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Andrzej Wroński

SEMANTIC NORMAL FORM

A b s t r a c t. The idea of semantic normal form originally developed by Jankov [17] for Brouwerian semilattices is made applicable to the variety of equivalential algebras and thereby, to a broader family of locally finite and permutable varieties obeying the conditions of Fregeanity i.e. point regularity and congruence orderability. It is proved that every term in the language of such a variety can be equivalently expressed with the help of a relatively small set of building blocks manufactured from so-called monolith assignments.

1. Introduction

Generally our notations and nomenclature follow Burris and Sankappanavar [8] but with three exceptions: first, we prefer to use German capitals for algebras, second, we seldom bother to make notational distinction

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between an algebra and its underlying set or between a term and the corresponding term operation, third, by subdirectly irreducible algebra we mean an algebra possessing a smallest non-trivial congruence (the monolith congruence). Thus, a one-element algebra will be denied subdirect irreducibility.

We assume that every similarity type has its corresponding discourse language and we use the symbol $\mathfrak{F}(X)$ for the absolutely free algebra of terms of the considered similarity type freely generated by the set X whose elements will be called variables.

The set of all variables occurring in terms of a set T will be denoted by $\mathsf{Var}(T)$ and we shall write $\mathsf{Var}(t)\,, \mathsf{Var}(t_1,\ldots,t_n)$ instead of $\mathsf{Var}(\{t\})\,$, $\mathsf{Var}(\{t_1,\ldots,t_n\})$ respectively.

Let $\mathbb{K}$ be a class of algebras of the similarity type of $\mathfrak{F}(X)$. By $\Theta_{\mathbb{K}}$ we will denote the fully invariant congruence of $\mathfrak{F}(X)$ determined by $\mathbb{K}$ i.e. the congruence defined by: $\langle t_1, t_2\rangle \in \Theta_{\mathbb{K}}$ iff $\mathbb{K} \models t_1 \approx t_2$, for all terms $t_1,\ t_2 \in \mathfrak{F}(X)$. The quotient algebra $\mathfrak{F}(X)/\Theta_{\mathbb{K}}$ — which is free in the variety generated by $\mathbb{K}$ — will be denoted by $\mathfrak{F}_{\mathbb{K}}(X)$. If $X = \{x_1,\ldots,x_n\}$ then we will write $\mathfrak{F}(n),\ \mathfrak{F}_{\mathbb{K}}(n)$ instead of $\mathfrak{F}(X),\ \mathfrak{F}_{\mathbb{K}}(X)$ respectively.

By an assignment in an algebra $\mathfrak{A}$ we mean a mapping whose domain is a set of variables and whose range is contained in $\mathfrak{A}$. The subalgebra of $\mathfrak{A}$ generated by the range of an assignment v will be called the target algebra of v and denoted by $[\![v]\!]$.

In general, to be sure that $[\![v]\!] \neq \emptyset$ one has to stipulate: $\mathsf{Dom}(v) \neq \emptyset$. Here, however, we can afford allowing $\mathsf{Dom}(v)$ to be empty because all algebras we are dealing with have constants among their basic operations.

Definition 1.[1] Assignments v and w are said to be equivalent if $\mathsf{Dom}(v) = \mathsf{Dom}(w)$ and there exists an isomorphism $[\![v]\!] \mapsto [\![w]\!]$ such that $v(x)$ is mapped to $w(x)$, for every $x \in \mathsf{Dom}(v)$.

[1] Earlier attempts at grasping the idea of semantic normal form were presented by the author in Karpacz, 24–27 April 1997 and 6–10 May 1998 to the Conferences "Applications of Logic in Mathematics" II and III. The author thanks Professor Piotr Wojtylak, whose remark at Karpacz Conference III helped him to correct an error in an earlier version of the Definition 1.

We adopt the convention of not distinguishing equivalent assignments and thus, each non-empty class of equivalent assignments will be perceived as a singleton.

Definition 2. An assignment v will be called a monolith assignment if $\mathsf{Dom}(v)$ is finite and the algebra $[\![v]\!]$ is subdirectly irreducible.

Let $\mathfrak{A}$ be an algebra whose discourse language has a constant term $\mathbf{1}$. The following two conditions on $\mathfrak{A}$ are of great importance for the present work:

(R) If $\mathbf{1}/\alpha = \mathbf{1}/\beta$ then $\alpha = \beta$, for every $\alpha, \beta \in \mathsf{Con}(\mathfrak{A})$,

($\leq$) If $\Theta(a, \mathbf{1}) = \Theta(b, \mathbf{1})$ then $a = b$, for every $a, b \in \mathfrak{A}$.

The condition (R) is known in the literature under several different names (see [14], [13], [23]). Here - following [2] - we shall call it 1-regularity or simply regularity. The condition (R) implies that the mapping $\alpha \mapsto \mathbf{1}/\alpha$ is an isomorphism of the congruence lattice $\mathsf{Con}(\mathfrak{A})$ and the lattice of filters $\mathsf{Fil}(\mathfrak{A}) \stackrel{\text{def}}{=} \{\mathbf{1}/\alpha \ : \ \alpha \in \mathsf{Con}(\mathfrak{A})\}$.

The condition ($\leq$) was considered by Büchi an Ovens in [7], where algebras obeying it were called fission free. It says that the mapping $a \mapsto \Theta(a, \mathbf{1})$ of the underlying set of $\mathfrak{A}$ into its congruence lattice is 1–1 which provides a kind of natural ordering of the underlying set of $\mathfrak{A}$ given by the obvious stipulation: $a \leq b$ iff $\Theta(a, \mathbf{1}) \subseteq \Theta(b, \mathbf{1})$.

A prominent place of ($\leq$) in algebraic logic is due to the work of D. Pigozzi and his collaborators (see [23], [9], [10], [12]). In the paper [23] of D. Pigozzi, classes of algebras obeying the condition ($\leq$) are named Fregean.

For reasons explained elsewhere (see [15]) our usage of the concept of Fregeanity will follow an earlier paper of Blok, Köhler and Pigozzi [2].

Thus, we say that algebra $\mathfrak{A}$ is Fregean if both conditions (R) and ($\leq$) are obeyed and by a Fregean class of algebras we mean a class $\mathbb{K}$ of similar algebras with a distinguished constant term $\mathbf{1}$ such that all members of $\mathbb{K}$ obey (R) and ($\leq$) wrt $\mathbf{1}$.

Among Fregean varieties are many important varieties connected with logic, such as Boolean algebras, Heyting algebras, Brouwerian semilattices, and equivalential algebras (see [15]). Note the following:

Fact 3. *(see Idziak, Słomczyńska, Wroński [15])*

(i) *If* $\mathfrak{A}$ *obeys* (R) *then* $\alpha \subseteq \Theta(\mathbf{1}/\alpha)$, *for every* $\alpha \in \mathsf{Con}(\mathfrak{A})$,

(ii) *If* $\mathfrak{A}$ *obeying* $(\leq)$ *is subdirectly irreducible and* μ *is the monolith congruence of* $\mathfrak{A}$ *then* $|\mathbf{1}/\mu| = 2$ *and all remaining* μ*-cosets are singletons,*

(iii) *If all members of* $\mathsf{H}(\mathfrak{A})$ *obey* $(\leq)$ *then* $\alpha \vee \Theta(a, \mathbf{1}) = \alpha \vee \Theta(b, \mathbf{1})$ *iff* $\langle a, b\rangle \in \alpha$, *for every* $\alpha \in \mathsf{Con}(\mathfrak{A})$ *and* $a, b \in \mathfrak{A}$.

Thus, if μ is a monolith congruence of a subdirectly irreducible Fregean algebra then the unique member of $\mathbf{1}/\mu$ which is distinct from $\mathbf{1}$ will be denoted by $\star$. It is easy to see that the monolith congruence of a subdirectly irreducible Fregean algebra always has the form $\Theta(\star, \mathbf{1})$. We also have the following nice looking:

Theorem 4. *(see Idziak, Słomczyńska, Wroński [15]) For every Fregean variety of algebras* $\mathbb{K}$ *the following conditions are equivalent:*

(i) $\mathbb{K}$ is congruence permutable

(ii) $\mathbb{K}$ has a binary term $\mathbf{e}$ satisfying any of the following three equivalent conditions:

  (1) $\mathbb{K} \models \mathbf{e}(x, x) \approx \mathbf{1}$ and $\mathbf{e}(\mathbf{1}, x) \approx x \approx \mathbf{e}(x, \mathbf{1})$,

  (2) $\mathbf{e}$ is a principal congruence term for $\mathbb{K}$ in the sense that $\Theta(a, b) = \Theta(\mathbf{e}(a, b), \mathbf{1})$, for every $a, b \in \mathfrak{A} \in \mathbb{K}$,

  (3) the $\mathbf{e}$-reduct of every $\mathfrak{A} \in \mathbb{K}$ is an equivalential algebra.

2. Equivalential algebras

In several places we employ a somewhat specialized knowledge about Brouwerian semilattices and equivalential algebras. For detailed information pertaining to Brouwerian semilattices one can consult [20] and for equivalential algebras [19] or better Słomczyńska [25]. Some amount of information about equivalential algebras — which are less well-known then they deserve — will be repeated here for the reader's convenience.

Equivalential algebras can be viewed as an algebraic counterpart of the purely equivalential fragment of the intuitionistic propositional logic in the same sense as Brouwerian semilattices correspond to the $(\wedge, \rightarrow)$-fragment of this logical system. They can be characterized as $\leftrightarrow$-subreducts of Brouwerian semilattices with $\leftrightarrow$ defined by the term $(x \rightarrow y) \wedge (y \rightarrow x)$.

It is known (see [19]) that equivalential algebras constitute a variety $\mathbb{E}$ of type $\langle 2, 0\rangle$ whose operations: $\leftrightarrow$ (binary) and $\mathbf{1}$ (nullary) satisfy the following identities:

- $(x \leftrightarrow x) = \mathbf{1}$,
- $(\mathbf{1} \leftrightarrow x) = x$,
- $((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z = (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z)$,
- $((x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z)) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) = x \leftrightarrow y$.

Equivalential algebras can also be characterized in a different manner. Let $\mathfrak{S} = \langle S, \vee, \mathbf{0}\rangle$ be an upper-semilattice with zero. Then for some $a, b \in S$ it can happen that the set $\{c \in S \ : \ a \vee c = b \vee c\}$ has a smallest element. Such an element – if it exists – will be denoted $a \oplus b$ and will be called a *symmetric pseudo–difference* of a and b in $\mathfrak{S}$.

Note the following:

Theorem 5. *(see Idziak, Słomczyńska, Wroński [15]) If $\mathfrak{S} = \langle S, \vee, \mathbf{0}\rangle$ is an upper-semilattice with zero and a nonempty $A \subseteq S$ is such that for every $a, b \in A$, the symmetric pseudo–difference of a and b in $\mathfrak{S}$ exists and belongs to A then $\langle A, \oplus, \mathbf{0}\rangle$ is an equivalential algebra.*

Note that the condition (iii) of Fact 3 says that if $\mathsf{H}(\mathfrak{A})$ obeys $(\leq)$ then for every $a, b \in \mathfrak{A}$, the symmetric pseudo-difference $\Theta(a, \mathbf{1})$ and $\Theta(b, \mathbf{1})$ in $\mathsf{Con}(\mathfrak{A})$ exists and equals $\Theta(a, b)$.

Thus, if only filters of principal congruences of $\mathfrak{A}$ were principal as filters i.e. if for every $a, b \in \mathfrak{A}$ we could find $c \in \mathfrak{A}$ such that $\mathbf{1}/\Theta(a, b) =$

$\mathbf{1}/\Theta(c, \mathbf{1})$ then the set of principal filters of $\mathfrak{A}$ could be endowed with the structure of an equivalential algebra.

The fact that such a situation occurs in permutable Fregean varieties clearly explains why the conditions (2) and (3) of Theorem 4 are equivalent.

Returning to technicalities we introduce the generalized equivalence operation $\bigoplus$ applicable to any finite sequence of elements of an equivalential algebra. The value of the operation $\bigoplus$ is the left associated equivalence of all successive elements of the sequence. Thus we put: $\bigoplus \emptyset \stackrel{\text{def}}{=} \mathbf{1}$ and $\bigoplus(a_1 \dots a_{n+1}) \stackrel{\text{def}}{=} (\bigoplus(a_1 \dots a_n)) \leftrightarrow a_{n+1}$.

Below we give a list of useful identities that hold in the variety of equivalential algebras. To enhance readability we adopt the convention of associating to the left and denoting $\leftrightarrow$ by juxtaposition.

1. $xy = yx$,
2. $xx = yy$,
3. $xyy(yxx) = xy$,
4. $xyyy = xy$,
5. $xyzz = xzz(yzz)$,
6. $xyyzz = xzzyy$,
7. $xyy(yzz)(yzz) = xyy$,
8. $xyy(yz)(yz) = xyyzz$,
9. $x(yz)y = xyzy$,
10. $xyyxzz(xyyx) = xzzxyy(xzzx)$,
11. $xyy(xzz)(xww) = xyy(xzz(xww))$.

For every finite set X of elements of an equivalential algebra we have a very useful unary polynomial $\&X$ defined by putting:

$$a\&\emptyset \stackrel{\text{def}}{=} a, \quad a\&(X \cup \{b\}) \stackrel{\text{def}}{=} (a\&X)bb$$

The identities 2.6 and 2.4 imply that the value of $\&X$ does not depend on permuting or repeating elements of X.

Important features of polynomials $\&X$ are listed below:

1. $a\&X \in \mathbf{1}/\Theta(a,\mathbf{1})$,
2. If $X \subseteq \mathbf{1}/\alpha$ then $\langle a\&X, a\rangle \in \alpha$,
3. $(ab)\&X = (a\&X)(b\&X)$,
4. $(a\&X)\&Y = a\&(X \cup Y)$,
5. $(a\&X)(a\&Y)(a\&Z) = (a\&X)((a\&Y)(a\&Z))$,
6. If $X \supseteq \{a_1,\ldots,a_n\}$ and $b_1,\ldots,b_n$ is a permutation of $a_1,\ldots,a_n$ then $(\bigoplus(a_1,\ldots,a_n))\&X = (\bigoplus(b_1,\ldots,b_n))\&X$.

Note that if X is a finite set of elements of an equivalential algebra and $\{a_1,\ldots,a_n\} \subseteq X$ then we can write $(\bigoplus\{a_1,\ldots,a_n\})\&X$ instead of $(\bigoplus(a_1,\ldots,a_n))\&X$ because order of elements of the sequence is irrelevant (see 2.6).

For the same reason, if $\{X_1,\ldots,X_n\}$ is a finite family of finite sets then we can write $\bigoplus\{a\&X_1,\ldots,a\&X_n\}$ instead of $\bigoplus(a\&X_1,\ldots,a\&X_n)$ (see 2.5).

In general, the operation $\leftrightarrow$ of an equivalential algebra does not need to be associative but every set of the form $\{a\&X \,:\, X \subseteq \mathfrak{A}, |X| < \aleph_0\}$ is an associative subalgebra of the equivalential algebra $\mathfrak{A}$ or, in other words, it is a Boolean group.

It is known (see [19]) that a filter of an equivalential algebra can be characterized as a subset Φ of the underlying set such that:

1. $\mathbf{1} \in \Phi$,
2. if $a, ab \in \Phi$ then $b \in \Phi$,
3. if $a \in \Phi$ then $a\&X \in \Phi$, for every finite subset X of the underlying set.

Equivalential algebras are locally finite, congruence permutable but not congruence distributive (see [16]). The important congruence extension property which is known to hold for Brouwerian semilattices fails to hold for equivalential algebras.

3. Brouwerian paradise

Let $\mathbb{B}$ be the variety of Brouwerian semilattices. It is known that $\mathbb{B}$ is a Fregean variety (see [2]) and moreover, for every subdirectly irreducible $\mathfrak{A} \in \mathbb{B}$, the set of the form $\mathfrak{A} \setminus \{*\}$ is a subuniverse of $\mathfrak{A}$. This fact can be used in a simple proof of local finiteness of $\mathbb{B}$ (see [22]). Thus, if v is a monolith assignment such that $[\![v]\!] = \mathfrak{A} \in \mathbb{B}$ then the algebra $\mathfrak{A}$ - being finitely generated - must be finite and one can define a so-called characteristic term of v putting:

$$\begin{aligned}\chi_v \stackrel{\text{def}}{=} (& \bigwedge((t_a \wedge t_b) \leftrightarrow t_{a \wedge_{\mathfrak{A}} b} : a, b \in \mathfrak{A}) \wedge \\ & \bigwedge((t_a \to t_b) \leftrightarrow t_{a \to_{\mathfrak{A}} b} : a, b \in \mathfrak{A}) \wedge \\ & \bigwedge(x \leftrightarrow t_{v(x)} : x \in \mathsf{Dom}(v)\,)) \to t_\star\end{aligned}$$

where for every $a \in \mathfrak{A}$, t_a is a fixed term such that $v(t_a) = a$ and $\mathsf{Var}(t_a) \subseteq \mathsf{Dom}(v)$.

The following fact is an immediate consequence of the definition:

Fact 6. *Let v, w be monolith assignments with the same domain and with Brouwerian semilattices as target algebras. Then the following conditions are equivalent:*

(i) $w(\chi_v) = \star$,

(ii) *v and w are equivalent (recall Definition 1).*

Let $\mathsf{M}^n_{\mathbb{B}}$ be the set of all monolith assignments having the set of variables $\{x_1, \ldots, x_n\}$ as the domain and Brouwerian semilattices as target algebras. For every term t of the discourse language of Brouwerian semilattices such that $\mathsf{Var}(t) \subseteq \{x_1, \ldots, x_n\}$ by $\mathsf{M}^n_{\mathbb{B}}(t)$ we denote the subset of $\mathsf{M}^n_{\mathbb{B}}$ containing all monolith assignments of t i.e.:

$$\mathsf{M}^n_{\mathbb{B}}(t) \stackrel{\text{def}}{=} \{v \in \mathsf{M}^n_{\mathbb{B}} \ : \ v(t) = \star\}.$$

Then we have the following:

Fact 7. *If v is a monolith assignment, $[\![v]\!] \in \mathbb{B}$, $\mathsf{Var}(t_1, t_2) \subseteq \mathsf{Dom}(v)$ and $\mathbb{B} \models t_1 \wedge t_2 \leq \chi_v$ then $\mathbb{B} \models t_i \leq \chi_v$, for some $i \in \{1, 2\}$.*

And next:

Theorem 8. *[Jankov] If t is a term of the language of Brouwerian semilattices such that $\mathsf{Var}(t) \subseteq \{x_1, \ldots, x_n\}$ then the following conditions are equivalent:*

(i) *$t/\Theta_{\mathbb{B}}$ is $\wedge$-irreducible in $\mathfrak{F}_{\mathbb{B}}(n)$,*

(ii) $|\mathsf{M}^n_{\mathbb{B}}(t)| = 1$,

(iii) *there exists $v \in \mathsf{M}^n_{\mathbb{B}}$ such that $\mathbb{B} \models \chi_v \approx t$.*

Theorem 8. *[Jankov normal form] $\mathbb{B} \models t \approx \bigwedge(\chi_v : v \in \mathsf{M}^n_{\mathbb{B}}(t))$.*

The idea of defining characteristic terms of monolith assignments and using them as building blocks, with a view to recovering all $\Theta_{\mathbb{B}}$ cosets of n-variable Brouwerian terms is due to V. A. Jankov [17] and the above presentation is a paraphrase of his work. The practical value of Jankov's approach seems to be the fact that the set of building blocks one needs for the job is relatively small and can be manufactured in a simple manner just from monolith assignments of $\mathsf{M}^n_{\mathbb{B}}$.

Moreover, by inspecting possible homomorphisms of target algebras one can endow the set $\mathsf{M}^n_{\mathbb{B}}$ with a structure (partial ordering relation) which yields complete information about the free algebra $\mathfrak{F}_{\mathbb{B}}(n)$ together with a practical method of counting the number of its elements (see [22]).

The reason why all this works so nicely is clear. In finite Brouwerian semilattices $\wedge$-irreducibility of elements is equivalent to join-irreducibility of corresponding filters and all filters are principal. Thus, a finite Brouwerian semilattice is in fact isomorphic with the lattice of its own filters, which is isomorphic with the congruence lattice. But a finite distributive lattice can always be recovered from the poset of its own join irreducibles (see [1]).

Virtually no changes are needed to extend the method of Jankov to Brouwerian semilattices with some additional operations that do not harm the local-finiteness, such – for example – as pseudo-complementation (see [18]). It is also possible – due to some tricks of Urquhart [27] – to apply the method to so-called Hilbert algebras i.e. groupoids arising from the purely implicational fragment of the intuitionistic propositional logic (see [11]). Hilbert algebras are not congruence permutable but they are congruence distributive.

In the next section we will show that the method of Jankov can be made usable beyond the Brouwerian paradise.

4. Malcev Purgatory

Let us consider the variety $\mathbb{E}$ of equivalential algebras. Let the sets of assignments $\mathsf{M}^n_{\mathbb{E}}, \mathsf{M}^n_{\mathbb{E}}(t)$, the congruence relation $\Theta_{\mathbb{E}}$ and the free algebra $\mathfrak{F}_{\mathbb{E}}(n)$ be defined analogously to their corresponding $\mathbb{B}$-variants.

To apply the method of Jankov, we shall need an equivalential analog of Jankov's theorem providing a means of recognizing cosets $t/\Theta_{\mathbb{E}}$ whose corresponding filters are join-irreducibles of the lattice $\mathsf{Fil}(\mathfrak{F}_{\mathbb{E}}(n))$ which — as we already know — is isomorphic with $\mathsf{Con}(\mathfrak{F}_{\mathbb{E}}(n))$.

We hope to be able to recognize $t/\Theta_{\mathbb{E}}$ by inspecting its corresponding set of monolith assignments $\mathsf{M}^n_{\mathbb{E}}(t)$ and thus, the following:

Definition 10. A Jankov-set (J-set) is a subset of $\mathsf{M}^n_{\mathbb{E}}$ having the form $\mathsf{M}^n_{\mathbb{E}}(t)$ where $\mathsf{Var}(t) \subseteq \{x_1, \dots, x_n\}$ and $\Theta(t/\Theta_{\mathbb{E}}, \mathbf{1})$ is join-irreducible in $\mathsf{Con}(\mathfrak{F}_{\mathbb{E}}(n))$.

Now, if $\Theta(t/\Theta_{\mathbb{E}}, \mathbf{1})$ is join-irreducible in $\mathsf{Con}(\mathfrak{F}_{\mathbb{E}}(n))$ then the set $\mathsf{M}^n_{\mathbb{E}}(t)$ no longer has to be a singleton, because of the lack of congruence distributivity.

Nevertheless, we shall see that all assignments in such a set must be in a precise sense similar and have certain, easy to recognize, distinctive qualities — in particular — all such assignments have the same target algebra.

The family of all J-sets can be equipped with a structure (this time richer than partial ordering) providing full information about the free algebra $\mathfrak{F}_{\mathbb{E}}(n)$ and a practical method of counting the number of its elements (see [26]).

This can be done by applying the representation theory of Słomczyńska and in particular, the fact that $\Theta(t/\Theta_{\mathbb{E}}, \mathbf{1})$ is join-irreducible in $\mathsf{Con}(\mathfrak{F}_{\mathbb{E}}(n))$ iff $t/\Theta_{\mathbb{E}}$ is irreducible in $\mathfrak{F}_{\mathbb{E}}(n)$ in the sense of Słomczyńska [24], [25].

We introduce a family of subsets of $\mathsf{M}^n_{\mathbb{E}}$ – to be called S-sets – by the following definition:

Definition 11. A set of assignments $V \subseteq \mathsf{M}^n_{\mathbb{E}}$ is an S-set if the following conditions (S1),(S2) and (S3) hold:

(S1) $V \neq \emptyset$ and for every $v, w \in V$, $[\![v]\!] = [\![w]\!]$

(S2) For every $v, w \in V$, for every $i = 1, \ldots, n$, $\langle v(x_i), w(x_i)\rangle \in \Theta(\star, \mathbf{1})$.

The condition (S2) implies that for every $v, w \in V$, $v^{-1}\{\star, \mathbf{1}\} = w^{-1}\{\star, \mathbf{1}\}$. This means that all assignments in V have the same large variables i.e. variables taking the value $\star$ or $\mathbf{1}$. Consequently, all assignments in V have the same set of small variables $S(V)$ containing those among $x_1, \ldots, x_n$ that are not large. The following observation is an important consequence of the condition (S2):

- All assignments of V agree on $S(V)$ i.e. $v|_{S(V)} = w|_{S(V)}$, for every $v, w \in V$.

To express the condition (S3) it will be convenient to define first an important set of variables $I(V)$, whose elements will be called internal variables of V.

We say that $x \in I(V)$ if x is a large variable of V and there exists $v \in V$ such that $v_x \notin V$, where v_x is the unique assignment such that $\{v(x), v_x(x)\} = \{\star, \mathbf{1}\}$ and $v(y) = v_x(y)$ for every $y \in \{x_1, \ldots, x_n\} \setminus \{x\}$.

Now the condition (S3) can be expressed in the following simple manner:

(S3) $I(V) \neq \emptyset$ and for every $w \in \mathsf{M}^n_{\mathbb{E}}$, if $|w^{-1}\{\star\} \cap I(V)|$ is odd and there exists $v \in V$ such that $v|_{S(V)} = w|_{S(V)}$ then $w \in V$.

A useful characterization of S-sets is given by the following:

Fact 12. *For every $V \subseteq \mathsf{M}^n_{\mathbb{E}}$, the following conditions are equivalent:*

(i) *V is an S-set,*

(ii) *There exist pairwise disjoint sets of variables S, I, O such that $S \cup I \cup O = \{x_1, \ldots, x_n\}$, $I \neq \emptyset$ and for some subdirectly irreducible $\mathfrak{A} \in \mathbb{E}$, there exist an assignment v_0 such that $\mathsf{Dom}(v_0) = S$, $[\![v_0]\!] = \mathfrak{A} \setminus \{\star\}$ and for every $w \in \mathsf{M}^n_{\mathbb{E}}$, $w \in V$ iff $w|_S = v_0$, $w^{-1}\{\star, \mathbf{1}\} = I \cup O$ and $|w^{-1}\{\star\} \cap I|$ is odd.*

Now we can state the sought after equivalential analog of Jankov's theorem:

Theorem 13. *If t is a term in the discourse language of $\mathbb{E}$ such that $\mathsf{Var}(t) \subseteq \{x_1, \ldots, x_n\}$ then the following conditions are equivalent:*

(i) *$t/\Theta_{\mathbb{E}}$ is S-irreducible in $\mathfrak{F}_{\mathbb{E}}(n)$,*

(ii) *$\mathsf{M}^n_{\mathbb{E}}(t)$ is an S-set,*

(iii) *$\mathsf{M}^n_{\mathbb{E}}(t)$ is J-set, i.e. $\Theta(t/\Theta_{\mathbb{E}}, \mathbf{1})$ is join-irreducible in $\mathsf{Con}(\mathfrak{F}_{\mathbb{E}}(n))$.*

To be able to recover the free algebra $\mathfrak{F}_{\mathbb{E}}(n)$ we have to endow the family of S-sets with a Słomczyńska-type structure comprising a partial ordering relation, an equivalence relation and a partial binary operation turning each equivalence class augmented with $\emptyset$ into a Boolean group (see [24], [25]).

We already know that all assignments of an S-set share the same set of small variables $S(V)$ and $v|_{S(V)} = w|_{S(V)}$, for every $v, w \in V$. One could say that all assignments of V are outgrowths of the same trunk, by which we mean the unique assignment v_0 such that $v_0 = w|_{S(V)}$, for every $w \in V$. It is important to remember that:

- the case: $v_0 = \emptyset$ is not excluded.

The concept of trunk can be used to define the equivalence relation we need for Słomczyńska-type structure.

Definition 14. Two S-sets $V_1, V_2 \subseteq \mathsf{M}^n_{\mathbb{E}}$, will be declared equivalent iff both have the same trunk.

Clearly, if S-sets V_1 and V_2 have the same trunk and the same set of internal variables then they are equal.

Thus a difference between V_1 and V_2 from the same equivalence class can only be due to different sets of internal variables. This means that members of the equivalence class of an S-set V are in 1-1 correspondence with non-empty subsets of the set of large variables of V i.e. with sets of internal variables of members of the equivalence class of V.

The operation of symmetric difference performed on sets of internal variables of S-sets equivalent to V is just what we need to turn the equivalence class of V augmented with Ø into a Boolean group.

It is clear that all assignments of an S-set V have the same target algebra $[\![V]\!]$ generated by the range of an arbitrary member of V. Moreover, if V_1 and V_2 are equivalent in the sense of Definition 14 then $[\![V_1]\!] = [\![V_2]\!]$.

Now the family $\{V \subseteq \mathsf{M}^n_{\mathbb{E}} : V \text{is an S-set}\}$ will be partially ordered, as required for a Słomczyńska-type structure.

Definition For every S-sets $V_1, V_2 \subseteq \mathsf{M}^n_{\mathbb{E}}$ we put: $V_1 \leq V_2$ iff there exists an epimorphism $h : [\![V_1]\!] \mapsto [\![V_2]\!]$ such that $h(V_1) \subseteq V_2$, where $h(V_1)$ is defined by the obvious stipulation: $(h(v))(x) = h(v(x))$, for every $v \in V_1, x \in \{x_1, \ldots, x_n\}$.

Let $V \subseteq \mathsf{M}^n_{\mathbb{E}}$ be an S-set and let $[\![V]\!] = \mathfrak{A}$. We will now proceed to define a characteristic term χ_V of V.

Let pairwise disjoint sets of variables S, I, O be such that $S = S(V)$, $I = I(V)$, $O = \{x_1, \ldots, x_n\} \setminus (S \cup I)$ and let an assignment $v_0 : S \mapsto \mathfrak{A} \setminus \{\star, \mathbf{1}\}$ be the trunk of V.

Since $[\![v_0]\!] = \mathfrak{A} \setminus \{\star\}$ then for every $a \in \mathfrak{A} \setminus \{\star\}$ we can pick a term t_a such that $v_0(t_a) = a$ and $\mathsf{Var}(t_a) \subseteq S$.

We shall also need a special term $t_\star$ such that $\mathsf{Var}(t_\star) = I$ and for every $v \in V, v(t_\star) = \star$. Such a term we get by putting:

$$t_\star \stackrel{\text{def}}{=} (\bigoplus I)\&I$$

Define now two auxiliary sets of terms Δ_0 and Δ_1 putting:

$$\Delta_0 \stackrel{\text{def}}{=} \{(t_a \leftrightarrow t_b) \leftrightarrow t_{a \leftrightarrow_{\mathfrak{A}} b} : a, b \in \mathfrak{A} \setminus \{\star, \mathbf{1}\}, a \neq b\}$$

$$\Delta_1 \stackrel{\text{def}}{=} \{x \leftrightarrow t_{v0(x)} \ : \ x \in S\}$$

Finally, we are ready to state the definition of the characteristic term of the S-set V.

Definition 16.

$$\chi_V \stackrel{\text{def}}{=} (\bigoplus \{t_\star \& X \ : \ X \subseteq S\})\&(\Delta_0 \cup \Delta_1 \cup O)$$

Note the following:

Fact 17. *If $V \subseteq \mathsf{M}^n_{\mathbb{E}}$ is an S-set then $v \in V$ iff $v(\chi_V) = \star$, for every assignment $v \in \mathsf{M}^n_{\mathbb{E}}$.*

It is easy to see that characteristic terms can be used to capture the partial ordering of S-sets introduced by the Definition 15. Indeed, for every S-sets $V_1, V_2 \subseteq \mathsf{M}^n_{\mathbb{E}}$:

$$V_1 < V_2 \text{ iff } \mathbb{E} \models \chi_{V_1} \leftrightarrow \chi_{V_2} \approx \chi_{V_2}.$$

Thus, $V_1 < V_2$ iff for every $v \in V_1$, $v(\chi_{V_2})$ is a small element of $[\![V_1]\!]$.

The above ordering is used to determine maximal S-sets mentioned in the following:

Theorem 18. *[normal form] For every term t of the discourse language of equivalential algebras such that $\mathsf{Var}(t) \subseteq \{x_1, \ldots, x_n\}$:*

$$\mathbb{E} \models t \approx \bigoplus \{\chi_V : \ V \textit{ is a maximal } \text{S-}\textit{set such that } V \subseteq \mathsf{M}^n_{\mathbb{E}}(t)\}$$

Let us consider now a permutable and locally finite variety $\mathbb{K}$ obeying the conditions of Fregeanity (R) and ($\leq$). We should not expect miracles because the only structure we can be sure to find in an algebra $\mathfrak{A} \in \mathbb{K}$ is that of an equivalential algebra, provided by a binary term $\mathbf{e}$ known to exist by Theorem 4. Taking advantage of this situation we shall use the notation $\langle \mathfrak{A}, \mathbf{e} \rangle$ for the $\mathbf{e}$-reduct of $\mathfrak{A} \in \mathbb{K}$.

First thing to realize is the fact that, in general, lattices: $\mathsf{Fil}(\mathfrak{F}_{\mathbb{K}}(n))$ and $\mathsf{Fil}(\langle \mathfrak{F}_{\mathbb{K}}(n), \mathbf{e} \rangle)$ are different and it happens that a filter of a coset $t/\Theta_{\mathbb{K}}$ is join-reducible in $\mathsf{Fil}(\mathfrak{F}_{\mathbb{K}}(n))$ but join-irreducible in $\mathsf{Fil}(\langle \mathfrak{F}_{\mathbb{K}}(n), \mathbf{e} \rangle)$.

Nevertheless, Słomczyńska's representation theory (see [25], [24]) works for every finite equivalential algebra and thus we can just single out all cosets $t/\Theta_{\mathbb{K}}$ whose corresponding filters are join-irreducible in the lattice $\mathsf{Fil}(\langle \mathfrak{F}_{\mathbb{K}}(n), \mathbf{e} \rangle)$ and successfully recover the whole of $\mathfrak{F}_{\mathbb{K}}(n)$ using them as building blocks just as we used terms χ_V in Theorem 18.

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Department of Logic, Jagiellonian University
Grodzka 52, PL-31-044 Kraków, Poland

uzwronsk@cyf-kr.edu.pl