only two elements, that is:

$$|\mathbf{F}_{\mathbb{V}_{J,K}}(x)| = 2, \text{ i.e. } \mathbf{F}_{\mathbb{V}_{J,K}}(x) = \{0, x\}. \tag{5}$$

Since J and K are fixed, we will from now on omit the subscripts and denote $\mathbb{V}_{J,K}$ by $\mathbb{V}$.

Let $\mathbb{Q}$ be the subquasivariety of $\mathbb{V}$ defined, relative to $\mathbb{V}$, by the quasi-identity:

$$\bigwedge_{i \in I} \Delta_i(x, y) \approx 0 \Longrightarrow x \approx y. \tag{6}$$

Quasi-identity (6) guarantees that the quasivariety $\mathbb{Q}$ is a relatively 0-regular quasivariety:

Proposition 3.1. *Let* $\mathbf{A}$ *be an algebra in* $\mathbb{Q}$.

(i) *For any* $\theta \in Con_{\mathbb{Q}}(\mathbf{A})$ *and* $a, b \in A$, $(a, b) \in \theta$ *iff* $\Delta_i^{\mathbf{A}}(a, b) \in 0/\theta$ *for all* $i \in I$.

(ii) *For any two relative congruences* θ *and* θ' *of* $\mathbf{A}$, $0/\theta = 0/\theta'$ *implies* $\theta = \theta'$, *i.e., the quasivariety* $\mathbb{Q}$ *is relatively* 0*-regular. Hence, any subvariety* $\mathbb{W}$ *of* $\mathbb{Q}$ *is* 0*-regular.*

Proof. If $\mathbf{A} \in \mathbb{Q}$, $\theta \in Con_{\mathbb{Q}}(\mathbf{A})$, and $a, b \in A$, then we have:

$$(a, b) \in \theta \Rightarrow (\forall i \in I)\Delta_i^{\mathbf{A}}(a, b) \overset{\theta}{\equiv} \Delta_i^{\mathbf{A}}(a, a) = 0 \Rightarrow (\forall i \in I)\Delta_i^{\mathbf{A}}(a, b) \in 0/\theta.$$

Conversely,

$$(\forall i \in I)\Delta_i^{\mathbf{A}}(a, b) \in 0/\theta \Rightarrow (\forall i \in I)\Delta_i^{\mathbf{A}}(a, b) \overset{\theta}{\equiv} 0 \overset{(6)}{\Rightarrow} a \overset{\theta}{\equiv} b,$$

so for all $a, b \in A$ we have $(a, b) \in \theta$ iff $\Delta_i^{\mathbf{A}}(a, b) \in 0/\theta$ for all $i \in I$.

Thus for any two relative congruences θ and θ' of $\mathbf{A}$, if $0/\theta = 0/\theta'$ then $\theta = \theta'$, so the quasivariety $\mathbb{Q}$ is relatively 0-regular and any subvariety of $\mathbb{Q}$ is a 0-regular variety.

Two algebras in $\mathbb{V}$ will play an important role in the remainder of this paper. The first one is the algebra $\mathbf{B_2}$ given by

$$\mathbf{B_2} = \langle \{0, a, b\}; \langle \Delta_i^{\mathbf{B_2}} \rangle_{i \in I}, 0 \rangle,$$

with, for $i \in I$,

$$\begin{aligned} \Delta_i^{\mathbf{B_2}}(x, 0) &= x \quad \text{if } i \in J, \\ \Delta_i^{\mathbf{B_2}}(0, x) &= x \quad \text{if } i \in K, \\ \Delta_i^{\mathbf{B_2}}(x, y) &= 0 \quad \text{otherwise.} \end{aligned}$$

The second one is the subalgebra of $\mathbf{B_2}$ with universe $\{0, a\}$, denoted by $\mathbf{B_1}$. Notice that $\mathbf{B_1}$ is also the homomorphic image of $\mathbf{B_2}$ under the homomorphism $\varepsilon : \mathbf{B_2} \longrightarrow \mathbf{B_1}$ given by $\varepsilon(0) = 0$, $\varepsilon(a) = \varepsilon(b) = a$.

Observe that the algebra $\mathbf{B_1}$ belongs to $\mathbb{Q}$. In fact, every non-trivial algebra $\mathbf{A}$ in $\mathbb{V}$ has a subalgebra isomorphic to $\mathbf{B_1}$. For if $a \in A$, $a \neq 0$, then the set $\{0, a\}$ is closed under the operations $\Delta_i^{\mathbf{A}}$ for all $i \in I$ and therefore a subuniverse of $\mathbf{A}$. Hence $\mathbf{Sg}^{\mathbf{A}}(a) \cong \mathbf{B_1}$. We will see in the proof of the following lemma that the algebra $\mathbf{B_1}$ has a reduct that is term equivalent to either the biconditional or the implicative reduct of the two-element Boolean algebra.

As a consequence of the observation above, we obtain:

Lemma 3.2. *The variety* $\mathsf{V}(\mathbf{B_1})$ *is the smallest nontrivial variety contained in* $\mathbb{V}$. *In addition,* $\mathsf{V}(\mathbf{B_1}) \subseteq \mathbb{Q}$.

Proof. Our previous argument shows that $\mathsf{V}(\mathbf{B_1})$ is the smallest nontrivial variety contained in $\mathbb{V}$. We will now show that any algebra $\mathbf{A} \in \mathsf{V}(\mathbf{B_1})$ satisfies quasi-identity (6). Since $J \cup K \neq \emptyset$, we can find $i_0 \in I$ such that $\Delta_{i_0}^{\mathbf{A}}(x, 0) \approx x$ or $\Delta_{i_0}^{\mathbf{A}}(0, x) \approx x$. To simplify the notation, we will denote $\Delta_{i_0}^{\mathbf{A}}(x, y)$ by xy. We have three possible cases:

(i) $x\,0 \approx 0\,x \approx x$, that is, $\mathbf{B_1}$ has a reduct isomorphic to the two-element Boolean group;

(ii) $x\,0 \approx x$ and $0\,x \approx 0$, i.e., $\mathbf{B_1}$ has a reduct isomorphic to the two-element BCK-algebra;

(iii) $x\,0 \approx 0$ and $0\,x \approx x$, that is, $\mathbf{B_1}$ has a reduct term equivalent to the two-element BCK-algebra.

Recall that the two-element BCK-algebra is term equivalent to the $\langle \rightarrow, 1\rangle$-reduct of the two-element Boolean algebra.

In case (i), $\mathbf{B_1}$ (and therefore also $\mathsf{V}(\mathbf{B_1})$) satisfies $x(xy) \approx y$, from which quasi-identity (6) follows. For if $\Delta_i^{\mathbf{B_1}}(a,b) = 0$ for all $i \in I$, then in particular $ab = 0$ and hence $a = a\,0 = a(ab) = b$.

In case (ii), $\mathsf{V}(\mathbf{B_1})$ satisfies $x(xy) \approx y(yx)$, which also implies quasi-identity (6).

Similarly, in case (iii) $\mathsf{V}(\mathbf{B_1})$ satisfies $(yx)x \approx (xy)y$, from which (6) follows.

Observe that the algebra $\mathbf{B_2}$ does not satisfy the quasi-identity

$$\bigwedge_{i\in I} \Delta_i(x,y) \approx 0 \Longrightarrow x \approx y,$$

and hence $\mathbf{B_2}$ does not belong to $\mathbb{Q}$. Moreover, we have the following lemma:

Lemma 3.3. *For any algebra $\mathbf{A} \in \mathbb{V}$, $\mathbf{A} \in \mathbb{Q}$ if and only if $\mathbf{A}$ does not contain a subalgebra isomorphic to $\mathbf{B_2}$.*

Proof. Clearly, as $\mathbf{B_2} \notin \mathbb{Q}$, if $\mathbf{B_2} \in \mathsf{S}(\mathbf{A})$, then $\mathbf{A} \notin \mathbb{Q}$. Conversely, if $\mathbf{A} \in \mathbb{V} \setminus \mathbb{Q}$, then $\mathbf{A}$ does not satisfy quasi-identity (6). That is, there are elements $a, b \in A$ such that $a \neq b$ and $\Delta_i^{\mathbf{A}}(a,b) = 0$ for all $i \in I$. The subalgebra $\mathbf{Sg}^{\mathbf{A}}(a,b)$ is then isomorphic to $\mathbf{B_2}$.

As a consequence of Lemma 3.3, it follows that the algebra $\mathbf{B_2}$ is the *splitting algebra* in $\mathbb{V}$ associated with $\mathbb{Q}$. More explicitly, the pair $\langle \mathsf{Q}(\mathbf{B_2}), \mathbb{Q}\rangle$ splits the lattice $\mathbf{L}^q(\mathbb{V})$ of all quasivarieties contained in $\mathbb{V}$ in the following way: for every quasivariety $\mathbb{K} \subseteq \mathbb{V}$, either $\mathbb{K} \subseteq \mathbb{Q}$ or $\mathsf{Q}(\mathbf{B_2}) \subseteq \mathbb{K}$ (and not both).

Notice that the algebra $\mathbf{B_2}$ is not 0-regular. Indeed, for the non-trivial principal congruence $\theta^{\mathbf{B_2}}(a,b)$ we have: $0/\theta^{\mathbf{B_2}}(a,b) = \{0\}$. We will show below that any variety $\mathbb{W} \subseteq \mathbb{V}$ which contains the algebra $\mathbf{B_2}$ fails to have many of the properties that one might hope for in a class of algebras. In particular, such a variety is not 0-regular, nor congruence modular, nor n-permutable for any $n \geq 2$. Since $\mathbf{B_2}$ is a splitting algebra, it follows that any variety not contained in $\mathbb{Q}$ fails to be 0-regular, congruence modular and congruence n-permutable for any $n \geq 2$.

Proposition 3.4.

(i) *The variety* $\mathsf{V}(\mathbf{B_2})$ *does not satisfy any non-trivial congruence identity.*

(ii) $\mathsf{V}(\mathbf{B_2})$ *is not congruence n-permutable for any integer* $n \geq 2$.

Proof. For each $n \in \omega$, let $\mathbf{B_n} = \langle \{0, a_1, \ldots, a_n\}; \langle \Delta_i^{\mathbf{B_n}} \rangle_{i \in I}, 0 \rangle$, with $|\mathbf{B_n}| = n+1$, $\Delta_i^{\mathbf{B_n}}(a_j, 0) = a_j$ for $i \in J$, $\Delta_i^{\mathbf{B_n}}(0, a_j) = a_j$ for $i \in K$ $(1 \leq j \leq n)$, $\Delta_i^{\mathbf{B_n}}(x, y) = 0$ otherwise. In particular, in $\mathbf{B_1}$ we have $a_1 = a$ and in $\mathbf{B_2}$ we have $a_1 = a$, $a_2 = b$. It is easy to see that $\mathbf{B_j} \in \mathsf{S}(\mathbf{B_k})$ whenever $j \leq k$.

The subalgebra of $(\mathbf{B_2})^n$ with universe $\{(0, \ldots, 0)\} \cup \{(c_1, \ldots, c_n) : c_i \neq 0 \text{ for } i = 1, 2, \ldots, n\}$ is isomorphic to $\mathbf{B_{2^n}}$. Since $\mathbf{B_j} \in \mathsf{S}(\mathbf{B_k})$ whenever $j \leq k$, it follows that $\mathbf{B_n} \in \mathsf{IS}(\mathbf{B_{2^n}}) \subseteq \mathsf{ISP}(\mathbf{B_2})$ for all $n \in \omega$.

Note that $\mathsf{H}(\mathbf{B_n}) = \{\mathbf{B_m} : m \leq n\}$. In fact, the congruences of $\mathbf{B_n}$ (other than the universal congruence) are in one to one correspondence with the partitions of the set $\{a_1, a_2, \ldots, a_n\}$. More precisely, any equivalence relation $\sim$ on the set $\mathrm{B_n}$ for which $0/\sim = \{0\}$ is a congruence of $\mathbf{B_n}$. Moreover, this correspondence preserves the lattice operations and the relational product operation (since the meet, join, and relational product of congruence relations coincide with the respective operations on the underlying equivalence relations). It is well known (see [16]) that for any non-trivial lattice identity there is a lattice of partitions on some finite set in which the identity does not hold. Therefore no nontrivial lattice identity is satisfied by all the congruence lattices $\mathbf{Con}(\mathbf{B_n}), 1 \leq n < \omega$.

Corollary 3.5. *Let* $\mathbb{W}$ *be a variety,* $\mathbb{W} \subseteq \mathbb{V}$. *Then*

(i) $\mathbb{W}$ *is* 0*-regular if and only if* $\mathbb{W} \subseteq \mathbb{Q}$.

(ii) $\mathbb{W}$ *is congruence modular if and only if* $\mathbb{W} \subseteq \mathbb{Q}$.

Proof. We have already observed that $\mathbb{Q}$ is a relatively 0-regular quasivariety. Any variety $\mathbb{W} \subseteq \mathbb{Q}$ will then be 0-regular and therefore also congruence modular [6]. Let $\mathbb{W} \subseteq \mathbb{V}$ be a variety, $\mathbb{W} \nsubseteq \mathbb{Q}$. Then, by Lemma 3.3 $\mathbf{B_2} \in \mathbb{W}$ and $\mathsf{V}(\mathbf{B_2}) \subseteq \mathbb{W}$. As $\mathsf{V}(\mathbf{B_2})$ is neither 0-regular nor congruence modular, neither is $\mathbb{W}$.

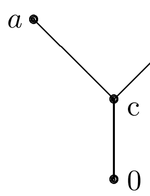

Figure 1. An algebra $\mathbf{A} \in \mathbb{Q}$ *with* $\mathsf{V}(\mathbf{A}) \nsubseteq \mathbb{Q}$.

The argument used in the proof of Lemma 3.2 is a simple instance of the main result of the following section, where we will characterize, by means of identities, those classes of algebras $\mathbb{K} \subseteq \mathbb{Q}$ which generate varieties contained in $\mathbb{Q}$. Given an algebra $\mathbf{A} \in \mathbb{Q}$, $\mathsf{V}(\mathbf{A})$ need not be contained in $\mathbb{Q}$, even if $\mathbf{A}$ is finite. Consider the algebra $\mathbf{A} = \langle \{0, a, b, c\}; \langle \Delta_i^{\mathbf{A}} \rangle_{i \in I}, 0 \rangle$, depicted in Figure 1, where $\Delta_i^{\mathbf{A}}(a, b) = \Delta_i^{\mathbf{A}}(b, a) = c$ for all $i \in I$, $\Delta_i^{\mathbf{A}}(a, 0) = \Delta_i^{\mathbf{A}}(a, c) = a$ for all $i \in J$, $\Delta_i^{\mathbf{A}}(b, 0) = \Delta_i^{\mathbf{A}}(b, c) = b$ for all $i \in J$, $\Delta_i^{\mathbf{A}}(0, a) = \Delta_i^{\mathbf{A}}(c, a) = a$ for all $i \in K$, $\Delta_i^{\mathbf{A}}(0, b) = \Delta_i^{\mathbf{A}}(c, b) = b$ for all $i \in K$, $\Delta_i^{\mathbf{A}}(x, y) = 0$ otherwise. The algebra $\mathbf{A}$ satisfies identities (4), (5), and quasi-identity (6). However, $\mathbf{A}/\theta^{\mathbf{A}}(0, c) \cong \mathbf{B_2} \notin \mathbb{Q}$. Observe that Figure 1 does not intend to suggest any partial order relation on the algebra $\mathbf{A}$.

4. A characterization of the subvarieties of $\mathbb{Q}$

Each subquasivariety of $\mathbb{Q}$ can be thought of as the equivalent quasivariety semantics for some deductive system over the language $\mathcal{L} = \{\Delta_i : i \in I\} \cup \{\top\}$. In particular, those corresponding to strongly algebraizable deductive systems will be subvarieties of $\mathbb{Q}$. Our goal in the remainder of this section will be to characterize those identities that, added to those of $\mathbb{V}$, define subvarieties of $\mathbb{Q}$, i.e. imply quasi-identity (6).

We will begin by proving two results which give some insight in the behavior of the term algebra $\mathbf{T}(x, y)$ in the variety $\mathbb{V}$.

Lemma 4.1. *Let* $u \in \mathrm{T}(x, y)$ *such that* $\models_{\mathbb{V}} u(x, x) \approx 0$. *Then* $u^{\mathbf{B_2}}(a, b) = 0$.

Proof. Let ε be the homomorphism $\varepsilon : \mathbf{B_2} \longrightarrow \mathbf{B_1}$ given by $\varepsilon(a) = \varepsilon(b) = a$. It is easy to see that ε is an onto homomorphism satisfying $\varepsilon^{-1}(\{0\}) = \{0\}$. We then have: $\varepsilon(u^{\mathbf{B_2}}(a, b)) = u^{\mathbf{B_1}}(a, a) = 0$, which implies $u^{\mathbf{B_2}}(a, b) = 0$.

Lemma 4.2. *Let* $t \in \mathrm{T}(x, y)$ *such that* $\models_{\mathbb{V}} t(x, x) \approx x$. *Then there exist* $n \in \omega$ *and terms* $s(x_0, x_1, \ldots, x_n) \in \mathrm{T}(x_0, x_1, \ldots, x_n)$, $u_i(x, y) \in \mathrm{T}(x, y)$ $\;1 \le i \le n$, *and* $z \in \{x, y\}$, *such that*

$\models_{\mathbb{V}} u_i(x, x) \approx 0 \quad 1 \le i \le n$,

$\models_{\mathbb{V}} s(x, 0, 0, \ldots, 0) \approx x$, *and*

$\models_{\mathbb{V}} t(x, y) \approx s(z, u_1(x, y), u_2(x, y), \ldots, u_n(x, y))$.

Proof. By induction on the complexity of t.

- Clearly $t \neq 0$, as $\models_{\mathbb{V}} t(x, x) \approx x$
- Suppose $t \in \{x, y\}$. Then we can choose $n = 0$, $s(x_0) = x_0$, and $z = t(x, y)$.
- Now assume t is complex, say $t = \Delta_{i_0}(t_1, t_2)$.

Since $\models_{\mathbb{V}} \Delta_{i_0}(t_1(x, x), t_1(x, x)) \approx 0$ and by assumption $\models_{\mathbb{V}} t(x, x) \approx x$ and hence $\not\models_{\mathbb{V}} t(x, x) \approx 0$, we conclude $\not\models_{\mathbb{V}} t_1(x, x) \approx t_2(x, x)$. We need to consider the following two cases, since $|\mathbf{F}_{\mathbb{V}}(x)| = 2$:

(a) $\models_{\mathbb{V}} t_1(x, x) \approx 0$ and $\models_{\mathbb{V}} t_2(x, x) \approx x$.

(b) $\models_{\mathbb{V}} t_1(x, x) \approx x$ and $\models_{\mathbb{V}} t_2(x, x) \approx 0$.

In case (a), since $\models_{\mathbb{V}} t_2(x, x) \approx x$, by inductive hypothesis we can find terms $u_1, u_2, \ldots, u_n \in \mathrm{T}(x, y)$, $s_2(x_0, x_1, \ldots, x_n)$, $z \in \{x, y\}$, such that $\models_{\mathbb{V}} u_i(x, x) \approx 0$ $(1 \le i \le n)$, $\models_{\mathbb{V}} s_2(x, 0, 0, \ldots, 0) \approx x$, and $\models_{\mathbb{V}} t_2(x, y) \approx s_2(z, u_1(x, y), \ldots, u_n(x, y))$.

We can define a new term s by

$$s(x_0, x_1, \ldots, x_n, x_{n+1}) = \Delta_{i_0}(x_{n+1}, s_2(x_0, \ldots, x_n)),$$

the terms $v_i(x,y)$ for $1 \le i \le n$ by $v_i(x,y) = u_i(x,y)$, and $v_{n+1}(x,y) = t_1(x,y)$.

Then

$$\begin{aligned}\models_{\mathbb{V}} & s(z, v_1(x,y), \ldots, v_n(x,y), v_{n+1}(x,y)) \\ & \approx \Delta_{i_0}(t_1(x,y), s_2(z, u_1(x,y), \ldots, u_n(x,y))) \\ & \approx \Delta_{i_0}(t_1(x,y), t_2(x,y)) \approx t(x,y),\end{aligned}$$

as desired.

The argument for case (b) is completely analogous.

We can now characterize the classes $\mathbb{K} \subseteq \mathbb{Q}$ that satisfy $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$.

Theorem 4.3. *Let $\mathbb{K}$ be a class of algebras, $\mathbb{K} \subseteq \mathbb{V}$. Then $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$ if and only if $\mathbb{K}$ satisfies an identity of the form*

$$s_1(x, u_1(x,y), \ldots, u_n(x,y)) \approx s_2(y, v_1(x,y), \ldots, v_m(x,y)) \tag{7}$$

for some $n, m \in \omega$, $s_1 \in \mathrm{T}(x_0, \ldots, x_n)$, $s_2 \in \mathrm{T}(x_0, \ldots, x_m)$, $u_i(x,y)$, $v_j(x,y) \in \mathrm{T}(x,y)$ for $1 \le i \le n$, $1 \le j \le m$, such that

$$\models_{\mathbb{V}} s_1(x, 0, 0, \ldots, 0) \approx x, \tag{8}$$

$$\models_{\mathbb{V}} s_2(x, 0, 0, \ldots, 0) \approx x, \text{ and} \tag{9}$$

$$\models_{\mathbb{V}} u_i(x,x) \approx 0 \approx v_j(x,x), \quad 1 \le i \le n,\ 1 \le j \le m. \tag{10}$$

Proof. $\Leftarrow$) Let $\mathbf{A} \in \mathsf{V}(\mathbb{K})$ and let $a, b \in A$ such that $\Delta_i^{\mathbf{A}}(a,b) = \Delta_i^{\mathbf{A}}(b,a) = 0$ for all $i \in I$. We prove by contradiction that $a = b$. Assume not. Then note that $\mathbf{Sg}^{\mathbf{A}}(\{a,b\}) \cong \mathbf{B_2}$. Since $\mathbf{A}$ satisfies an identity of the form

$$s_1(x, u_1(x,y), \ldots, u_n(x,y)) \approx s_2(y, v_1(x,y), \ldots, v_m(x,y))$$

for some terms as above, it follows that

$$s_1^{\mathbf{A}}(a, u_1^{\mathbf{A}}(a,b), \ldots, u_n^{\mathbf{A}}(a,b)) = s_2^{\mathbf{A}}(b, v_1^{\mathbf{A}}(a,b), \ldots, v_m^{\mathbf{A}}(a,b)).$$

Then, by Lemma 4.1,

$$a = s_1^{\mathbf{A}}(a, 0, \ldots, 0) = s_2^{\mathbf{A}}(b, 0, \ldots, 0) = b,$$

and hence $a = b$, a contradiction.

$\Rightarrow$) Let $\mathbf{F} = \mathbf{F}_{\mathsf{V}(\mathbb{K})}(x, y)$ be the free algebra on two generators in the variety generated by $\mathbb{K}$. Let $\mathbf{T}(x, y)$ be the term algebra generated by x and y. Consider the homomorphism $\eta : \mathbf{T}(x, y) \longrightarrow \mathbf{F}$ given by $\eta(x) = x,\ \eta(y) = y$. Let $\mu : \mathbf{T}(x, y) \longrightarrow \mathbf{B_2}$ be the homomorphism given by $\mu(x) = a,\ \mu(y) = b$, and let ε be the homomorphism given in Lemma 4.1. Recall that $\varepsilon(a) = \varepsilon(b) = a$. Since $\mathbf{B_1} \in \mathsf{V}(\mathbb{K})$, there exists a homomorphism $\pi : \mathbf{F} \longrightarrow \mathbf{B_1}$ such that $\pi(x) = \pi(y) = a$.

We have then the commuting diagram given in Figure 2.

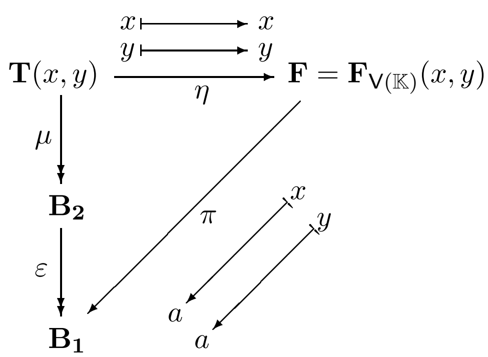

Figure 2. A commuting diagram in $\mathbb{V}$.

Since $\mathbf{B_2} \notin \mathbb{Q}$, then $\mathbf{B_2} \notin \mathsf{H}(\mathbf{F})$ and thus there is no homomorphism from $\mathbf{F}$ onto $\mathbf{B_2}$. Hence Ker$\eta \not\subseteq$ Kerμ; say we have then terms $t_1(x, y)$, $t_2(x, y) \in \mathrm{T}(x, y)$, such that $\mu(t_1) \neq \mu(t_2)$, while $\eta(t_1) = \eta(t_2)$. That is, $t_1^{\mathbf{F}}(x, y) = t_2^{\mathbf{F}}(x, y)$, and hence, in particular, $t_1^{\mathbf{B_1}}(a, a) = t_2^{\mathbf{B_1}}(a, a)$, but $t_1^{\mathbf{B_2}}(a, b) \neq t_2^{\mathbf{B_2}}(a, b)$.

Since $t_1^{\mathbf{B_1}}(a, a) = t_2^{\mathbf{B_1}}(a, a)$, it is impossible for one of $t_1^{\mathbf{B_2}}(a, b)$, $t_2^{\mathbf{B_2}}(a, b)$ to be 0 and the other one a or b. Thus, we must have $\{t_1^{\mathbf{B_2}}(a, b), t_2^{\mathbf{B_2}}(a, b)\} = \{a, b\}$. Without loss of generality, we can assume that $t_1^{\mathbf{B_2}}(a, b) = a$ and $t_2^{\mathbf{B_2}}(a, b) = b$.

Applying ε, we obtain $t_1^{\mathbf{B_1}}(a,a) = t_2^{\mathbf{B_1}}(a,a) = a$. Since $a \neq 0$, it follows that $\models_{\mathbb{V}} t_1(x,x) \approx x \approx t_2(x,x)$. By Lemma 4.2, we can write

$$t_1(x,y) \approx s_1(z, u_1(x,y), \dots, u_n(x,y)), \tag{11}$$

$$t_2(x,y) \approx s_2(w, v_1(x,y), \dots, v_m(x,y)), \tag{12}$$

for some terms $s_1(x_0,\dots,x_n)$, $s_2(x_0,\dots,x_m)$, $u_i(x,y)$, $v_j(x,y)$, $1 \leq i \leq n$, $1 \leq j \leq m$, with $\models_{\mathbb{V}} s_1(x,0,\dots,0) \approx x$, $\models_{\mathbb{V}} s_2(x,0,\dots,0) \approx x$, $\models_{\mathbb{V}} u_i(x,x) \approx v_j(x,x) \approx 0$, $1 \leq i \leq n$, $1 \leq j \leq m$, and $z, w \in \{x,y\}$. Then it follows from Lemma 4.1 that for $1 \leq i \leq n$, $1 \leq j \leq m$, $u_i^{\mathbf{B_2}}(a,b) = 0$ and $v_j^{\mathbf{B_2}}(a,b) = 0$. That is, $\mu(u_i(x,y)) = 0 = \mu(v_j(x,y))$. Applying now μ to the equation (11), we obtain:

$$\begin{aligned} a = t_1^{\mathbf{B_2}}(a,b) &= \mu(t_1(x,y)) = s_1^{\mathbf{B_2}}(\mu(z), \mu(u_1(x,y)), \dots, \mu(u_n(x,y))) \\ &= \mu(z), \end{aligned}$$

and similarly, from equation (12),

$$\begin{aligned} b = t_2^{\mathbf{B_2}}(a,b) &= \mu(t_2(x,y)) = s_2^{\mathbf{B_2}}(\mu(w), \mu(v_1(x,y)), \dots, \mu(v_m(x,y))) \\ &= \mu(w). \end{aligned}$$

It then follows that $z = x$ and $w = y$. Finally, since $t_1^{\mathbf{F}}(x,y) = t_2^{\mathbf{F}}(x,y)$, we obtain from equations (11) and (12):

$$\vDash_{\mathbf{F}} s_1(x, u_1(x,y), \dots, u_n(x,y)) = s_2(y, v_1(x,y), \dots, v_m(x,y)),$$

and hence

$$\vDash_{\mathsf{V}(\mathbb{K})} s_1(x, u_1(x,y), \dots, u_n(x,y)) \approx s_2(y, v_1(x,y), \dots, v_m(x,y)).$$

Corollary 4.4. *Let $\mathbb{V}'$ be a subvariety of $\mathbb{V}$, $\mathbb{Q}' = \mathbb{Q} \cap \mathbb{V}'$ the corresponding relative subvariety of $\mathbb{Q}$, and $\mathbb{K}$ a class of algebras, $\mathbb{K} \subseteq \mathbb{V}'$. Then $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}'$ if and only if $\mathbb{K}$ satisfies an identity of the form*

$$s_1(x, u_1(x,y), \dots, u_n(x,y)) \approx s_2(y, v_1(x,y), \dots, v_m(x,y)) \tag{13}$$

for some $n, m \in \omega$, $s_1 \in \mathrm{T}(x_0,\dots,x_n)$, $s_2 \in \mathrm{T}(x_0,\dots,x_m)$, $u_i(x,y)$, $v_j(x,y) \in \mathrm{T}(x,y)$ for $1 \leq i \leq n$, $1 \leq j \leq m$, such that

(14) $\models_{\mathbb{V}'} s_1(x, 0, 0, \dots, 0) \approx x,$

(15) $\models_{\mathbb{V}'} s_2(x, 0, 0, \dots, 0) \approx x,$ *and*

(16) $\models_{\mathbb{V}'} u_i(x, x) \approx 0 \approx v_j(x, x), \quad 1 \le i \le n, 1 \le j \le m.$

Proof. The result is immediate if the algebra $\mathbf{B_2}$ does not belong to $\mathbb{V}'$, as in this case $\mathbb{V}' \subseteq \mathbb{Q}$. For then $\mathbb{Q}' = \mathbb{V}'$ and $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{V}' = \mathbb{Q}'$. Also, since $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}' \subseteq \mathbb{Q}$, the class $\mathbb{K}$ must satisfy an equation of the form (13) for some terms $s_1, s_2, u_i, v_j,\ 1 \le i \le n,\ 1 \le j \le m$, for which conditions (8), (9), and (10) hold. We can therefore restrict ourselves to the case $\mathbf{B_2} \in \mathbb{V}'$ (i.e., $\mathbb{V}' \not\subseteq \mathbb{Q}$). Observe that then $\mathbb{V}'$ is a non-trivial variety.

$\Rightarrow$) If $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}'$, then also $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$ and, by Theorem 4.3, the class $\mathbb{K}$ satisfies an identity of the form (13) for some terms $s_1, s_2, u_i, v_j,\ 1 \le i \le n,\ 1 \le j \le m$, for which conditions (8), (9), and (10) hold. In particular, these terms also satisfy conditions (14), (15) and (16).

$\Leftarrow$) Assume now that $\mathbb{K}$ satisfies an identity of the form (13) for some terms $s_1, s_2, u_i, v_j,\ 1 \le i \le n,\ 1 \le j \le m$, satisfying (14), (15), and (16). Since $\mathbb{V}'$ is non-trivial and $|\mathbf{F}_{\mathbb{V}}(x)| = 2$, we have $\mathbf{F}_{\mathbb{V}}(x) \cong \mathbf{F}_{\mathbb{V}'}(x)$. Therefore, the terms $s_1, s_2, u_i, v_j,\ 1 \le i \le n,\ 1 \le j \le m$, satisfy conditions (8), (9), and (10). Hence, by Theorem 4.3, $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$ and, since $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{V}'$, it follows that $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}'$.

We will refer to identities of the type (13) satisfying (14)–(16), as *Komori identities for* $\mathbb{Q}'$ *relative to* $\mathbb{V}'$ (or Komori identities for $\mathbb{Q}'$, for short, if the variety $\mathbb{V}'$ is clear from the context). We will call a set of Komori identities for $\mathbb{Q}'$ a *complete set of Komori identities for* $\mathbb{Q}'$ if the conclusion of Corollary 4.4 holds with respect to the set. Thus, a set of Komori identities is complete for $\mathbb{Q}'$ if every variety $\mathbb{K} \subseteq \mathbb{Q}'$ satisfies at least one of the identities in the set.

5. Applications

The characterization obtained in Theorem 4.3 allows us to settle some natural algebraic questions about subvarieties of $\mathbb{Q}$. We begin by studying the permutability of congruences in such varieties.

We have seen that the variety $\mathbb{V}$ is not congruence permutable. More generally, any variety $\mathbb{W}$ not contained in $\mathbb{Q}$ will not be congruence n-permutable for any n, as $\mathsf{V}(\mathbf{B_2}) \subseteq \mathbb{V}$ and we have already observed that any variety containing $\mathsf{V}(\mathbf{B_2})$ is not congruence n-permutable for any n (Proposition 3.4). In fact, even if the variety $\mathbb{W}$ is contained in $\mathbb{Q}$ it need not be congruence permutable, as we will later see in examples 6.1.2 and 6.2.1. However, we will show below that any variety contained in $\mathbb{Q}$ is at least congruence 3-permutable.

We will make use of the following lemma, proved in the more general setting of semicongruences in [1].

Lemma 5.1. *Let $\mathbb{W}$ be any variety whose type contains a constant* 0 *and a binary operation Δ satisfying $\Delta(x,x) \approx 0$ and one of the identities $\Delta(0,x) \approx x$, $\Delta(x,0) \approx x$. If $\mathbf{A} \in \mathbb{W}$, then:*

(i) *for any tolerance τ on $\mathbf{A}$, $n \in \omega$, $n \geq 1$, $0/\tau = 0/\tau^n$.*

(ii) *for any $X \subseteq A$, $0/\theta^{\mathbf{A}}(X) = 0/\tau^{\mathbf{A}}(X)$.*

Proof. (i) We will show that if τ is a tolerance of $\mathbf{A}$ then $0/\tau^2 = 0/\tau$, that is, tolerance relations are *transitive at* 0. Since $\mathbf{A} \in \mathbb{W}$, $\mathbf{A}$ satisfies $\Delta(x,0) \approx x$ or $\Delta(0,x) \approx x$. In the first case, if $a, b \in A$, $a \overset{\tau}{\sim} b \overset{\tau}{\sim} 0$, then $a = \Delta^{\mathbf{A}}(a,0) = \Delta^{\mathbf{A}}(a, \Delta^{\mathbf{A}}(b,b)) \overset{\tau}{\sim} \Delta^{\mathbf{A}}(b, \Delta^{\mathbf{A}}(b,0)) = \Delta^{\mathbf{A}}(b,b) = 0$. In the second case, if $a, b \in A$, $a \overset{\tau}{\sim} b \overset{\tau}{\sim} 0$, then $a = \Delta^{\mathbf{A}}(0,a) = \Delta^{\mathbf{A}}(\Delta^{\mathbf{A}}(b,b),a) \overset{\tau}{\sim} \Delta^{\mathbf{A}}(\Delta^{\mathbf{A}}(0,b),b) = \Delta^{\mathbf{A}}(b,b) = 0$. Hence $a \in 0/\tau^2$ implies $a \in 0/\tau$. Thus for any $n \geq 1$, $0/\tau = 0/\tau^n$.

(ii) Since $\theta^{\mathbf{A}}(X)$ is the transitive closure of $\tau^{\mathbf{A}}(X)$, we have:

$$x \in 0/\theta^{\mathbf{A}}(X) \qquad \text{iff} \qquad (x,0) \in \theta^{\mathbf{A}}(X) \qquad \text{iff}$$

$$(x,0) \in (\tau^{\mathbf{A}}(X))^n \text{ for some } n \qquad \text{iff} \qquad x \in 0/(\tau^{\mathbf{A}}(X))^n = 0/\tau^{\mathbf{A}}(X).$$

Theorem 5.2. *Let $\mathbb{K} \subseteq \mathbb{Q}$ be a class of algebras such that $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$. Then the variety $\mathsf{V}(\mathbb{K})$ is congruence 3-permutable.*

Proof. We will make use in this proof of the following result by Raftery and Sturm [15]: *If an algebra $\mathbf{A}$ satisfies $tn(\mathbf{A}) \leq n$, then $\mathbf{A}$ is congruence $(n+1)$-permutable.*

Our goal is to show that, for any algebra $\mathbf{A} \in \mathsf{V}(\mathbb{K})$ and for any tolerance relation τ on $\mathbf{A}$, $\tau^2 = \tau^3$. So let $\mathbf{A} \in \mathsf{V}(\mathbb{K})$, τ a tolerance on $\mathbf{A}$, and $a, b \in A$. Assume that $(a,b) \in \tau^3$. We will show that $(a,b) \in \tau^2$.

Let $\Delta[a,b] = \{\Delta_i^{\mathbf{A}}(a,b) : i \in I\}$ and $(\Delta[a,b],0) = \{(x,0) : x \in \Delta[a,b]\}$. By Lemma 5.1, if $(a,b) \in \tau^3$, then $\Delta[a,b] \subseteq 0/\tau^3 = 0/\tau$, so $(\Delta[a,b],0) \subseteq \tau$ and the tolerance relation generated by $(\Delta[a,b],0)$ satisfies $\tau^{\mathbf{A}}((\Delta[a,b],0)) \subseteq \tau$. By Lemma 5.1, we have $0/\theta^{\mathbf{A}}((\Delta[a,b],0)) = 0/\tau^{\mathbf{A}}((\Delta[a,b],0)) \subseteq 0/\tau$.

Since $\mathbf{A} \in \mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$, the algebra $\mathbf{A}$ is not only relatively 0-regular, but 0-regular. It follows from Proposition 3.1 that $\theta^{\mathbf{A}}(a,b) = \theta^{\mathbf{A}}((\Delta[a,b],0))$. Hence $0/\theta^{\mathbf{A}}(a,b) \subseteq 0/\tau$.

Since $\mathsf{V}(\mathbb{K}) \subseteq \mathbb{Q}$, there exist $n, m \in \omega$, and terms $s_1(x_0,\ldots,x_n)$, $s_2(x_0,\ldots,x_m)$, $u_i(x,y)$, $v_j(x,y)$, $1 \leq i \leq n, 1 \leq j \leq m$, as in Theorem 4.3, such that $\models_{\mathbb{V}} s_1(x,0,\ldots,0) \approx x$, $\models_{\mathbb{V}} s_2(x,0,\ldots,0) \approx x$, $\models_{\mathbb{V}} u_i(x,x) \approx v_j(x,x) \approx 0$, $1 \leq i \leq n$, $1 \leq j \leq m$, and

$$\models_{\mathbb{V}} s_1(x, u_1(x,y), \ldots, u_n(x,y)) \approx s_2(y, v_1(x,y), \ldots, v_m(x,y)).$$

We then have:

$$0 = u_i^{\mathbf{A}}(a,a) \overset{\theta^{\mathbf{A}}(a,b)}{\equiv} u_i^{\mathbf{A}}(a,b) \quad 1 \leq i \leq n, \quad \text{and similarly,}$$
$$0 = v_j^{\mathbf{A}}(a,a) \overset{\theta^{\mathbf{A}}(a,b)}{\equiv} v_j^{\mathbf{A}}(a,b) \quad 1 \leq j \leq m,$$

so $u_i^{\mathbf{A}}(a,b) \in 0/\theta^{\mathbf{A}}(a,b) \subseteq 0/\tau$, $\quad v_j^{\mathbf{A}}(a,b) \in 0/\theta^{\mathbf{A}}(a,b) \subseteq 0/\tau$, $\quad$ for all i, j. Finally, we obtain

$$\begin{aligned} a = s_1^{\mathbf{A}}(a,0,\ldots,0) \ &\tau\ s_1^{\mathbf{A}}(a, u_1^{\mathbf{A}}(a,b), \ldots, u_n^{\mathbf{A}}(a,b)) \\ &=\ s_2^{\mathbf{A}}(b, v_1^{\mathbf{A}}(a,b), \ldots, v_m^{\mathbf{A}}(a,b)) \\ &\tau\ s_2^{\mathbf{A}}(b,0,\ldots,0) = b, \end{aligned}$$

that is, $a\ \tau^2\ b$, so $(a,b) \in \tau^2$. We have thus shown that $\tau^3 \subseteq \tau^2$. This implies that, for any tolerance τ on $\mathbf{A}$, τ^2 is a congruence. That is, the tolerance number of $\mathbf{A}$ satisfies $\mathrm{tn}(\mathbf{A}) \leq 2$, from which we can conclude that $\mathbf{A}$ is congruence 3-permutable.

Given any variety $\mathbb{W}$, the class of all subvarieties of $\mathbb{W}$, ordered by inclusion, can be viewed as a lattice in which the lattice operations are defined by $\mathbb{W} \wedge \mathbb{W}' = \mathbb{W} \cap \mathbb{W}'$, $\mathbb{W} \vee \mathbb{W}' = \mathsf{V}(\mathbb{W} \cup \mathbb{W}')$. If $\mathbb{K}$ is a quasivariety, however, this is in general not true, since for $\mathbb{W}, \mathbb{W}' \subseteq \mathbb{K}$ we have no guarantee that $\mathsf{V}(\mathbb{W} \cup \mathbb{W}')$ will be contained in $\mathbb{K}$. In the following result we use the characterization obtained in Theorem 4.3 to show that the subvarieties of $\mathbb{Q}$ do indeed form a lattice, and thus an ideal in the lattice of subvarieties of $\mathbb{V}$.

Theorem 5.3. *If* $\mathbb{W}, \mathbb{W}'$ *are varieties,* $\mathbb{W}, \mathbb{W}' \subseteq \mathbb{Q}$*, then* $\mathbb{W} \vee \mathbb{W}' = \mathsf{V}(\mathbb{W} \cup \mathbb{W}') \subseteq \mathbb{Q}$.

Proof. The result is immediate if any of $\mathbb{W}, \mathbb{W}'$ is a trivial variety. We will restrict ourselves to the case in which both varieties are non-trivial. By Theorem 4.3 there exist terms $s_1(x_0, \ldots, x_n)$, $s_2(x_0, \ldots, x_m)$, $s'_1(x_0, \ldots, x_k)$, $s'_2(x_0, \ldots, x_l)$, $u_i(x,y)$, $1 \leq i \leq n$, $v_i(x,y)$, $1 \leq i \leq m$, $u'_i(x,y)$, $1 \leq i \leq k$, $v'_i(x,y)$, $1 \leq i \leq l$, such that

$$\mathbb{V} \models s_1(x, 0, \ldots, 0) \approx x,$$

$$\mathbb{V} \models s_2(x, 0, \ldots, 0) \approx x,$$

$$\mathbb{V} \models s'_1(x, 0, \ldots, 0) \approx x,$$

$$\mathbb{V} \models s'_2(x, 0, \ldots, 0) \approx x,$$

$$\mathbb{V} \models u_i(x,x) \approx v_j(x,x) \approx u'_{i'}(x,x) \approx v'_{j'}(x,x) \approx 0,$$

for $1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq i' \leq k$, $1 \leq j' \leq l$,

$$\mathbb{W} \models s_1(x, u_1(x,y), \ldots, u_n(x,y)) \approx s_2(y, v_1(x,y), \ldots, v_m(x,y)),$$

and

$$\mathbb{W}' \models s'_1(x, u'_1(x,y), \ldots, u'_k(x,y)) \approx s'_2(y, v'_1(x,y), \ldots, v'_l(x,y)).$$

We abbreviate the left-hand sides and right-hand sides of the identities above by $r_1(x,y)$, $r_2(x,y)$, $r_1'(x,y)$, $r_2'(x,y)$. More explicitly:

$$r_1(x,y) = s_1(x, u_1(x,y), \ldots, u_n(x,y)),$$

$$r_2(x,y) = s_2(y, v_1(x,y), \ldots, v_m(x,y)),$$

$$r_1'(x,y) = s_1'(x, u_1'(x,y), \ldots, u_k'(x,y)),$$

$$r_2'(x,y) = s_2'(y, v_1'(x,y), \ldots, v_l'(x,y)).$$

We claim that

$$\models_{\mathbb{W}\cup\mathbb{W}'} r_1(r_1'(x,y), r_2'(x,y)) \approx r_2(r_1'(x,y), r_2'(x,y)).$$

Since $\models_{\mathbb{W}} r_1(x,y) \approx r_2(x,y)$, in particular we have that

$$\models_{\mathbb{W}} r_1(r_1'(x,y), r_2'(x,y)) \approx r_2(r_1'(x,y), r_2'(x,y)).$$

Also, as $\models_{\mathbb{W}'} r_1'(x,y) \approx r_2'(x,y)$, to show that $\models_{\mathbb{W}'} r_1(r_1'(x,y), r_2'(x,y)) \approx r_2(r_1'(x,y), r_2'(x,y))$ it suffices to show that

$$\models_{\mathbb{W}'} r_1(r_1'(x,y), r_1'(x,y)) \approx r_2(r_1'(x,y), r_1'(x,y)).$$

Since both $\mathbb{W}, \mathbb{W}'$ are non-trivial subvarieties of $\mathbb{V}$ and $|\mathbf{F}_{\mathbb{V}}(x)| = 2$, it follows that $\mathbf{F}_{\mathbb{W}}(x) \cong \mathbf{F}_{\mathbb{W}'}(x) \cong \mathbf{F}_{\mathbb{V}}(x)$. As $\models_{\mathbb{W}} r_1(x,y) \approx r_2(x,y)$, we have $\mathbf{F}_{\mathbb{W}}(x) \models r_1(x,x) \approx r_2(x,x)$ and hence $\mathbf{F}_{\mathbb{W}'}(x) \models r_1(x,x) \approx r_2(x,x)$. Thus $\models_{\mathbb{W}'} r_1(x,x) \approx r_2(x,x)$, and therefore $\models_{\mathbb{W}'} r_1(r_1'(x,y), r_1'(x,y)) \approx r_2(r_1'(x,y), r_1'(x,y))$.

It only remains to show that $r_1(r_1'(x,y), r_2'(x,y)) \approx r_2(r_1'(x,y), r_2'(x,y))$ is an identity of the form (7). Expanding the terms $r_1(x,y)$, $r_2(x,y)$, $r_1'(x,y)$, $r_2'(x,y)$, and omitting the variables x, y to improve readability, we can rewrite $r_1(r_1'(x,y), r_2'(x,y)) \approx r_2(r_1'(x,y), r_2'(x,y))$ in the form:

$$\begin{gathered} s_1(s_1'(x, u_1', \ldots, u_k'), u_1(r_1', r_2'), \ldots, u_n(r_1', r_2')) \approx \\ s_2(s_2'(y, v_1', \ldots, v_l'), v_1(r_1', r_2'), \ldots, v_m(r_1', r_2')). \end{gathered} \tag{17}$$

We can define new terms t_1 and t_2 on $k+n+1$ and $l+m+1$ variables, respectively by:

$$t_1(x_0, x_1, \dots, x_k, y_1, \dots, y_n) = s_1(s_1'(x_0, \dots, x_k), y_1, \dots, y_n),$$
$$t_2(x_0, x_1, \dots, x_m, y_1, \dots, y_l) = s_2(s_2'(x_0, \dots, x_l), y_0, \dots, y_m).$$

It can be easily verified that equation (17) can then be written as

$$t_1(x, u_1', \dots, u_k', u_1(r_1', r_2'), \dots, u_n(r_1', r_2')) \approx$$
$$t_2(y, v_1', \dots, v_l', v_1(r_1', r_2'), \dots, v_m(r_1', r_2')).$$

In addition,

$$\models_{\mathbb{V}} r_1'(x,x) \;\approx\; s_1'(x, u_1'(x,x), \dots, u_k'(x,x)) \approx$$
$$x \;\approx\; s_2'(x, v_1'(x,x), \dots, v_l'(x,x)) \;\approx\; r_2'(x,x),$$

so, if $1 \le i \le n$, $1 \le j \le m$,

$$\models_{\mathbb{V}} u_i(r_1'(x,x), r_2'(x,x)) \;\approx\; u_i(x,x) \;\approx\; 0, \text{ and}$$
$$\models_{\mathbb{V}} v_j(r_1'(x,x), r_2'(x,x)) \;\approx\; v_j(x,x) \;\approx\; 0.$$

Hence, again by Theorem 4.3, $\mathbb{W} \vee \mathbb{W}' = \mathsf{V}(\mathbb{W} \cup \mathbb{W}') \subseteq \mathbb{Q}$.

6. Examples

We examine now some classes of algebras which are examples of subvarieties of a quasivariety $\mathbb{Q}$ of the type defined above. Many of these classes are equivalent quasivariety semantics for well-known deductive systems, such as fragments of classical propositional logic, intuitionistic propositional logic, and Łukasiewicz many-valued logics. The first two examples are classes of algebras over a language with two binary operations and a constant, while in the remaining examples the language consists of a single binary operation (which for simplicity we will denote by $\star$) and a constant.

6.1. The non-commutative case.

6.1.1. *Biresiduation algebras.* The class $\mathbb{BR}$ of *biresiduation algebras* is the quasivariety over the language $\mathcal{L} = \{\Delta_1, \Delta_2, 1\}$, where $\Delta_1(x, y)$ will be denoted by $x\diagdown y$ and $\Delta_2(x, y)$ by $x\diagup y$, defined by the following identities and quasi-identity:

$$x\diagdown x \approx 1, \quad x\diagup x \approx 1 \tag{A1}$$
$$x\diagdown 1 \approx 1, \quad 1\diagup x \approx 1 \tag{A2}$$
$$(x\diagdown y)\diagdown((z\diagdown x)\diagdown(z\diagdown y)) \approx 1, \quad ((y\diagup z)\diagup(x\diagup z))\diagup(y\diagup x) \approx 1 \tag{A3}$$
$$1\diagdown x \approx x, \quad x\diagup 1 \approx x \tag{A4}$$
$$(x\diagdown y)\diagup z \approx x\diagdown(y\diagup z) \tag{A5}$$
$$x\diagdown((y\diagup x)\diagdown y) \approx 1 \tag{A6}$$
$$(y\diagup(x\diagdown y))\diagup x \approx 1 \tag{A7}$$
$$x\diagdown y \approx 1 \ \wedge \ x\diagup y \approx 1 \Longrightarrow x \approx y \tag{A8}$$

Let $\leq_1$ be given by $x \leq_1 y$ if $x\diagdown y = 1$ and let $\leq_2$ be given by $x \leq_2 y$ if $y\diagup x = 1$. Axioms (A1)–(A4) guarantee that $\leq_1$ and $\leq_2$ are reflexive and transitive relations with $x \leq_1 1$ and $x \leq_2 1$ for all x. Axioms (A6) and (A7) ensure that $\leq_1 = \leq_2$, while axiom (A8) shows that these two relations are in fact partial orders. Axiom (A5) is needed to guarantee that the algebras are subreducts of residuated partially ordered monoids.

Notice that in this class we have $J = \{\diagup\}$ and $K = \{\diagdown\}$. Identities (A1), (A2), (A4), and quasi-identity (A8) show that the variety $\mathbb{V}^r$ defined by identities (A1)—(A7) and the quasivariety $\mathbb{BR}$ are examples of a variety $\mathbb{V}'$ and a quasivariety $\mathbb{Q}'$ of the kind described in Corollary 4.4.

Van Alten [17] has verified that a class $\mathbb{K} \subseteq \mathbb{BR}$ generates a variety contained in $\mathbb{BR}$ if and only if $\mathbb{K}$ satisfies an identity of the form

$$((\prod_{i=1}^{n_1} u_i(x, y))\diagdown x)\diagup \prod_{j=1}^{n_2} u'_j(x, y) \approx ((\prod_{k=1}^{m_1} v_k(x, y))\diagdown y)\diagup \prod_{l=1}^{m_2} v'_l(x, y), \tag{18}$$

where $\mathbb{BR} \models u_i(x, x) \approx u'_j(x, x) \approx v_k(x, x) \approx v'_l(x, x) \approx 0$, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, $1 \leq k \leq m_1$, $1 \leq l \leq m_2$. Here we have used the abbreviations $(s \cdot t)\diagdown x := s\diagdown(t\diagdown x)$ and $y\diagup(s \cdot t) := (y\diagup t)\diagup s$. Clearly all identities of form (18) are Komori identities for $\mathbb{BR}$ relative to $\mathbb{V}^r$. Van Alten's result guarantees that these identities form a complete set of Komori identities for $\mathbb{BR}$.

A similar result was obtained for it left residuation algebras in [18].

6.1.2. *BCK-algebras.* The class $\mathbb{BCK}$ of BCK-algebras was introduced by Iséki [10]. It constitutes the equivalent algebraic semantics for the deductive system $\mathcal{BCK}$ of Meredith (see [14],[2]). An interesting aspect of this deductive system is that is not strongly algebraizable, that is, the class $\mathbb{BCK}$ is a quasivariety, but not a variety ([7], [19]).

It is not difficult to see that the class $\mathbb{BCK}$ is term equivalent to the relative subvariety of $\mathbb{BR}$ defined by the identity $x \diagdown y \approx y \diagup x$. Therefore one of the binary operations can be eliminated; we denote the remaining one by $\rightarrow$. Quasi-identity (A8) then becomes

$$x \rightarrow y \approx 1 \;\wedge\; y \rightarrow x \approx 1 \Longrightarrow x \approx y.$$

The roles of the variety $\mathbb{V}^r$ and the quasivariety $\mathbb{BR}$ in the previous example are now played by the subvariety of $\mathbb{V}^r$ defined, relative to $\mathbb{V}^r$, by the extra identity $x \diagup y \approx y \diagdown x$ and the quasivariety $\mathbb{BCK}$ of BCK-algebras respectively. Once again, Corollary 4.4 guarantees the existence of Komori identities for any variety of BCK-algebras.

In the usual treatment of BCK-algebras, dual to the one given here, BCK-algebras are presented as algebras of type $\langle 2, 0\rangle$ satisfying the following axioms:

$$\begin{aligned}((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) &\approx 0\\ x \dot{-} 0 &\approx x\\ 0 \dot{-} x &\approx 0\\ x \dot{-} y \approx 0 \wedge y \dot{-} x \approx 0 &\Longrightarrow x \approx y.\end{aligned}$$

In this formulation, it can be shown that the set of all identities of the form

$$x \dot{-} u_1(x,y) \dot{-} \ldots \dot{-} u_n(x,y) \;\approx\; y \dot{-} v_1(x,y) \dot{-} \ldots \dot{-} v_m(x,y),$$

where $\mathbb{BCK} \models u_i(x,x) \approx v_j(x,x) \approx 0$, $1 \le i \le n$, $1 \le j \le m$, is a complete set of Komori identities for $\mathbb{BCK}$.

The fact that a class $\mathbb{K} \subseteq \mathbb{BCK}$ generates a variety contained in $\mathbb{BCK}$ if and only if $\mathbb{K}$ satisfies an identity of the type above was first mentioned in [8] and attributed to Komori. See [3] for a generalization and proof of this result.

The class $\mathbb{BCK}$ is relatively 0-regular and relatively congruence distributive. Note that even the smallest non-trivial subvariety of $\mathbb{BCK}$, the one generated by the two-element algebra, is not congruence permutable. As a consequence of Theorem 5.2, first proved for BCK-algebras in [8], any variety contained in $\mathbb{BCK}$ is congruence 3-permutable. In addition, any such variety is congruence distributive (see [13]).

6.2. The commutative case. We will consider now classes of algebras over the language $\mathcal{L} = \{\star, 0\}$, where the binary operation $\star$ is commutative.

Let $\mathbb{V}^c$ be the variety defined by the following identities:

$$x \star x \approx 0, \tag{19}$$

$$x \star 0 \approx x, \tag{20}$$

$$x \star y \approx y \star x. \tag{21}$$

The quasivariety $\mathbb{Q}^c$ will be defined, relative to $\mathbb{V}^c$, by the quasi-identity

$$x \star y \approx 0 \Longrightarrow x \approx y \tag{22}$$

The variety $\mathbb{V}^c$ and the quasivariety $\mathbb{Q}^c$ are examples of classes of algebras of the kind we have been studying in this paper. In particular, in $\mathbb{V}^c$ we have $J = K = \{\star\}$.

The class $\mathbb{Q}^c$ is the equivalent quasivariety semantics for the deductive system $\mathcal{S}$ given by the following axioms and rules:

1. $\vdash p \leftrightarrow p$,
2. $\vdash (p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$,

3. $\vdash p \leftrightarrow (p \leftrightarrow \top)$,
4. $p \leftrightarrow q \vdash (p \leftrightarrow r) \leftrightarrow (q \leftrightarrow r)$ (or, equivalently, $p \leftrightarrow q,\ q \leftrightarrow r \vdash p \leftrightarrow r$),
5. $p,\ p \leftrightarrow q \vdash q$. (Detachment)

It is easy to see that the system $\mathcal{S}$ also satisfies the so-called G-rule: $p, q \ \vdash p \leftrightarrow q$. As a result (see [2]), $\mathcal{S}$ is algebraizable, with equivalence formula $p \leftrightarrow q$ and defining equation $p \approx \top$.

To simplify the notation, we will often drop the symbol $\star$ and write xy for $x \star y$. Moreover, to avoid the use of unnecessary parentheses, we will associate to the left. Thus xyz and $x \star y \star z$ will both denote $(x \star y) \star z$.

In this "commutative" case, an immediate application of Corollary 4.4 shows that the identities of the form

$$x \star u_1(x,y) \star \ldots \star u_n(x,y) \approx y \star v_1(x,y) \star \ldots \star v_m(x,y), \tag{23}$$

where $\mathbb{V}^c \vDash u_i(x,x) \approx v_j(x,x) \approx 0,\ 1 \leq i \leq n,\ 1 \leq j \leq m$, constitute a complete set of Komori identities for $\mathbb{Q}^c$ relative to $\mathbb{V}^c$ (see [12]).

We will now look at examples of varieties contained in $\mathbb{Q}^c$ and the Komori identities that these varieties satisfy.

6.2.1. *Łukasiewicz Algebras.* The purely equivalential fragments of multivalued Łukasiewicz's logics are algebraizable and their equivalent quasivariety semantics are generated by equivalential subreducts of Łukasiewicz algebras. More explicitly, let

$$\mathbf{Ł}_\mathbf{n}^\star = \langle \{i \ : \ 0 \leq i < n\}; \star, 0 \rangle,$$

$$\mathbf{Ł}_\omega^\star = \langle \omega; \star, 0 \rangle,$$

where the operation $\star$ is defined by $x \star y := |x - y|$. Then the equivalent quasivariety semantics of the equivalential fragment of the infinite-valued logic is the quasivariety $\mathbb{L} = \mathsf{Q}(\{\mathbf{Ł}_\mathbf{n}^\star \ : \ n \leq \omega\})$, while $\mathsf{Q}(\mathbf{Ł}_\mathbf{n})$ provides an equivalent semantics for the equivalential fragment of the n-valued logic.

It is clear that all algebras $\mathbf{Ł}_\mathbf{n}^\star$, $n \leq \omega$, satisfy identities (19), (20), (21), and quasi-identity (22), so the class $\mathbb{L}$ is a subclass of the quasivariety $\mathbb{Q}^c$.

Moreover, simple computations yield, in $\mathbf{Ł}^{\star}_{\omega}$,

$$xyy = \begin{cases} x, & \text{if } x \leq y, \\ 2y - x, & \text{if } y < x < 2y, \\ x - 2y, & \text{if } 2y < x; \end{cases}$$

where $\leq$ denotes the usual order relation on ω. It follows that for any x, y, we have $xyy \leq x$, so all algebras in $\mathbb{L}$ satisfy

$$xyyxx \approx xyy.$$

Observe that this identity can also be written as $(xyyx)x \approx (xy)y$ or, using commutativity, $x(xyyx) \approx y(xy)$, a Komori identity for $\mathbb{Q}^c$. Hence any class of equivalential reducts of Łukasiewicz algebras generates a variety which is contained in $\mathbb{Q}^c$.

Notice that $\mathsf{V}(\mathbf{Ł_n}) = \mathsf{Q}(\mathbf{Ł_n})$ is a congruence 3-permutable variety (by Corollary 5.2), but is not congruence permutable if $n \geq 4$ (see [12]).

The smallest subvariety of $\mathbb{L}$ is the variety generated by the algebra $\mathbf{Ł_2} \cong \mathbf{B_1}$, which coincides with the variety $\mathbb{BG}$ of Boolean Groups. This class, which can be axiomatized relative to $\mathbb{V}^c$ by the identity

$$x \star (y \star z) \approx (x \star y) \star z,$$

is the equivalent algebraic semantics for the equivalential fragment of classical propositional logic. The class $\mathbb{BG}$ satisfies a simpler Komori identity, namely

$$xyy \approx x.$$

6.2.2. *Equivalential Algebras.* Let $\mathcal{IPC}^{\leftrightarrow}$ be the $\langle\leftrightarrow\rangle$-fragment of the intuitionistic propositional calculus, the *intuitionistic equivalential calculus.* $\mathcal{IPC}^{\leftrightarrow}$ is strongly algebraizable with defining equation $p \approx p \leftrightarrow p$, and equivalence formula $\Delta(p, q) := p \leftrightarrow q$. Its equivalent quasivariety semantics is the variety $\mathbb{EA}$ of equivalential algebras, introduced in [11] by Kabziński and Wroński. Equivalential algebras were originally defined as algebras with a single binary operation $\star$. However, since $\mathbb{EA}$ also satisfies

$x \star x \approx y \star y$, this variety is term equivalent to a subvariety of $\mathbb{Q}^c$ with 0 defined as $0^{\mathbf{A}} = x \star^{\mathbf{A}} x$.

It can be shown (see [11]) that $\mathbb{EA}$ satisfies the following identity:

$$xyyxx \approx xyy,$$

which can be rewritten as $x(xyyx) \approx y(xy)$, a Komori identity for $\mathbb{Q}^c$. This identity then witnesses the fact that $\mathbb{EA} \subseteq \mathbb{Q}^c$.

$\mathbb{EA}$ is congruence permutable, with Mal'cev term

$$p(x, y, z) = (xyz)(yzx)(xzy)$$

(see [9]).

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