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IMPLICATION SYSTEMS FOR MANY-DIMENSIONAL LOGICS

A b s t r a c t. The main result of the present paper is equivalence of the following conditions, for any k-dimensional logic $\mathcal{L}$:

(i) $\mathcal{L}$ has a *full-replacement implication system*, i.e., a finite set of k-dimensional formulas with $2k$ variables that in a natural way adopts the Identity axiom and the *Modus Ponens* rule for the ordinary implication connective;

(ii) $\mathcal{L}$ has an *unary-replacement implication system*, i.e., a finite set of k-dimensional formulas with $k+1$ variables that in a different way adopts the Identity axiom and the *Modus Ponens* rule for the ordinary implication connective;

(iii) $\mathcal{L}$ has a *parameterized local deduction theorem*;

(iv) $\mathcal{L}$ has the *syntactic correspondence property* that is essentially the restriction of the filter correspondence property to deductive $\mathcal{L}$-filters over the formula algebra alone;

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(v) $\mathcal{L}$ is *protoalgebraic* in the sense that the Leibniz operator is monotonic on the set of deductive $\mathcal{L}$-filters over every algebra;

(vi) $\mathcal{L}$ has a *system of equivalence formulas with parameters* that defines the Leibniz operator on deductive $\mathcal{L}$-filters over every algebra.

We also present a family of specific examples which collectively show that the above metaequivalence doesn't remain true when in (i) "$2k$" (resp., in (ii) "$k+1$") is replaced by "$2k-1$" (resp., by "k"). This, in particular, disproves the statement of [4], Theorem 13.2.

1. Introduction

Many-dimensional propositional systems are introduced in [3] and examined in [4] to provide a common approach to both ordinary propositional systems (viz. *sentential logics* or *standard systems* in the terminology of [13]) and *quasivarieties* in Mal'cev's sense [11]. Formally speaking, given any $k > 0$, a *k-dimensional propositional logic* (*k-logic* for short) is a finitary and structural consequence operation on the set of all *k-dimensional propositional formulas* (*k-formulas* for short), that is, k-tuples constituted by ordinary propositional formulas. In this way the standard systems in Wojcicki's sense become exactly the 1-dimensional propositional logics. As for Mal'cev's quasivarieties, their equational consequence operations (cf. [2]) are exactly those 2-dimensional logics which satisfy the rules for equality. [1] Besides these two basic kinds of many-dimensional logics there are also some interesting systems that can equally be covered by the formalism under consideration. These are mainly (but certainly not exclusively) strict universal Horn theories without equality of binary relations like those of partial orderings, of quasi-orderings, etc. (see [4] for more detail).

Appearance of this general conception of deductive system raises a number of metalogical issues concerning, in particular, those which have

[1] Equations are treated as pairs of propositional formulas.

been studied within the context of the ordinary (i.e., 1-dimensional) propositional logics. Being a natural expansion of the concept of standard system, the one of many-dimensional logic can certainly be studied along the line of Sentential Logic Theory [13] without any significant problem. A part of such a straightforward elaboration is presented in [4]. This concerns mainly the concept of logical matrix, the problems of completeness, axiomatizability, etc. In this connection, there have been found no surprising behaviour of many-dimensional logics in comparison with 1-dimensional ones. However there are still certain points relevant to standard systems (though not completely involved in [13]) whose extension to many-dimensional logics has proved far from being unambiguous. One of them concerns the problem of existence of what we call here an *implication system*, that is, a finite set of many-dimensional formulas which satisfies, with respect to a given many-dimensional logic, proper analogs of the *Modus Ponens* rule and the Identity axiom for the ordinary implication connective. The importance of this issue within the context of sentential logics has been realized mainly due to the works [4] and [8], though its roots go back to the papers [6], [7] where the condition of existence of an implication system appears as a prerequisite in the formulations of theorems that provide characterizations of standard systems with a (uniform or local, respectively) deduction theorem through behaviour of the lattices of their deductive filters. [2] In [8] it has been shown that any standard system has an implication system iff it has a so called *parameterized local deduction theorem*. A quite different insight into the conception of implication system for sentential logics can be found in [Section 13][4] where it has been proved that any standard system has an implication system iff it is *protoalgebraic* in the sense of [1] (cf. [4], Theorem 13.2, for the case $k = 1$). Protoalgebraicity of a given sentential logic is also proved to be equivalent to existence of a *system of equivalence formulas with parameters* that defines the *Leibniz operator* (in the sense of [2]) on deductive filters of the logic (cf. [4], Theorem 13.10). In addition, the property of being protoalgebraic has far-reaching consequences.

[2] In [7] standard systems with an implication system are even referred to as *non-pathological*.

In particular, many important results that have been obtained for quasivarieties (this concerns mainly the problem of finite axiomatizability, various forms of deduction theorem and related issues) can easily be extended to protoalgebraic logics. We regret of not having the space here to describe the great amount of work done in this connection in detail and to present the full list of references relevant to the issue. We believe the paper [4] as well as its bibliography can provide the reader with, at least, a preliminary comprehension of the issue involved.

It appears that, being true for the case $k = 1$, the statement of Theorem 13.2 in [4] is erroneous when $k > 1$, as we argue in Section 5. [3] On the other hand, the extrapolation of the concept of protoalgebraicity studied in [4] seems to be perfectly right. This implies that the extrapolation of the concept of implication system from 1- to k-dimensional logics with $k > 1$ chosen in Definition 13.1 in [4] is incorrect.[4] It is worth mentioning that Blok and Pigozzi suggested such schema of *Modus Ponens* for many-dimensional logics that replaces just a single propositional formula in a many-dimensional premise and leaves the rest unchanged. We suggest here another approach to a "many-dimensional *Modus Ponens*" that consists in dealing with the following schema

$$\vec{\varphi}, \Delta(\vec{\varphi}, \vec{\psi}) \vdash \vec{\psi} \tag{1}$$

where $\vec{\varphi}$ and $\vec{\psi}$ are many-dimensional formulas and $\Delta(\vec{x}, \vec{y})$ is a finite set of many-dimensional formulas with variables $\vec{x}, \vec{y}$. In this way, the identity law is expressed as follows

$$\vdash \Delta(\vec{\varphi}, \vec{\varphi}). \tag{2}$$

Any set Δ of such a kind satisfying the conditions (1) and (2) with respect to the consequence $\vdash$ of a given many-dimensional logic is called a *full-replacement implication system* for the logic (cf. Definition 3.1.). We prove

[3] As a consequence, the proofs and, in some cases, even statements presented further in [4], Section 13, become incorrect as well, because they rely upon Theorem 13.2 of [4].

[4] To be more precise, Blok and Pigozzi use the term *system of equivalence formulas* for their extension of the concept involved to many-dimensional logics.

that a many-dimensional logic has a full-replacement implication system iff it has a parameterized local deduction theorem iff it is protoalgebraic iff it has a system of equivalence formulas with parameters (cf. Theorem 4.1.). In this way, the main results on the issue that have been proved for 1-dimensional logics become generalized to k-dimensional ones. As a by-product, we prove that the statement (but not the proof!) of Theorem 13.10 of [4] is still correct. This implies that the statements of further results of Section 13 in [4] are correct as well, for their proofs are based upon Theorem 13.10 of [4] alone in the sense that they do not involve Theorem 13.2 of [4] directly. Furthermore, the issue of parameterized local deduction theorem has been examined for quasivarieties in [9]. Actually, our main result gives new proofs to some results in Universal Algebra (for more detail, see the comments after Theorem 4.1.).

In order to trace more close connections with Blok and Pigozzi's conception of implication system and to show how the statement of Theorem 13.2 in [4] can be corrected "gently", we consider also a proper "unary-replacement *Modus Ponens* schema" and a corresponding notion of an *unary-replacement implication system* (cf. Definition 3.2.). Unlike Blok and Pigozzi's systems of equivalence formulas, our unary-replacement implication systems contain, generally speaking, more than two variables. We prove that full-replacement and unary-replacement implication systems are definable in terms of one another (cf. Proposition 3.3.).

The rest of the paper is as follows. Section 2 provides a brief summary of basic concepts concerning the general topic of many-dimensional logics. In Section 3 we specify and examine key notions of the present paper. Next, in Section 4 we prove the main result of our paper (see Theorem 4.1.). Finally, in Section 5 we consider a family of counterexamples which, in particular, disprove the statement of Theorem 13.2, [4].

2. Preliminaries

Throughout the paper, we fix a countable set Var of *(propositional) letters* or *variables*. Small Italic letters p, q, u, v, w, x, y and z (possibly with indices) are used as metavariables ranging over the set Var.

A *(propositional) language* is any set L of *(propositional) connectives* of arbitrary finite arity. These are treated naturally as operation symbols. Then, the L-terms with variables in the set Var are referred to as *(propositional) formulas over L*. The set of all formulas over L will be denoted by Fm_L.

The notion of an L-algebra and related concepts are standard. In order to unify notations, algebras will be denoted by Calligraphic letters, their carriers being denoted by the corresponding capital Italic letters.

The absolutely free L-algebra freely generated by the set Var (notice that its carrier coincides with the set Fm_L) is called the *formula L-algebra* and is denoted by $\mathcal{F}m_L$. The endomorphisms of $\mathcal{F}m_L$ are called *substitutions over L*. The set of all substitutions over L is denoted by Sb_L. Given a sequence of distinct variables $\overline{x}$ and a sequence of formulas $\overline{\varphi}$ over L of the same length, by $[\overline{\varphi}/\overline{x}]$ we denote the substitution over L defined by $[\overline{\varphi}/\overline{x}]\overline{x} := \overline{\varphi}$ and $[\overline{\varphi}/\overline{x}]v := v$, for all $v \in \mathrm{Var} \setminus \{\overline{x}\}$.

A *dimension* is any number $k > 0$. We will follow the general rule, according to which a k-tuple $\langle a_1, \ldots, a_k \rangle$ is briefly written as $\vec{a}$. Likewise, $f\vec{a}$ is used as an abbreviation for the k-tuple $\langle fa_1, \ldots, fa_k \rangle$.

A *k-dimensional (propositional) formula*, or briefly *k-formula*, *over L* is any k-tuple constituted by propositional formulas over L. The set of all k-formulas over L is denoted by Fm_L^k. Given a set Γ consisting of formulas and many-dimensional formulas, $\mathrm{Var}(\Gamma)$ denotes the set of all variables occurring in Γ. As usual, we will also write $\Gamma(\overline{x})$ to express the fact that $\mathrm{Var}(\Gamma) \subseteq \{\overline{x}\}$.

Let A be a non-empty set. By $\wp(A)$ we denote the set of all subsets of A. We also write $X \subseteq_\omega A$ for "X is a finite subset of A".

A *closure operator on A* is any unary operation C on $\wp(A)$ such that, for all $X \subseteq Y \subseteq A$, it holds that $X \cup C(C(X)) \subseteq C(X) \subseteq C(Y)$. A closure operator C on A is said to be *finitary* provided, for all $X \cup \{a\} \subseteq A$, from $a \in C(X)$ it follows that $a \in C(Y)$ for some $Y \subseteq_\omega X$.

Let L be a propositional language and k a dimension. A closure operator C on Fm_L^k is said to be *structural* provided, for all $\Gamma \subseteq \mathrm{Fm}_L^k$ and all $\sigma \in \mathrm{Sb}_L$, it holds that $\sigma C(\Gamma) \subseteq C(\sigma\Gamma)$. A *k-dimensional (propositional) logic*, or simply *k-logic*, *over* L is any triple of the form $\mathcal{L} = \langle L, k, \mathrm{Cn}_{\mathcal{L}}\rangle$ where $\mathrm{Cn}_{\mathcal{L}}$ is a finitary and structural closure operator on Fm_L^k, called the *consequence (operation) of* $\mathcal{L}$. In order to make notations more natural, we will normally write $\Gamma \vdash_{\mathcal{L}} \Delta$ for $\Delta \subseteq \mathrm{Cn}_{\mathcal{L}}\Gamma$. Related abbreviations like $\vec{\varphi}, \Gamma \vdash_{\mathcal{L}} \vec{\psi}$ for $\{\vec{\varphi}\} \cup \Gamma \vdash_{\mathcal{L}} \{\vec{\psi}\}$ are then supposed to be clear as well.

A *k-dimensional (propositional) rule*, or briefly *k-rule*, *over* L is any couple $\langle \Gamma, \vec{\varphi}\rangle$, normally written as $\Gamma \vdash \vec{\varphi}$, constituted by a finite set Γ of k-formulas over L and by a single k-formula $\vec{\varphi}$ over L. A k-rule over L of the form $\varnothing \vdash \vec{\varphi}$ is called a *k-axiom over* L, and is identified with the k-formula $\vec{\varphi}$. An *L-instance* of a k-rule $\Gamma \vdash \vec{\varphi}$ over L is any k-rule over L of the form $\sigma\Gamma \vdash \sigma\vec{\varphi}$ where $\sigma \in \mathrm{Sb}_L$.

A *k-calculus over* L is a set of k-rules over L. Any k-calculus $\mathcal{D}$ over L defines a k-logic $\mathcal{L}$ over L in the standard proof-theoretical manner as follows. For all $\Gamma \cup \{\vec{\varphi}\} \in \mathrm{Fm}_L^k$, put $\Gamma \vdash_{\mathcal{L}} \vec{\varphi}$ iff $\vec{\varphi}$ is *derivable from* Γ *by means of rules in* $\mathcal{D}$ in the sense that there exists a non-empty sequence $\vec{\psi}_1, \ldots, \vec{\psi}_n \in \mathrm{Fm}_L^k$, called a *$\mathcal{D}$-derivation of* $\vec{\varphi}$ *from* Γ, such that $\vec{\varphi} = \vec{\psi}_n$ and, for all $1 \leq i \leq n$, either $\vec{\psi}_i \in \Gamma$ or there exist $k \geq 0$ and $j_1, \ldots, j_k < i$ such that $\vec{\psi}_{j_1}, \ldots, \vec{\psi}_{j_k} \vdash \vec{\varphi}_i$ is an L-instance of a rule in $\mathcal{D}$. The so-defined logic $\mathcal{L}$ is said to be *axiomatized by* $\mathcal{D}$.

Let $\mathcal{L}$ be a k-logic over L and $\mathcal{A}$ an L-algebra. A set $F \subseteq A^k$ is called a *(deductive) $\mathcal{L}$-filter over* $\mathcal{A}$ provided, for all $\Gamma \cup \{\vec{\varphi}\} \subseteq \mathrm{Fm}_L^k$ and all $h \in \mathrm{Hom}(\mathcal{F}m_L, \mathcal{A})$,

$$\Gamma \vdash_{\mathcal{L}} \vec{\varphi} \;\Rightarrow\; (h\Gamma \subseteq F \Rightarrow h\vec{\varphi} \in F).$$

The set of all $\mathcal{L}$-filters over L is denoted by $\mathrm{Fi}_{\mathcal{L}}\mathcal{A}$.

3. Basic notions and results

Throughout this and the next sections we fix a dimension k, a propositional language L and a k-logic $\mathcal{L}$ over L.

Definition 3.1. Let $\vec{x}, \vec{y}$ be $2k$ distinct variables. A finite set $\Delta(\vec{x}, \vec{y})$ of k-formulas over L is called a *full-replacement implication system for* $\mathcal{L}$ provided the following conditions are satisfied:

$$\vec{x}, \Delta(\vec{x}, \vec{y}) \vdash_{\mathcal{L}} \vec{y} \tag{3}$$

$$\vdash_{\mathcal{L}} \Delta(\vec{x}, \vec{x}) \tag{4}$$

Below (see Lemma 4.3.) we show how full-replacement implication systems arise naturally in logics having a deduction theorem of a general form (cf. Definition 3.4.).

Definition 3.2. Let $x, y, z_1, \ldots, z_{k-1}$ be $k+1$ distinct variables. A finite set of k-formulas $\Delta(x, y, z_1, \ldots, z_{k-1})$ over L is called an *unary-replacement implication system for* $\mathcal{L}$ provided the following conditions are satisfied:

$$\begin{aligned} &\langle z_1, \ldots, z_{i-1}, x, z_i, \ldots, z_{k-1}\rangle, \\ &\qquad \Delta(x, y, z_1, \ldots, z_{k-1}) \vdash_{\mathcal{L}} \langle z_1, \ldots, z_{i-1}, y, z_i, \ldots, z_{k-1}\rangle, \\ &\qquad\qquad \text{for all } 1 \leq i \leq k, \end{aligned} \tag{5}$$

$$\vdash_{\mathcal{L}} \Delta(x, x, z_1, \ldots, z_{k-1}) \tag{6}$$

As opposed to full-replacement implication systems, unary-replacement ones arise naturally in *equivalential logics* (cf. Definition 3.7. and Proposition 3.8.). However, in view of the following statement (whose proof is straightforward and, for this reason, is omitted), both kinds of implication system are essentially equivalent.

Proposition 3.3.

(i) If $\Delta(\vec{x}, \vec{y})$ is a full-replacement implication system for $\mathcal{L}$ then

$$\bigcup_{1 \leq i \leq k} \Delta(z_1, \ldots, z_{i-1}, x, z_i, \ldots, z_{k-1}, z_1, \ldots, z_{i-1}, y, z_i, \ldots, z_{k-1})$$

is an unary-replacement implication system for $\mathcal{L}$.

(ii) If $\Delta(x, y, z_1, \ldots, z_{k-1})$ is an unary-replacement implication system for $\mathcal{L}$ then

$$\bigcup_{1 \leq i \leq k} \Delta(x_i, y_i, y_1, \ldots, y_{i-1}, x_{i+1}, \ldots, x_k)$$

is a full-replacement implication system for $\mathcal{L}$. In particular, $\mathcal{L}$ has a full-replacement implication system iff it has an unary-replacement one.

The concept defined below has been introduced and examined in [8] for the case $k = 1$. In case $k = 2$ and $\mathcal{L}$ is the 2-logic defined by a quasivariety, we also get exactly the concept of parameterized local deduction theorem for quasivarieties specified in [9].

Definition 3.4. Let $\vec{x}, \vec{y}$ be $2k$ distinct variables. Then $\mathcal{L}$ is said to *have the parameterized local $(\vec{x}, \vec{y})$-deduction theorem with respect to* a system Φ of finite sets of k-formulas over L provided, for all $\Gamma \cup \{\vec{\varphi}, \vec{\psi}\} \subseteq \mathrm{Fm}_L^k$, it holds that

$$\Gamma, \vec{\varphi} \vdash_{\mathcal{L}} \vec{\psi} \;\Leftrightarrow\; \Gamma \vdash_{\mathcal{L}} \Xi(\vec{\varphi}, \vec{\psi}, \overline{\xi}) \text{ for some } \Xi(\vec{x}, \vec{y}, \overline{z}) \in \Phi \text{ and some } \overline{\xi} \in \mathrm{Fm}_L.$$

If, in addition, $\mathrm{Var}(\Xi) \subseteq \{\vec{x}, \vec{y}\}$, for all $\Xi \in \Phi$, then $\mathcal{L}$ is said to *have the local deduction theorem with respect to* Φ (cf. [7], [3]). If, moreover, $\Phi = \{\Xi\}$ is a singleton, then $\mathcal{L}$ is said to *have the uniform deduction theorem with respect to* Ξ (cf. [6], [3]).

What is defined below is essentially the restriction of the *filter correspondence property* (cf. [Definition 7.5][4]) to deductive filters over the formula algebra alone.

Definition 3.5. $\mathcal{L}$ is said to *have the syntactic correspondence property* provided, for every surjective $\sigma \in \mathrm{Sb}_L$ and all $\Gamma \cup \{\vec{\varphi}\} \subseteq \mathrm{Fm}_L^k$, it holds that

$$\sigma^{-1}\mathrm{Cn}_{\mathcal{L}}(\sigma\vec{\varphi}, \Gamma) = \mathrm{Cn}_{\mathcal{L}}(\vec{\varphi}, \sigma^{-1}\mathrm{Cn}_{\mathcal{L}}\Gamma).$$

Finally, we recall several concepts that are mainly due to Blok and Pigozzi [4].

Let $\mathcal{A}$ be an L-algebra and $F \subseteq A^k$. A congruence θ of $\mathcal{A}$ is said to be *compatible with* F provided, for all $\vec{a}, \vec{b} \in A^k$ such that $\langle a_i, b_i \rangle \in \theta$ for each $1 \leq i \leq k$, it holds that $\vec{a} \in F \Leftrightarrow \vec{b} \in F$. The binary relation $\Omega^{\mathcal{A}} F$ on A defined by

$$\langle a, b\rangle \in \Omega^{\mathcal{A}} F \overset{\text{def}}{\Longleftrightarrow} \vec{\psi}^{\mathcal{A}}(a, \overline{c}) \in F \;\Leftrightarrow\; \vec{\psi}^{\mathcal{A}}(b, \overline{c}) \in F,$$
$$\text{for all } \vec{\psi}(p, \overline{q}) \in \mathrm{Fm}_L^k \text{ and all } \overline{c} \in A, \qquad (7)$$

for all $a, b \in A$, is called the *Leibniz* (or *indiscernability*) *relation of* F *over* $\mathcal{A}$. This is the greatest congruence of $\mathcal{A}$ compatible with F (cf. [Section 5][4]). The function $\Omega^{\mathcal{A}}$ with domain $\wp(A)$ is called the *Leibniz operator over* $\mathcal{A}$.

Definition 3.6. $\mathcal{L}$ is said to be *protoalgebraic* provided, for each L-algebra $\mathcal{A}$, the Leibniz operator $\Omega^{\mathcal{A}}$ over $\mathcal{A}$ is *monotonic on* $\mathrm{Fi}_{\mathcal{L}}\mathcal{A}$ in the sense that

$$F \subseteq G \Rightarrow \Omega^{\mathcal{A}} F \subseteq \Omega^{\mathcal{A}} G,$$

for all $F, G \in \mathrm{Fi}_{\mathcal{L}}\mathcal{A}$.

Definition 3.7. Let x and y be two distinct variables. A set $\Theta \subseteq \mathrm{Fm}_L^k$ is called a *system of* (x, y)*-equivalence formulas*, or *congruence formulas* according to [4], *with parameters for* $\mathcal{L}$ provided, for each L-algebra $\mathcal{A}$, each $F \in \mathrm{Fi}_{\mathcal{L}}\mathcal{A}$ and all $a, b \in A$, it holds that

$$\langle a, b\rangle \in \Omega^{\mathcal{A}} F \Leftrightarrow \vec{\varphi}^{\mathcal{A}}(a, b, \overline{c}) \in F,$$
$$\text{for all } \vec{\varphi}(x, y, \overline{z}) \in \Theta \text{ and all } \overline{c} \in A. \qquad (8)$$

If, in addition, $\Theta = \Theta(x, y)$ then Θ is called a *system of equivalence formulas*, or *congruence formulas* according to [4], *without parameters for* $\mathcal{L}$. A logic is said to be (resp., *finitely*) *equivalential*, or *weakly congruential* (resp., *congruential*) according to [4], whenever it has a (resp., finite) system of equivalence formulas without parameters.

Due to [5] it is well known that a 1-logic is (resp., finitely) equivalential in the above sense iff it is (resp., finitely) equivalential in the original sense of Prucnal and Wroński [12]. [5]

Proposition 3.8. *Assume $\mathcal{L}$ is (resp., finitely) equivalential and $\Theta(x, y)$ is a (resp., finite) system of equivalence formulas without parameters for $\mathcal{L}$. Then some set $\Delta \subseteq_\omega \Theta$ (resp., the set Θ itself) is an unary-replacement implication system for $\mathcal{L}$.*

Proof. Observe that Θ satisfies the conditions (5) and (6) *mutatis mutandis*. In case Θ is infinite, by finitariness of $\mathrm{Cn}_\mathcal{L}$ there is some $\Delta \subseteq_\omega \Theta$ which satisfies the mentioned conditions as well. ■

In case $k = 2$ and $\mathcal{L}$ is the 2-logic defined by a quasivariety, the pair $\langle x, y\rangle$ alone is well-known to be a single equivalence formula without parameters for $\mathcal{L}$. Hence $\{\langle x, y\rangle\}$ is an implication system for such a logic. In that case a logic always possesses an implication system *with at most two variables*. However this is not, generally speaking, the case for any dimension $k > 1$, as it is shown in Section 5. [6]

4. Main issues

Now we are in a position to prove the main result of the present paper.

Theorem 4.1. *The following are equivalent:*

(i) $\mathcal{L}$ has a full-replacement implication system;

(ii) $\mathcal{L}$ has an unary-replacement implication system;

(iii) $\mathcal{L}$ has a parameterized local deduction theorem;

(iv) $\mathcal{L}$ has the syntactic correspondence property;

(v) $\mathcal{L}$ is protoalgebraic;

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[5] This is why we have preferred here the standard terminology that is different from the one adopted in [4].

[6] This is why the statement of [Theorem 13.2][4] is erroneous.

(vi) $\mathcal{L}$ has a system of equivalence formulas with parameters.

Proof. The metaequivalence (i) $\Leftrightarrow$ (ii) is by Proposition 3.3.. The metaimplication (vi) $\Rightarrow$ (v) is obvious. The metaimplication (v) $\Rightarrow$ (iv) is a particular case of [Theorem 7.6][4]. The metaimplications (i) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (i) are, respectively, by the following two lemmas, which are also interesting in their own right.

Lemma 4.2. *Let $\vec{x}, \vec{y}$ be $2k$ distinct variables. Assume $\Delta(\vec{x}, \vec{y})$ is a full-replacement implication system for $\mathcal{L}$. Then, for any distinct $u, v \in \mathrm{Var}$, the set*

$$\Theta := \{\vec{\varphi}(\vec{\psi}[u/w], \vec{\psi}[v/w]) : \vec{\varphi} \in \Delta, \vec{\psi} \in \mathrm{Fm}_L^k, w \in \mathrm{Var}, u, v \notin \mathrm{Var}(\vec{\psi})\}$$

is a system of (u, v)-equivalence formulas with parameters for $\mathcal{L}$.

Proof. Take any L-algebra $\mathcal{A}$, any $F \in \mathrm{Fi}_{\mathcal{L}}\mathcal{A}$ and arbitrary $a, b \in A$.

First, assume $\langle a, b\rangle \in \Omega^{\mathcal{A}}F$. Then, for every $\vec{\varphi}(\vec{x}, \vec{y}) \in \Delta$, every $\vec{\psi}(w, \overline{z}) \in \mathrm{Fm}_L^k$ and all $\overline{c} \in A$, we have

$$\vec{\varphi}^{\mathcal{A}}(\vec{\psi}^{\mathcal{A}}(a, \overline{c}), \vec{\psi}^{\mathcal{A}}(b, \overline{c})) \in F \;\Leftrightarrow\; \vec{\varphi}^{\mathcal{A}}(\vec{\psi}^{\mathcal{A}}(a, \overline{c}), \vec{\psi}^{\mathcal{A}}(a, \overline{c})) \in F.$$

By (6) we get

$$\vec{\varphi}^{\mathcal{A}}(\vec{\psi}^{\mathcal{A}}(a, \overline{c}), \vec{\psi}^{\mathcal{A}}(b, \overline{c})) \in F. \tag{9}$$

Thus the right part of (8) holds.

Conversely, assume the right part of (8) holds. Then (9) holds for every $\vec{\varphi}(\vec{x}, \vec{y}) \in \Delta$, every $\vec{\psi}(w, \overline{z}) \in \mathrm{Fm}_L^k$ and all $\overline{c} \in A$. Take any $\vec{\psi}(p, \overline{q}) \in \mathrm{Fm}_L^k$ and arbitrary $\overline{c} \in A$. By (9) and (3) we get

$$\vec{\psi}^{\mathcal{A}}(a, \overline{c}) \in F \;\Rightarrow\; \vec{\psi}^{\mathcal{A}}(b, \overline{c}) \in F.$$

For proving the converse, observe that, for all $\vec{\phi}(\vec{x}, \vec{y}) \in \Delta$, by (9) we have

$$\vec{\phi}^{\mathcal{A}}(\vec{\varphi}^{\mathcal{A}}(\vec{\psi}^{\mathcal{A}}(a, \overline{c}), \vec{\psi}^{\mathcal{A}}(a, \overline{c})), \vec{\varphi}^{\mathcal{A}}(\vec{\psi}^{\mathcal{A}}(b, \overline{c}), \vec{\psi}^{\mathcal{A}}(a, \overline{c}))) \in F.$$

By (4) and (3) we then get

$$\vec{\varphi}^{\mathcal{A}}(\vec{\psi}^{\mathcal{A}}(b, \overline{c}), \vec{\psi}^{\mathcal{A}}(a, \overline{c})) \in F.$$

Applying (3) once more, we eventually conclude that

$$\vec{\psi}^{\mathcal{A}}(b, \overline{c}) \in F \;\Rightarrow\; \vec{\psi}^{\mathcal{A}}(a, \overline{c}) \in F.$$

Thus $\langle a, b\rangle \in \Omega^{\mathcal{A}}F$. ∎

Lemma 4.3. *Let $\vec{x}, \vec{y}$ be $2k$ distinct variables. Assume $\mathcal{L}$ has the parameterized local $(\vec{x}, \vec{y})$-deduction theorem with respect to a system Φ of finite sets of k-formulas over L. Then there are some $\Gamma(\vec{x}, \vec{y}, \overline{z}) \in \Phi$ and some $\overline{\eta}(\vec{x}, \vec{y}) \in \mathrm{Fm}_L$ such that $\Gamma(\vec{x}, \vec{y}, \overline{\eta})$ is a full-replacement implication system for $\mathcal{L}$.*

Proof. As $\vec{x} \vdash_{\mathcal{L}} \vec{x}$, there are some $\Gamma(\vec{x}, \vec{y}, \overline{z}) \in \Phi$ and some $\overline{\xi} \in \mathrm{Fm}_L$ such that $\vdash_{\mathcal{L}} \Gamma(\vec{x}, \vec{x}, \overline{\xi})$. On the other hand, $\Gamma(\vec{x}, \vec{y}, \overline{\xi}) \vdash_{\mathcal{L}} \Gamma(\vec{x}, \vec{y}, \overline{\xi})$, so $\vec{x}, \Gamma(\vec{x}, \vec{y}, \overline{\xi}) \vdash_{\mathcal{L}} \vec{y}$. Let σ be the substitution over L defined by $\sigma\vec{x} := \vec{x}$, $\sigma\vec{y} := \vec{y}$ and $\sigma v := x_1$ for all $v \in \mathrm{Var} \setminus \{\vec{x}, \vec{y}\}$. Put $\overline{\eta}(\vec{x}, \vec{y}) := \sigma\overline{\xi}$. By structurality of $\mathrm{Cn}_{\mathcal{L}}$ we then get $\vec{x}, \Gamma(\vec{x}, \vec{y}, \overline{\eta}) \vdash_{\mathcal{L}} \vec{y}$ and $\vdash_{\mathcal{L}} \Gamma(\vec{x}, \vec{x}, \overline{\eta}(\vec{x}, \vec{x}))$. ∎

Finally, we prove (iv) ⇒ (iii). Suppose $\mathcal{L}$ has the syntactic correspondence property. Put

$$\Phi := \{\Xi \subseteq_\omega \mathrm{Fm}_L^k : \vec{x}, \Xi \vdash_{\mathcal{L}} \vec{y}\}.$$

We claim that $\mathcal{L}$ has the parameterized local $(\vec{x}, \vec{y})$-deduction theorem with respect to Φ. For take any $\Gamma \cup \{\vec{\varphi}, \vec{\psi}\} \subseteq \mathrm{Fm}_L^k$. Assume $\vec{\varphi}, \Gamma \vdash_{\mathcal{L}} \vec{\psi}$. Consider any surjective $\sigma \in \mathrm{Sb}_L$ such that $\sigma\vec{x} = \vec{\varphi}$ and $\sigma\vec{y} = \vec{\psi}$. Then, by the syntactic correspondence property, we have $\vec{x}, \sigma^{-1}\mathrm{Cn}_{\mathcal{L}}\Gamma \vdash_{\mathcal{L}} \vec{y}$. By finitariness of $\mathrm{Cn}_{\mathcal{L}}$ there is some $\Xi(\vec{x}, \vec{y}, \overline{z}) \in \Phi$ such that $\Xi \subseteq \sigma^{-1}\mathrm{Cn}_{\mathcal{L}}\Gamma$. Setting $\overline{\xi} := \sigma\overline{z}$, we then get $\Gamma \vdash_{\mathcal{L}} \Xi(\vec{\varphi}, \vec{\psi}, \overline{\xi})$. Conversely, assume $\Gamma \vdash_{\mathcal{L}} \Xi(\vec{\varphi}, \vec{\psi}, \overline{\xi})$ for some $\Xi(\vec{x}, \vec{y}, \overline{z}) \in \Phi$ and some $\overline{\xi} \in \mathrm{Fm}_L$. By structurality of $\mathrm{Cn}_{\mathcal{L}}$ one can easily see that $\vec{\varphi}, \Gamma \vdash_{\mathcal{L}} \vec{\psi}$. ∎

By this theorem we immediately conclude that the 2-logic defined by any quasivariety has a parameterized local deduction theorem (cf. the comments in the end of Section 3). This fact has been noticed independently in [9] as a direct corollary of [Theorem 2.1] [9] that states that every quasivariety has *locally definable principal congruences* (see also [p. 206] [10]) and [Theorem 3.1] [9] that states that any quasivariety has locally definable principal congruences with respect to a given family of finite sets of equations (viz. 2-formulas in our terms) iff the 2-logic defined by the quasivariety has the parameterized local deduction theorem with respect to the same family (cf. the comments after the formulation of the mentioned theorem). On the other hand, Theorem 4.1. collectively with [Theorem 3.1] [9] gives a new proof to Gorbunov's result [p. 206] [10] explicitly formulated in [Theorem 2.1] [9].

As for the item (iv) of Theorem 4.1., remark that, as opposed to the *filter* correspondence property [Definition 7.5][4], the syntactic one does not imply protoalgebraicity *immediately.* The problem here is that a quotient of a formula algebra need not be isomorphic to the same formula algebra. Therefore the arguments used in proving [Theorem 7.6][4] cannot be applied to proving the metaimplication Theorem 4.1. (iv) $\Rightarrow$ (v) directly. So this metaimplication does give a new insight into the concept of protoalgebraic logic.

It is also worth to notice that Theorem 4.1. solves positively the problem whether a many-dimensional logic having a local deduction theorem is protoalgebraic. [7]

In case $\mathcal{L}$ is protoalgebraic, it does not however possess, generally speaking, a full-replacement (resp., unary-replacement) implication system with less than $2k$ (resp., $k+1$) variables. We show it in the next section. This, in particular, disproves the statement of [Theorem 13.2][4].

5. Examples

Take an arbitrary dimension k. Consider the propositional language $L_{\supset} := \{\supset\}$ where $\supset$ is an infix binary connective. Let $\mathcal{L}^k_{\supset}$ be the k-logic over $L_{\supset}$ axiomatized by the calculus $\mathcal{D}^k_{\supset}$ constituted by the following k

[7] The fact that the problem whether a k-logic having a uniform deduction theorem is protoalgebraic was open for the case $k > 1$ was pointed out to the author by Don Pigozzi.

axioms and k rules:

$$\vdash \langle z_1, \ldots, z_{i-1}, x \supset x, z_i, \ldots, z_{k-1} \rangle,$$
$$\langle z_1, \ldots, z_{i-1}, x, z_i, \ldots, z_{k-1} \rangle,$$
$$\langle z_1, \ldots, z_{i-1}, x \supset y, z_i, \ldots, z_{k-1} \rangle \vdash \langle z_1, \ldots, z_{i-1}, y, z_i, \ldots, z_{k-1} \rangle,$$

for all $1 \leq i \leq k$.

The proof of the following statement is straightforward and, for this reason, is omitted.

Proposition 5.1. *The set*

$$\{\langle z_1, \ldots, z_{i-1}, x \supset y, z_i, \ldots, z_{k-1} \rangle\}_{1 \leq i \leq k}$$

is an unary-replacement implication system for $\mathcal{L}^k_\supset$. In particular, $\mathcal{L}^k_\supset$ is protoalgebraic.

On the other hand, we have

Theorem 5.2.

(i) Assume $\Delta(\vec{x}, \vec{y})$ is a full-replacement implication system for $\mathcal{L}^k_\supset$. Then $\mathrm{Var}(\Delta) = \{\vec{x}, \vec{y}\}$.

(ii) Assume $\Delta(x, y, z_1, \ldots, z_{k-1})$ is an unary-replacement implication system for $\mathcal{L}^k_\supset$. Then $\mathrm{Var}(\Delta) = \{x, y, z_1, \ldots, z_{k-1}\}$.

Proof. We start from proving the following lemma.

Lemma 5.3. *Let $\Gamma \cup \{\vec{\varphi}\} \subseteq \mathrm{Fm}^k_{L_\supset}$. Assume $\Gamma \vdash_{\mathcal{L}^k_\supset} \vec{\varphi}$. Then either $\vec{\varphi}$ is an $L_\supset$-instance of one of the axioms in $\mathcal{D}^k_\supset$ or there is some $\vec{\psi} \in \Gamma$ such that* $\mathrm{Var}(\vec{\varphi}) \subseteq \mathrm{Var}(\vec{\psi})$.

Proof. By induction on *the minimal* length n of a $\mathcal{D}^k_\supset$-derivation $\vec{\phi}_1, \ldots, \vec{\phi}_n$ of $\vec{\varphi}$ from Γ. If $n = 1$ then either $\vec{\varphi}$ is an $L_\supset$-instance of one of the axioms in $\mathcal{D}^k_\supset$ or $\vec{\varphi} \in \Gamma$, in which case the claim of the lemma is obvious. If $n > 1$ then, by minimality of the length of the $\mathcal{D}^k_\supset$-derivation under consideration, $\vec{\varphi} \notin \Gamma$ and $\vec{\varphi}$ is not an $L_\supset$-instance of any axiom in $\mathcal{D}^k_\supset$. In particular, there are $0 < l, m < n$ such that, for some $\zeta \in \mathrm{Fm}_{L_\supset}$ and some $1 \le i \le k$, $\vec{\phi}_l = \langle \varphi_1, \ldots, \varphi_{i-1}, \zeta, \varphi_{i+1}, \ldots, \varphi_k \rangle$ and $\vec{\phi}_m = \langle \varphi_1, \ldots, \varphi_{i-1}, \zeta \supset \varphi_i, \varphi_{i+1}, \ldots, \varphi_k \rangle$. By minimality of the length of the $\mathcal{D}^k_\supset$-derivation involved, $\vec{\phi}_l \ne \vec{\varphi}$. Hence $\zeta \ne \varphi_i$, so $\vec{\phi}_m$ is not an $L_\supset$-instance of any axiom in $\mathcal{D}^k_\supset$. Therefore, by induction hypothesis there is some $\vec{\psi} \in \Gamma$ such that $\mathrm{Var}(\vec{\phi}_m) \subseteq \mathrm{Var}(\vec{\psi})$. On the other hand, $\mathrm{Var}(\vec{\varphi}) \subseteq \mathrm{Var}(\vec{\phi}_m)$. ∎

Return to proving the theorem as such.

(i) As $\vec{y}$ is not an $L_\supset$-instance of any axiom in $\mathcal{D}^k_\supset$ and $\mathrm{Var}(\vec{y}) \not\subseteq \mathrm{Var}(\vec{x})$, by (3) and Lemma 5.3. we conclude that $y_i \in \mathrm{Var}(\Delta)$ for all $1 \le i \le k$. Let us prove by contradiction that $x_i \in \mathrm{Var}(\Delta)$ for every $1 \le i \le k$. For suppose $x_i \notin \mathrm{Var}(\Delta)$ for some $1 \le i \le k$. Then $\Delta(\vec{y}, \vec{y}) = \Delta(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k, \vec{y})$. In that case by (4)-(3) we would have

$$\langle y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k \rangle \vdash_{\mathcal{L}^k_\supset} \vec{y}.$$

However $\vec{y}$ is not an $L_\supset$-instance of any axiom in $\mathcal{D}^k_\supset$ and

$$y_i \notin \mathrm{Var}(\langle y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k \rangle).$$

This would contradict Lemma 5.3..

(ii) Take any $1 \le i \le k$. As $\langle z_1, \ldots, z_{i-1}, y, z_i, \ldots, z_{k-1} \rangle$ is not an $L_\supset$-instance of any axiom in $\mathcal{D}^k_\supset$ and $y \notin \mathrm{Var}(\langle z_1, \ldots, z_{i-1}, x, z_i, \ldots, z_{k-1} \rangle)$, by (5) and Lemma 5.3. we conclude that $y, z_1, \ldots, z_{k-1} \in \mathrm{Var}(\Delta)$. Let us prove by contradiction that $x \in \mathrm{Var}(\Delta)$ too. For suppose $x \notin \mathrm{Var}(\Delta)$. Then $\Delta = \Delta(y, y, z_1, \ldots, z_{k-1})$. In that case by (6)-(5) we would have

$$\langle z_1, \ldots, z_{i-1}, x, z_i, \ldots, z_{k-1} \rangle \vdash_{\mathcal{L}^k_\supset} \langle z_1, \ldots, z_{i-1}, y, z_i, \ldots, z_{k-1} \rangle.$$

This would contradict Lemma 5.3.. Therefore $x \in \mathrm{Var}(\Delta)$ as required. ∎

Thus, in particular, the statement of [Theorem 13.2][4] is indeed erroneous (except for the case $k = 1$, of course).

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