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PRETABULAR VARIETIES OF EQUIVALENTIAL ALGEBRAS

A b s t r a c t. It is shown that there are precisely two pretabular varieties of equivalential algebras.

1. Minimal preliminaries

A variety $\mathcal{V}$ is called *pretabular* iff all its proper subvarieties are finitely generated, but $\mathcal{V}$ itself is not. Pretabular varieties have been investigated mostly in connection to so-called varieties of logic. Thus, [1] and [5] dealt with modal algebras, [6] with Heyting algebras and [7, 8] with Brouwerian semilattices, the last being the case closest to ours.

An *equivalential algebra* is a groupoid $\mathbf{A} = \langle A; \cdot \rangle$ such that the operation $\cdot$ (called *equivalence*) satisfies the following identities:

(i) $(x \cdot x) \cdot y = y$;

(ii) $((x \cdot y) \cdot z) \cdot z = (x \cdot z) \cdot (y \cdot z)$;

(iii) $\big((x \cdot y) \cdot ((x \cdot z) \cdot z)\big) \cdot ((x \cdot z) \cdot z) = x \cdot y$.

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Following common practice we will henceforth associate the parentheses to the left, and leave out the operation symbol '$\cdot$'. Thus, the above identities become: $xxy = y$, $xyzz = xz(yz)$, and $xy(xzz)(xzz) = xy$, respectively. Since $xx = yy$ is true in all equivalential algebras, we can define a constant 1 by putting $xx = 1$. Congruences on equivalential algebras are 1-regular with respect to this constant, and thus, we can define a natural partial order among the elements of an algebra $\mathbf{A}$ as follows: $a \leq_E b$ iff $\Theta(b,1) \subseteq \Theta(a,1)$. Then, subdirectly irreducible (henceforth, si, as usual) algebras can be characterised as these which have the largest non-unit element, traditionally denoted by $\star$. The monolith μ of an si equivalential algebra has only one non-singleton congruence class, namely, $1/\mu = \{1, \star\}$. Although principal congruences in equivalential algebras are not, in general, first-order definable, the property of being si is, namely, by the formula $(\exists x \neq 1)(\forall y \notin \{x,1\})\ xy = y$.

The basic tool employed in our argument will be the representation theorem (see [9]), later on referred to as Słomczyńska representation. Let us briefly recall how this can be arrived at.

An element $a \in A \setminus \{1\}$ is *irreducible* iff $axx \in \{a,1\}$ for all $x \in A$. The set of all irreducible elements of an algebra $\mathbf{A}$ will be denoted by $\mathrm{Irr}(\mathbf{A})$, and if $\mathbf{A}$ is generated by $\mathrm{Irr}(\mathbf{A})$, we will call it *irreducibly generated.* All finite equivalential algebras are irreducibly generated. Recall, that a *boolean group* is a group that satisfies: $x^{-1} = x$. Every associative (i.e., satisfying $xyz = x(yz)$) equivalential algebra is (term equivalent to) a boolean group, upon defining $x^{-1} = x$.

Let now $\langle P; \leq \rangle$ be any poset. We define an equivalence relation $\sim$ on P by: $a \sim b$ iff for every $x \in P$: $x < a \Leftrightarrow x < b$. Let 1 be any element not in P. We write $\overline{U}$ for $P \cup \{1\}$ and similarly for an equivalence class U of $\sim$. Let $\circ$ be a partial binary operation on $\overline{P}$, whose domain is $\bigcup\{\overline{P} \times \overline{U} : U \in P/\sim\}$. Then, we say that $\mathcal{P} = \langle \overline{P}; \leq, \circ \rangle$ is an *equivalential frame* if:

(i) $\langle \overline{U}; \circ \rangle$ is a boolean group with unit 1, for each $\sim$-class U.

(ii) $x \circ y < z$ implies that $x < z$ or $y < z$, for all $x, y, z \in P$ with $x \sim y$.

It turns out that the set $\mathrm{Irr}(\mathbf{A})$ with the ordering relation $\leq_E$ defined a few lines above (i.e., with the $\sim$ relation induced), and the original equivalence operation $\cdot$ restricted to $\sim$-classes is an equivalential frame. Conversely, if $\mathcal{P}$ is an equivalential frame, we define $A(\mathcal{P})$ to be the set $\big\{a \in \prod_{j\in J} \overline{U}_j \mid \{a_j : j \in J\} \setminus \{1\}$ is a finite $\leq$-antichain$\big\}$, where J is a suitable set indexing $\sim$-classes. Then, we define an operation $\cdot$ on $A(\mathcal{P})$ coordinatewise, by:

$$(a \cdot b)_i = \begin{cases} a_i \circ b_i, & \text{if this is a minimal element of } \{a_j \circ b_j : j \in J\} \\ 1, & \text{otherwise.} \end{cases}$$

Then, $\langle A(\mathcal{P}); \cdot\rangle$ is an equivalential algebra.

The above gives rise to two natural maps: $\mu : \mathbf{A} \longrightarrow \langle \mathrm{Irr}(\mathbf{A}); \leq_E, \cdot\rangle$ and $\nu : \mathcal{P} \longrightarrow \langle A(\mathcal{P}); \circ\rangle$. Thus, we have:

Representation Theorem. *[Słomczyńska] The maps μ and ν are mutually inverse and establish a one-to-one correspondence between the class of irreducibly generated equivalential algebras and the class of equivalential frames.*

For more information on equivalential algebras the reader is referred to [4], where they were first introduced, or to [9] and [10], where the representation theorem was proved and many other essential aspects of their theory were developed.

2. Pretabular varieties

In what follows we will show that, precisely as in the case of Brouwerian semilattices (see [7, 8]), there are two pretabular varieties of equivalential algebras.

Two equivalential algebras will play a special role in the sequel. One is the countable equivalential chain, i.e., the algebra $\mathbf{\Omega} = \langle \omega; \cdot, 0\rangle$, where the operation $\cdot$ is defined by:

$$n \cdot m = \begin{cases} \max\{n, m\}, & \text{if } n \neq m \\ 0, & \text{if } n = m \end{cases}$$

with the maximum taken with respect to the natural ordering. The resulting equivalence ordering is the converse of the former, with 0 as the largest element. The other is the algebra $\mathbf{\Sigma} = \langle \sigma \cup \{\star\}; \cdot, \emptyset \rangle$, where σ is any countably infinite family of sets closed under symmetric difference. The operation $\cdot$ is defined on σ as the very symmetric difference, whereas $x \cdot \star = \star \cdot x = x$ for $\sigma \ni x \neq \emptyset$, $\star \cdot \star = \emptyset$, and $\star \cdot \emptyset = \emptyset \cdot \star = \star$.

We begin with the following:

Fact 1. *If $\mathcal{V}$ is a variety of equivalential algebras that is not finitely generated, then $\mathcal{V}$ contains an irreducibly generated infinite si algebra.*

Proof. Let $\mathcal{V}$ be as in the assumption. Then, $\mathcal{V}$ contains an infinite si algebra $\mathbf{A}$. This follows, for instance, from the fact that in the variety of equivalential algebras the property of being si carries over to ultraproducts, which in turn follows by Łoś Theorem from the first-order definability of that property (cf. [2] or any standard textbook on model theory for the properties of ultraproducts; we shall use them throughout the proof without further ado).

Thus, if we had only finite si algebras in $\mathcal{V}$, then there must have been a finite bound for their cardinalities and $\mathcal{V}$ would have been finitely generated which it is not.

The infinite si algebra $\mathbf{A}$, does not have to be irreducibly generated, but we will soon show that there must be an irreducibly generated infinite si algebra in $\mathcal{V}$.

Firstly, we will show that an ultraproduct of finitely generated subalgebras of $\mathbf{A}$, into which $\mathbf{A}$ is embeddable, itself is si. To see this, let I be the set of all finite subsets of A and $X(a)$ the set of all finite subsets of A that contain a, for some $a \in A$. Consider the family F of subsets of I defined by: $F = \{X(a) : a \in A\}$. This clearly has the finite intersection property in the powerset of I and hence can be extended to a free ultrafilter $\mathcal{F}$. Now, let $\mathbf{B} = \prod_{i \in I} \mathbf{A}_i / \mathcal{F}$, where $\mathbf{A}_i$ is the subalgebra of $\mathbf{A}$ generated by i, and the desired embedding can be defined in the canonical way by putting $e(a) = x/\mathcal{F}$, where $x(i) = a$ if $a \in A_i$ and is arbitrary otherwise.

To prove that $\mathbf{B}$ is si it suffices to show that $\{i \in I : A_i \text{ is si}\} \in \mathcal{F}$. This, however, contains the set $\{i \in I : \star \in A_i\}$ and this in turn is identical with $X(\star)$, since $\star$ cannot be generated by anything except itself. Now, the ultrafilter $\mathcal{F}$ has been so chosen as to contain $X(a)$, for all $a \in A$; in particular it contains $X(\star)$. Hence, $\{i \in I : A_i \text{ is si}\} \in \mathcal{F}$ as needed.

Secondly, the set $\mathrm{Irr}(\mathbf{B})$ of all irreducible elements in $\mathbf{B}$ must be infinite. To see that, remind that irreducibility is a first-order property, and thus we have $b \in \mathrm{Irr}(\mathbf{B})$ iff $\{i \in I : b(i) \in \mathrm{Irr}(\mathbf{A}_i)\} \in \mathcal{F}$. Moreover, for any $n \in \omega$ there is an algebra $\mathbf{A}_i$ with $|\mathrm{Irr}(\mathbf{A}_i)| \geq n$. If it were otherwise, the cardinalities of algebras $\mathbf{A}_i$, and hence of their ultraproduct, would be bounded by a finite number $f(n)$ where f is the spectrum function for the variety of equivalential algebras, and this would contradict $\mathbf{A}$'s being infinite. Now, by an appropriate renumbering, we can ensure that $|\mathrm{Irr}(\mathbf{A}_i)| \geq |\mathrm{Irr}(\mathbf{A}_j)|$ iff $i \geq j$. Let, $\mathrm{card}(i)$ stand for $|\mathrm{Irr}(\mathbf{A}_j)|$. Take any sequence $a = \langle a(i) : i \in I\rangle$ with $a(i) \in \mathrm{Irr}(\mathbf{A}_i)$. Starting from this sequence, we can proceed in the following recursive manner: We put $a_0 = a$, and define $a_{n+1} = \langle a_{n+1}(i) : i \in I\rangle$ as the sequence $a_{n+1}(i) \in \mathrm{Irr}(\mathbf{A}_i)$, such that: $a_{n+1}(i) = a_n(i)$, if $\mathrm{card}(i) \leq \{a_k(i) : k \leq n\}$; and $a_{n+1}(i)$ is an arbitrary member of $\mathrm{Irr}(\mathbf{A}_i) \setminus \{a_k(i) : k \leq n\}$, otherwise. Then, it is easily seen that for all $k \neq n$, $a_n(i) = a_k(i)$ on at most finite number of coordinates. Thus, clearly, $a_k/\mathcal{F} \neq a_n/\mathcal{F}$ iff $k \neq n$, and for all $n \in \omega$, a_n is irreducible in $\mathbf{B}$.

Then, we let $\mathbf{C}$ be the subalgebra of $\mathbf{B}$ generated by the set of irreducibles in $\mathbf{B}$, and thus we obtain the algebra with all the desired properties. ■

Fact 2. *If $\mathcal{V}$ is a variety of equivalential algebras that is not finitely generated then $\mathbf{\Omega} \in \mathcal{V}$ or $\mathbf{\Sigma} \in \mathcal{V}$.*

Proof. By Fact 1. we have that $\mathcal{V}$ contains an irreducibly generated infinite si algebra $\mathbf{C}$. Its equivalential frame $\mathcal{P}(\mathbf{C})$ is also infinite, and thus must contain an infinite chain or an infinite antichain. If the former, then, by the properties of Słomczyńska representation, any countable subchain S with the unit element 1 of $\mathbf{C}$ belonging to S is the universe of a subalgebra of $\mathbf{C}$ isomorphic to $\mathbf{\Omega}$.

Suppose now that $\mathcal{P}(\mathbf{C})$ contains an infinite antichain. Then, we take a countable antichain $D \subseteq \mathcal{P}(\mathbf{C})$, and, by the properties of Słomczyńska representation again (cf. [9], Proposition 1.1), it follows that the subalgebra generated by D is associative, and its cardinalty equals to this of the set of all finite subsets of D. Thus, the subalgebra generated by D is countable. Moreover, as $\mathbf{C}$ is si, there is a unique element $\star \in \mathcal{P}(\mathbf{C})$ such that augmenting $\overline{W}$ by $\star$ we obtain the universe of an algebra isomorphic to $\mathbf{\Sigma}$. ■

Corollary. *The only pretabular varieties of equivalential algebras are* $\mathcal{V}(\mathbf{\Omega})$ *and* $\mathcal{V}(\mathbf{\Sigma})$*.*

Proof. In view of Fact 2. it suffices to show that $\mathcal{V}(\mathbf{\Omega})$ and $\mathcal{V}(\mathbf{\Sigma})$ are pretabular. Firstly, we show that neither is finitely generated. For suppose $\mathcal{V}$ (with $\mathcal{V}$ being either of $\mathcal{V}(\mathbf{\Omega})$, $\mathcal{V}(\mathbf{\Sigma})$) is finitely generated. Then, it is also generated by a single finite algebra $\mathbf{A}$. Combining an extension of Jónsson Lemma for congruence modular varieties with the fact that the monolith μ of any si equivalential algebra is Abelian (see [3], Theorem 10.1, for the former, and [12] for the latter), we obtain that for any si algebra $\mathbf{B}$ from $\mathcal{V}$ we have: $\mathbf{B}/\mu \in HS(\mathbf{A})$. Thus, $\mathbf{B}/\mu$ must be finite. However, as the monolith of an si equivalential algebra has precisely one non-singleton congruence class—the coset of 1, of cardinality 2—neither $\mathbf{\Omega}/\mu$, nor $\mathbf{\Sigma}/\mu$ is finite. A contradiction.

Secondly, suppose $\mathcal{W}$ is a proper subvariety of $\mathcal{V}(\mathbf{\Omega})$, yet $\mathcal{W}$ is not finitely generated. By Fact 2, $\mathcal{W}$ contains $\mathbf{\Omega}$ or $\mathbf{\Sigma}$. The latter is impossible as $\mathcal{V}(\mathbf{\Omega})$ satisfies the linearity condition (see [4], [11]), i.e., the identity:

$$(\lambda)\quad (x(yzz)(yzz))(x(zyy)(zyy))(x(yz)(yz)) = x,$$

which is falsified by $\mathbf{\Sigma}$. Thus, $\mathbf{\Omega} \in \mathcal{W}$, and thus $\mathcal{W} = \mathcal{V}(\mathbf{\Omega})$ contradicting the assumption.

Suppose now that $\mathcal{W}$ is a proper subvariety of $\mathcal{V}(\mathbf{\Sigma})$, but $\mathcal{W}$ is not finitely generated. By Fact 2 again, $\mathcal{W}$ contains $\mathbf{\Omega}$ or $\mathbf{\Sigma}$. However, the former is impossible since $\mathcal{V}(\mathbf{\Sigma})$ satisfies the height 3 condition (see [4], [11]), i.e., the identity:

$$(\chi_3) \quad x(yzzy)(yzzy)x = 1,$$

which is falsified by $\mathbf{\Omega}$. Thus, $\mathbf{\Sigma} \in \mathcal{W}$, and thus $\mathcal{W} = \mathcal{V}(\mathbf{\Sigma})$ contradicting the assumption. This finishes the argument. ■

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