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# LATTICE OF SUBSTITUTIONS

A b s t r a c t. A lattice whose elements are sets of substitutions is defined in this paper. We prove that the lattice is distributive and it has the upper and lower bound.

## 1. Preliminaries

Let $A$ and $B$ be two fixed sets. By a substitution we mean any partial function $e : A \to B$. All further results would remain valid if substitutions were defined as finite functions. A motivation for such a restriction is given at the end of the third section. Note that our definition of a substitution is more general than the usual one.

We adopt the following notation:

$\mathrm{PF}(A, B)$ — the set of all partial functions from the set $A$ into the set $B$,

$\mathrm{Fin}X$ — the family of all finite subsets of the set $X$,

$f_\emptyset$ — the empty function.

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Capitals $C, D, E$ will denote finite subsets of $\mathrm{PF}(A, B)$ while small letters shall be usually partial functions (substitutions).

## 2. Basic operations on sets of substitutions

**Definition 1.**

$$\mathcal{V} := \Big\{ C \in \mathrm{FinPF}(A, B) : \\ : \forall_{s,t \,\in\, \mathrm{PF}(A,B)} \big( (s \in C \wedge t \in C \wedge s \subseteq t) \Rightarrow s = t \big) \Big\}.$$

Evidently, $\mathcal{V}$ is non-empty. Moreover, it always has at least two elements, namely $\emptyset$ and $\{f_\emptyset\}$.

**Definition 2.** We say that functions $s : X \to Y$ and $t : Z \to W$ *tolerate* each other ($s \approx t$), if they agree on their domains.

$$s \approx t \Leftrightarrow \forall_{x \,\in\, X \cap Z}\ s(x) = t(x).$$

**Definition 3.** Let $C, D$ be elements of $\mathrm{FinPF}(A, B)$. Operation $*$ is defined as follows:

$$C * D := \{ s \cup t : s \in C \ \wedge\ t \in D \ \wedge\ s \approx t \}.$$

Clearly, the operation is commutative and associative. The following two lemmata are obvious:

**Lemma 1.** *If $C \subseteq D$, then $C * E \subseteq D * E$.*

**Lemma 2.** $C * (D \cup E) = (C * D) \cup (C * E)$.

Any formula can be seen as a tree-like structure with atomic formulas as its leaves. Suppose that to every leaf a list of possible substitutions is given. Our tree determines an operation on the sets of substitutions, if logical operators are associated with elementary operation on sets of substitutions. The operations $\cup$ and $*$ correspond in a natural way to logical conjunction and disjunction.

One can easily notice that the result of the operation $\cup$ as well as $*$ applied to the pair of elements of $\mathcal{V}$ may not belong to $\mathcal{V}$. It is enough to consider singletons of different functions $f$ and $g$ with $f \subseteq g$. To close this set under the union and $*$ operation, we have to apply additionally an operator $\mu$ defined as follows:

**Definition 4.** For any $C \in \mathrm{FinPF}(A, B)$

$$\mu(C) := \Big\{t \in C : \forall_{s \in C} \ \big(s \subseteq t \Rightarrow s = t\big)\Big\}.$$

Two next lemmata are simple consequences of Definition 4.

**Lemma 3.** $\forall_{a \in \mu(C)} \forall_{b \in C}\ (b \subseteq a \Rightarrow b = a)$.

**Lemma 4.** $\mu(C) \subseteq C$.

One can easily check that $\mu(C) \in \mathcal{V}$.

The $\mu$ operator leaves from any chain in $C$ the minimal, wrt. inclusion, element.

**Lemma 5.** $\forall_{b \in C}\ \exists_c\ c \subseteq b \wedge c \in \mu(C)$.

We now state some properties of the operator $\mu$.

**Lemma 6.** $\forall_{C \in \mathcal{V}}\ \mu(C) = C$.

**Lemma 7.** $\mu(C \cup D) \subseteq \mu(C) \cup D$.

**Proof.** Let $a \in \mu(C \cup D)$. We obtain $a \in C \cup D$ by Lemma 4.

We consider now two cases:

(1) Let us assume that $a \in C$. We shall show that $a \in \mu(C)$. Let $b$ be an element of $\mathrm{PF}(A, B)$ such that $b \in C$. Hence $b \in C \cup D$, and $b = a$ by Lemma 3 and 4.

(2) We assume that $a \in D$. So $a \in \mu(C) \cup D$ and the conclusion is evident. ■

**Lemma 8.** $\mu\big(\mu(C) \cup D\big) = \mu(C \cup D)$.

**Proof.** We shall prove first that $\mu\big(\mu(C) \cup D\big) \subseteq \mu(C \cup D)$. Assume that $a \in \mu\big(\mu(C) \cup D\big)$. If so, then by Lemma 4 we have $a \in \mu(C) \cup D$. Let us consider now an element $b \in C \cup D$ such that $b \subseteq a$.

If $b \in C$, then we are able to show, using Lemma 5, that there is an element $c$ such that $c \subseteq b$ and $c \in \mu(C)$. Then $c \in \mu(C) \cup D$ as well as $c \subseteq a$. Hence, by Lemma 3, $c = a$ and finally $b = a$.

Obviously, if $b \in D$, we have $b \in \mu(C) \cup D$, and using Lemma 3 we have $b = a$.

By the definition of $\mu$ we have $a \in \mu(C \cup D)$.

There remains to show that $\mu(C \cup D) \subseteq \mu\big(\mu(C) \cup D\big)$. Let $a \in \mu(C \cup D)$. If $b \subseteq a$ for some $b \in \mu(C) \cup D$, then $b \in C \cup D$ by Lemma 4 and hence $a = b$. Thus $a \in \mu\big(\mu(C) \cup D\big)$. ■

Nonmonotonicity of the operator $\mu$ is noteworthy. If $C$ contains only one non-empty substitution, we have obviously $\mu(C) = C$. If we take $D = C \cup \{f_\emptyset\}$, we can easily check that $\mu(D) = \{f_\emptyset\}$. So the inclusion $\mu(C) \subseteq \mu(D)$ does not hold.

Idempotency of $\mu$ follows from Lemma 8 (by putting $D = \emptyset$).

The proof of the lemma below is based mainly on Lemma 5 so it can be omitted.

**Lemma 9.** $\forall_{a \in C * D}\ \exists_b\ b \subseteq a \wedge b \in \mu(C) * D$.

The following property of the operation $*$ is also left without an easy proof:

**Lemma 10.** $\forall_{a \in C * D}\ \exists_{b \in D}\ b \subseteq a$.

**Lemma 11.** $\mu\big((C * D) \cup D\big) = \mu(D)$.

**Proof.** $\subseteq$: Assume $a \in \mu\big((C * D) \cup D\big)$. From Lemma 3 we obtain $a \in C * D$ or $a \in D$. Let us assume $a \in C * D$. Lemma 10 enables us to choose an element $b \in D$ such that $b \subseteq a$. Hence $b \in (C * D) \cup D$ and so, by Lemma 3 we have $a \in D$. Let $b \in D$. Since $b \in (C * D) \cup D$, we obtain $b \subseteq a \Rightarrow b = a$ by our assumptions and Lemma 3. Hence $a \in \mu(D)$, by the definition of $\mu$.

$\supseteq$: Assume $a \in \mu(D)$. Clearly $a \in (C * D) \cup D$. Assume that $b \in (C * D) \cup D$ and $b \subseteq a$. Suppose that $b \in C * D$. We choose (Lemma 10) $c \in D$ such that $c \subseteq b$. Hence $c \subseteq a$ and finally $c = a$ by the definition of $\mu$. Hence $b \in D$. If $b \in D$, we obtain $b = a$ by Lemma 3.

By the definition of $\mu$ we have $a \in \mu\big((C * D) \cup D\big)$. ■

**Lemma 12.** $\mu(C * C) = \mu(C)$.

**Proof.** Note that $C \subseteq C * C$. Henceforth $\mu(C * C) = \mu\big((C * C) \cup C\big)$ and by Lemma 11 we complete the proof. ■

**Lemma 13.** $\mu(C * D) \subseteq \mu(C) * D$.

**Proof.** Assume that $a \in \mu(C*D)$. Surely, $a \in C*D$ by Lemma 4. Let us take $b \subseteq a$ with $b \in \mu(C) * D$ by Lemma 9. In view of $\mu(C) * D \subseteq C * D$ (see Lemma 1), we obtain $b = a$ by Lemma 3 and eventually, $a \in \mu(C) * D$ which was to be proved. ■

**Lemma 14.** $\mu\big(\mu(C) * D\big) = \mu(C * D)$.

**Proof.** $\subseteq$: Let us assume that $a \in \mu\big(\mu(C) * D\big)$. Using Lemma 4 we obtain $a \in \mu(C) * D$.

Let us assume now $b \in C * D$. Using Lemma 9, we can choose a substitution $c \subseteq b$ such that $c \in \mu(C) * D$.

Now we shall use the definition of $\mu$. Let us assume that $b \subseteq a$. Clearly, $c \subseteq a$ and according to Lemma 3 we have $c = a$, which implies $b = a$. We get $a \in \mu(C * D)$ as required.

$\supseteq$: Assume $a \in \mu(C * D)$. We use Lemma 13 to show the implication: $\big(b \in \mu(C) * D \wedge b \subseteq a\big) \Rightarrow b = a$, and subsequently $a \in \mu\big(\mu(C) * D\big)$ by the definition of $\mu$. ■

Now we formulate and prove the lemma needed to show distributivity of operations.

**Lemma 15.** *For every* $P, Q, R \in \mathcal{V}$ :

$$\mu\big(P * \mu(Q \cup R)\big) = \mu\big(\mu(P * Q) \cup \mu(P * R)\big).$$

**Proof.**

$$\begin{aligned}\mu\big(P * \mu(Q \cup R)\big) &= \mu\big(P * (Q \cup R)\big) \text{ by Lemma 14} \\ &= \mu\big((P * Q) \cup (P * R)\big) \text{ by Lemma 2} \\ &= \mu\big(\mu(P * Q) \cup \mu(P * R)\big) \text{ by Lemma 8.}\end{aligned}$$

■

## 3. Definition of the structure

**Theorem 1.** *For every* $P, Q \in \mathcal{V}$ :

$$\mu(P \cup Q) = \mu(Q \cup P) \quad \textit{and} \quad \mu(P * Q) = \mu(Q * P).$$

**Proof.** Straightforward from the commutativity of $\cup$ and $*$. ■

**Theorem 2.** *For every* $P, Q, R \in \mathcal{V}$ :

$$\mu\big(P \cup \mu(Q \cup R)\big) = \mu\big(\mu(P \cup Q) \cup R\big) \wedge \mu\big(P * \mu(Q * R)\big) = \mu\big(\mu(P * Q) * R\big).$$

**Proof.** We use the associativity of $*$, Lemma 8 and 14. ■

**Theorem 3.** *For every* $P, Q \in \mathcal{V}$ :

$$\mu\big(P \cup \mu(P * Q)\big) = P \quad \textit{and} \quad \mu\big(P * \mu(P \cup Q)\big) = P.$$

**Proof.** Let us prove the first conjunct.

$$\begin{aligned}\mu\big(P \cup \mu(P * Q)\big) &= \mu\big(P \cup (P * Q)\big) \text{ by Lemma 8} \\ &= P \text{ in view of Lemma 6 and 11.}\end{aligned}$$

To prove the second conjunct we use Lemma 15. We have

$$\begin{aligned}\mu\big(P * \mu(P \cup Q)\big) &= \mu\big(\mu(P * P) \cup \mu(P * Q)\big)\\ &= \mu\big(P \cup \mu(P * Q)\big) \text{ by Lemma 12}\\ &= P \text{ from the previous considerations.}\end{aligned}$$

■

Now we are ready to define the lattice operations.

$$P \sqcap Q = \mu(P * Q), \tag{1}$$

$$P \sqcup Q = \mu(P \cup Q) \tag{2}$$

for every $P, Q \in \mathcal{V}$.

The commutativity, associativity and absorption laws for (1) and (2) has been shown. Hence

$$P \sqcup Q = Q \sqcup P, \quad P \sqcap Q = Q \sqcap P,$$
$$(P \sqcup Q) \sqcup R = P \sqcup (Q \sqcup R), \quad (P \sqcap Q) \sqcap R = P \sqcap (Q \sqcap R),$$
$$(P \sqcap Q) \sqcup P = P, \quad (P \sqcup Q) \sqcap P = P,$$

**Definition 5.** By *the lattice of substitutions* we mean the structure

$$\mathcal{K}_{\mathcal{V}} := \langle \mathcal{V}, \sqcap, \sqcup \rangle.$$

**Theorem 4.** *The lattice $\mathcal{K}_{\mathcal{V}}$ is distributive.*

**Proof.** Straightforward in view of Lemma 15 and definitions (1) and (2). ■

**Theorem 5.** $0_{\mathcal{K}_{\mathcal{V}}} = \emptyset$.

**Proof.** Assume that $P \in \mathcal{V}$. Hence $P \sqcup \emptyset = \mu(P \cup \emptyset) = \mu(P) = P$. ■

**Theorem 6.** $1_{\mathcal{K}_{\mathcal{V}}} = \{f_\emptyset\}$.

**Proof.** We shall use the equation $C * \{f_\emptyset\} = C$ true for any finite subset of $\mathrm{PF}(A, B)$ (note that $f_\emptyset \approx f$ for any function $f$).

Let $P \in \mathcal{V}$. Then $P \sqcap \{f_\emptyset\} = \mu(P * \{f_\emptyset\}) = \mu(P) = P$. ■

Therefore, the lattice $\mathcal{K}_{\mathcal{V}}$ has the lower and upper bound.

Let us show that $\mathcal{K}_{\mathcal{V}}$ may be not complemented, nor pseudo-complemented. Let $A$ be the set of natural numbers and $B = \{a, b\}$ be a two-element set. Consider the function $g$ which for any element of $A$ yields $a$ as its value. It is clear that all functions $f_n$ with $b$ as a value do not tolerate $g$, so $\{f_n\} \sqcap \{g\} = \emptyset$ for any natural $n$. But the infinite join of singletons $\{f_n\}$ does not belong to $\mathcal{V}$. Consequently, there is no greatest element in the set of all $x$'s such that $\{g\} \sqcap x = 0_{\mathcal{K}_{\mathcal{V}}}$.

The above counter-example fails if we restrict our research to finite partial functions only.

## 4. Examples

**Example [1].** As the simplest example, we can consider the case $A = \{a\}$ and $B = \{b\}$. Obviously, $\mathrm{PF}(A, B) = \{f_\emptyset, f\}$, where $f$ is function with the domain equal to $\{a\}$ and $f(a) = b$.

The lattice is a chain with the length equal to 2.

**Example [2].** Consider a two-element set $A = \{a, b\}$ and a singleton $B = \{0\}$. Let us simplify the notation and write $a \to 0$ instead of $\{(a, 0)\}$ and $\substack{a\to 0\\ b\to 0}$ in place of $\{(a, 0), (b, 0)\}$.

Obviously, there are four partial functions from $A$ into $B$, namely the empty function $f_\emptyset$, two singletons: $f_1 = a \to 0$ and $f_2 = b \to 0$ and a doubleton $f_3 = \substack{a\to 0\\ b\to 0}$. The elements of $\mathcal{V}$ are: $\emptyset$, $\{f_\emptyset\}, \{f_1\}, \{f_2\}, \{f_3\}$ and $\{f_1, f_2\}$.

It is clear that the supremum of $\{f_1\}$ and $\{f_2\}$ is $\{f_1, f_2\}$ and the infimum is $\{f_3\}$. Taking into account the theorems stating that $\emptyset$ is the least and $\{f_\emptyset\}$ — the greatest element of $\mathcal{K}_\mathcal{V}$, we present the diagram of the whole structure below (Figure 1).

**Example [3].** *Let* $A = \{a\}$ *and* $B = \{0, 1\}$. *Using the notation from the previous example we will write:* $\mathrm{PF}(A, B) = \{f_\emptyset,\ a \to 0,\ a \to 1\}$ ($\{f_\emptyset,\ f_1,\ f_2\}$ *for short) so* $\mathcal{V}$ *contains five elements:* $\emptyset$, $\{f_\emptyset\}$, $\{f_1\}, \{f_2\}$ *and* $\{f_1, f_2\}$. *Since* $f_1$ *and* $f_2$ *do not tolerate each other, the corresponding infimum is equal to* $\emptyset$ *(see Figure 2).*

## 5. On the order on substitution sets

The question may arise, how to describe the ordering relation on the sets of substitutions. It can be defined (cf. [1]) for any well-defined lattice in the same way. Namely we have $a \leq b$ if and only if $a \sqcup b = b$ (or dually $a \leq b$ if and only if $a \sqcap b = a$).

Clearly, with the lattice operations given as above, the ordering of substitution sets is not the ordinary set-theoretical inclusion.

The ordering relation can be defined as follows:

$\{f_1, \ldots, f_p\} \leq \{g_1, \ldots, g_q\}$ if and only if
for every $i \leq p$ there exists an $j \leq q$ such that $g_j \subseteq f_i$.

Formally we have the following result:

**Theorem 7.** *For every* $C, D \in \mathcal{V}$ :

$$C \leq D \iff \forall_{f \in C} \exists_{g \in D}\, g \subseteq f.$$

**Proof.** $\Rightarrow$: Assume $C \leq D$. Clearly we have $C \sqcap D = C$. Let $f \in C$. In view of Lemma 4 and our definition of the lattice operations we get $f \in C * D$. Then (by Lemma 10), $g \subseteq f$ for some $g \in D$.

$\Leftarrow$: Let us assume that for any $f \in C$ there exists $g \in D$ such that $g \subseteq f$. We have to show $\mu(C * D) = C$. First we shall prove $\mu(C * D) \subseteq C$. Let $x \in \mu(C * D)$. Of course $x \in C * D$. Using the definition of the operator $*$ we may find elements $a, b$ of $C$ and $D$ respectively such that $x = a \cup b$. From our assumptions it follows that there is an $a_1 \in D$ such that $a_1 \subseteq a$. Of course, $a = a \cup a_1$, which implies $a \in C * D$. But $a \subseteq x$ and hence we obtain $x = a \in C$.

We complete our proof if we show that $C \subseteq \mu(C * D)$. Let $a \in C$. We can choose an $b \in D$ such that $b \subseteq a$. Both $a$ and $b$ may be treated as partial functions which tolerate each other (the domain of $b$ includes in that of $a$). Consequently $a = a \cup b \in C * D$ by the definition of $*$. There remains to show that the following implication holds for any $x : (x \in C * D \wedge x \subseteq a) \Rightarrow x = a$. So, let $x \in C * D$ and $x \subseteq a$. By Lemma 10 we can take an $c \in C$ such that $c \subseteq x$ and consequently $c \subseteq a$ by our assumptions. Since $C$ is in the carrier of $\mathcal{K}_{\mathcal{V}}$, we have $\mu(C) = C$ (Lemma 6) and both $c$ and $a$ belong to $C$ so then, by Lemma 3, $c = a$ and $x = a$ as well. ■

## References

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