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Teodor STĘPIEŃ

# DERIVABILITY

A b s t r a c t. The basic notion connected with the structural completness is the notion of derivability (see [1] and [2]). Thus in this paper we give a sufficient and necessary condition of derivability of any structural and permissible rules in an arbitrary but fixed over-system of the classical predicate calculus. In consequence we estabilish the scope of the structural completeness in the class of all over-system of the classical predicate calculus.

## Section 1

Let $C$, $N$, $A$, $K$, $E$ denote connectives of implication, negation, disjunction, conjunction and equivalence respectively. Symbols $x_1$, $x_2$, ... are individual variables. Symbols $P^n_k$ ( $n, k \in \mathcal{N} = \{1, 2, ...\}$) are $n$-ary predicate letters. The set of all atomic formulas of the form $P^n_k(x_1, ..., x_n)$ is denoted by $At_1$. The symbols $\bigwedge x_k$, $\bigvee x_k$ are called quantifiers. The set $S_1$ of all well-formed formulas is constructed in the usual manner from the symbols listed above. By $C(\alpha)$ we denote the set of all connectives occuring

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in $\alpha$ ($\alpha \in S_1$). Hence $S_1^+ = \{\alpha \in S_1 : C(\alpha) \subseteq \{C, A, K, E\}\}$. Next $vf(\phi)$ denotes the set of all free variables occuring in $\phi$. Hence $\hat{\alpha} = \bigwedge x_1, ..., x_k \alpha$, if $vf(\alpha) = \{x_1, ..., x_k\}$. By $At_0 = \{p_1^1, p_2^1, ..., p_1^2, p_2^2, ..., p_1^k, p_2^k, ...\}$ we denote the set of all propositional variables. Hence $S_0$ is the set of all well-formed formulas that are built in the usual manner from propositional variables by means of logical connectives. Next $R_1$ denotes the set of all rules over $S_1$ (see [3], [4] for definition). For any $X \subseteq S_1$, $Cn(R, X)$ is the smallest subset of $S_1$ containing $X$ and closed under the rules of $R \subseteq R_1$. Next the couple $\langle R, X \rangle$ is called a system, whenever $X \subseteq S_1$ and $R \subseteq R_1$. By $r_*$ we denote the rule of simultaneous substitution for predicate letters. Namely $\langle \{\alpha\}, \beta \rangle \in r_* \Leftrightarrow \beta = h^e(\alpha)$ for some endomorphism $h^e : S_1 \longrightarrow S_1$ which is an extension of the function $e : At_1 \longrightarrow S_1$, $e \in \mathcal{E}_*$ (for details see [3], [4]). Hence $Sb(X) = Cn(\{r_*\}, X)$. Next $r_0$ denotes Modus Ponens and $r_+$ denotes the generalization rule. $\{r_0, r_+\} = R_{0+}$. We use $\Rightarrow$, $\neg$, $\Leftrightarrow$, $\wedge$, $\vee$, $\forall$, $\exists$ as metalogical symbols and write $X \subset Y$ for $X \subseteq Y$ and $Y \neq X$. By $\mathfrak{M}_2$ we denote the classical two-valued matrix. In this paper we assume that for each $\phi \in S_1$, $\check{\phi} = \bigvee x_1, ..., x_n N\phi$, if $vf(\phi) = \{x_1, ..., x_n\}$. Hence $\check{\phi} = N\phi$, if $vf(\phi) = \emptyset$.

We define function $i : S_1 \longrightarrow S_0$ as follows:

(a) $i(P_k^n(x_1, ..., x_n)) = p_k^n$, ($p_k^n \in At_0$),

(b) $i(N\phi) = Ni(\phi)$,

(c) $i(F\phi\psi) = Fi(\phi)i(\psi)$, $F \in \{C, K, A, E\}$,

(d) $i(\bigwedge x_k \phi) = i(\bigvee x_k \phi) = i(\phi)$.

Let $Z_2$ denote the set of all formulas valid in the classical propositional calculus and let $L_2$ denote the set of all formulas valid in the classical predicate calculus. Next $Z_2^* = \{\phi \in S_1 : i(\phi) \in Z_2\}$ and $\overline{S}_1 = \{\phi \in S_1 : vf(\phi) = \emptyset\}$ and $\overline{Z}_2 = Z_2^* \cap \overline{S}_1$. $S_1^+$ denotes the set of all formulas from $S_1$ in which the negation sign does not occur. Hence $L_2^+ = L_2 \cap S_1^+$. Finally, $S_1^*$ denotes the set of all well-formed formulas that are in prenex - conjunctive normal form (see [2] pp. 130-132).

Now we repeat some well known definitions (see [3]). Let $R \subseteq R_1$ and $X \subseteq S_1$. Then

$$r \in Perm(R, X) \quad \text{iff}$$

$$(\forall \pi \subseteq S_1)(\forall \phi \in S_1)[\langle \pi, \phi \rangle \in r \wedge \pi \subseteq Cn(R, X) \ \Rightarrow \ \phi \in Cn(R, X)]$$

$$r \in Der(R, X) \quad \text{iff}$$

$$(\forall \pi \subseteq S_1)(\forall \phi \in S_1)[\langle \pi, \phi \rangle \in r \ \Rightarrow \ \phi \in Cn(R, X \cup \pi)]$$

$$r \in Struct_{s_1} \quad \text{iff}$$

$$(\forall \pi \subseteq S_1)(\forall \phi \in S_1)(\forall e \in \mathcal{E}_*)[\langle \pi, \phi \rangle \in r \ \Rightarrow \ \langle h^e(\pi), h^e(\phi) \rangle \in r]$$

$$\langle R, X \rangle \in SCpl \quad \text{iff} \quad Struct_{S_1} \cap Perm(R, X) \subseteq Der(R, X)$$

$$\langle R, X \rangle \in Inw \text{ iff } R \subseteq Struct_{S_1} \text{ and } X = Sb(X)$$

$$\langle R', X' \rangle < \langle R, X \rangle \text{ iff } [X' \subseteq Cn(R, X) \wedge R' \subseteq Der(R, X)] \wedge \\ \wedge [Cn(R', X') \subset Cn(R, X) \vee Der(R', X') \subset Der(R, X)]$$

Finally,

**Definition 1.1.** Let $\phi \in S_1$ and $\alpha \in At_1$ and let $v : At_0 \longrightarrow |\mathfrak{M}_2|$ be an arbitrary but fixed valuation in the matrix $\mathfrak{M}_2$ such that $h^v(i(\phi)) = 1$. Then

$$e_\phi(\alpha) = \begin{cases} K\hat{\phi}\alpha, & \text{if } v(i(\alpha)) = 0 \\ C\hat{\phi}\alpha, & \text{if } v(i(\alpha)) = 1 \end{cases}$$

## Section 2

It is a well known fact that on the ground of the classical predicate calculus the following theorems are valid (see [2] and [3]).

**Theorem I.** *Let* $\alpha \in \overline{S}_1$ *and* $X \subseteq S_1$. *Then*

$$\beta \in Cn(R_{0+}, L_2 \cup X \cup \{\alpha\}) \Rightarrow C\alpha\beta \in Cn(R_{0+}, L_2 \cup X).$$

**Theorem II.** *Let* $\phi$, $\psi$, $\delta$, $\alpha$, $\beta \in S_1$ *and* $Q_n \in \{\bigwedge x_n, \bigvee x_n\}$. *Then the following formulas are valid on the ground of the classical predicate calculus:*

(1) $C\hat{\phi}\phi$

(2) $ENN\phi\phi$

(3) $CC\phi\delta CC\phi\psi C\phi K\delta\psi$

(4) $ENK\phi\psi C\phi N\psi$

(5) $EQ_1...Q_k C\phi\psi C\phi Q_1...Q_k\psi$, if $x_1, ..., x_k \notin vf(\phi)$

(6) $E\bigwedge x_k C\phi\psi C\bigvee x_k\phi\psi$, if $x_k \notin vf(\psi)$

(7) $EN\bigvee x_k N\phi NN\bigwedge x_k\phi$

(8) $EN\bigwedge x_k\phi\bigvee x_k N\phi$

(9) $EC\phi\psi CN\psi N\phi$

(10) $EN\bigvee x_k\phi\bigwedge x_k N\phi$

(11) $C\alpha C\beta K\alpha\beta$

(12) $C\alpha\bigvee x_k\alpha$

(13) $C\alpha A\alpha\beta$

(14) $C\alpha A\beta\alpha$

(15) $CC\alpha\beta C\alpha A\delta\beta$

(16) $CC\alpha\beta C\alpha A\beta\delta$

(17) $ECCC\alpha\beta\beta\beta C\alpha\beta$

(18) $E\bigvee x_1...x_k N\phi N\bigwedge x_1...x_k\phi$

(19) $C\alpha C\beta\alpha$

(20) $ENC\alpha\beta K\alpha N\beta$

(21) $CK\alpha\beta\beta$

(22) $CK\alpha\beta\alpha$.

**Theorem III.** $Cn(R_{0+} \cup \{r_*\}, L_2) = L_2$.

**Theorem IV.** *Let* $\pi \subseteq S_1$, $\phi \in S_1$ *and* $\overline{\overline{\pi}} <\aleph_0$. *Then* $\langle R, X\rangle \in SCpl$ *iff* $(\forall e \in \mathcal{E}_*)[h^e(\pi) \subseteq Cn(R,X) \Rightarrow h^e(\phi) \in Cn(R,X)] \Rightarrow \phi \in Cn(R, X \cup \pi)$.

## Section 3

At first we prove

**Lemma 3.1.** *Let* $\phi \in S_1^*$ *and* $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)h^v(i(\phi)) = 1$. *Then* $h^{e_\phi}(\phi) \in L_2$.

**Proof.** Assume that

(1) $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)h^v(i(\phi)) = 1$ and $\phi \in S_1^*$.

Let

(1.1) $\phi \in At_1$.

Hence by (1) and Definition 1.1 we have

(2) $h^{e_\phi}(\phi) = C\hat{\phi}\phi$.

By means of Theorem II (1) we obtain

(3) $h^{e_\phi}(\phi) \in L_2$.

Let

(1.2) $\phi = NP_k^n(x_1, ..., x_n)$.

Hence, from (1) and Definition 1.1 it follows that

(4) $h^{e_\phi}(\phi) = NK\hat{\phi}P_k^n(x_1, ..., x_n)$.

Therefore, by Theorem II (4) and (1.2) we have

(5) $Eh^{e_\phi}(\phi)C\hat{\phi}\phi \in L_2$.

So, using Theorem II (1) and Theorem III we get

(6) $h^{e_\phi}(\phi) \in L_2$.

Let

(1.3) $\phi = A\phi_1\phi_2$

and assume inductively that

$(a_1)$ $h^{e_\phi}(\phi_1) \in L_2$ or $(a_2)$ $h^{e_\phi}(\phi_2) \in L_2$.

From Definition 1.1 it follows that

(7) $h^{e_\phi}(A\phi_1\phi_2) = Ah^{e_\phi}(\phi_1)h^{e_\phi}(\phi_2)$.

Hence, by Theorem II (13) and (14), Theorem III and (1.3), in both cases $(a_1)$ and $(a_2)$, we obtain

(8) $h^{e_\phi}(\phi) \in L_2$.

Let

(1.4) $\phi = K\phi_1\phi_2$

and assume inductively that

(9) $h^{e_\phi}(\phi_1),\ h^{e_\phi}(\phi_2) \in L_2$.

Hence, using Definition 1.1, by Theorem II (11), Theorem III and (1.4) we get

(10) $h^{e_\phi}(\phi) \in L_2$.

Let

(1.5) $\phi = Q_k\phi'$, where $Q_k \in \{\bigwedge x_k, \bigvee x_k\}$

and assume inductively that

(11) $h^{e_\phi}(\phi') \in L_2$.

Using Definition 1.1, by Theorem II (12), Theorem III and (1.5) we obtain

(12) $h^{e_\phi}(\phi) \in L_2$ ,

which completes the proof. ■

**Lemma 3.2.** *Let* $\phi, \psi \in S_1^*$, $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)h^v(i(\phi)) = 1$, $Cn(R_{0+}, L_2 \cup X) = Z_3$ *and* $(\forall e \in \mathcal{E}_*)[h^e(\phi) \in Z_3 \Rightarrow h^e(\psi) \in Z_3]$. *Then* $C\hat{\phi}\psi \in Z_3$.

**Proof.** Assume that

(1) $\phi, \psi \in S_1^*$,

(2) $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)h^v(i(\phi)) = 1$,

(3) $Cn(R_{0+}, L_2 \cup X) = Z_3$,

(4) $(\forall e \in \mathcal{E}_*)[h^e(\phi) \in Z_3 \Rightarrow h^e(\psi) \in Z_3]$.

We have two cases:

$(a_1)$ $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(i(\phi)) = 1 \wedge h^v(i(\psi)) = 0]$

or

$(a_2)$ $(\forall v : At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(i(\phi)) = 1 \Rightarrow h^v(i(\psi)) = 1]$ .

From $(a_1)$ it follows that

(5) $(\exists e_1 \in \mathcal{E}_*)(\forall v : At_0 \longrightarrow |\mathfrak{M}_2|)[h^v(i(h^{e_1}(\phi))) = 1 \wedge h^v(i(h^{e_1}(\psi))) = 0]$

Hence, from (1)–(3), by means of Lemma 3.1 and Theorem III we obtain

(6) $(\forall e \in \mathcal{E}_*)[h^e(h^{e_{\phi'}}(\phi')) \in Z_3]$, where $\phi' = h^{e_1}\phi$.

Hence and from (4) we have

(7) $Sb(h^{e_{\phi'}}(h^{e_1}\psi)) \subseteq Z_3$.

Hence, by Definition 1.1 and (1)–(6) we have (see [3] p.102)

(8) $Cn(R_{0+}, L_2 \cup X) = Z_3 = S_1$.

So

(9) $C\hat{\phi}\psi \in Z_3$ ,

which ends the proof in case $(a_1)$.

In case $(a_2)$, from (1)–(3) by means of Lemma 3.1 we have

(10) $h^{e_\phi}(\phi) \in Z_3$.

Therefore by (4) we get

(11) $h^{e_\phi}(\psi) \in Z_3$ .

Let

(1.1) $\psi \in At_1$ .

Hence, from (1)–(4) in case $(a_2)$ by means of Definition 1.1 we obtain

(12) $h^{e_\phi}(\psi) = C\hat{\phi}\psi$ .

Therefore by (11) we have

(13) $C\hat{\phi}\psi \in Z_3$ .

Let

(1.2) $\psi = NP^n_k(x_1, ..., x_n)$ .

Hence, from (1) - (4) in case $(a_2)$ by means of Definition 1.1 we obtain

(14) $h^{e_\phi}(\psi) = NK\hat{\phi}P_k^n(x_1, ..., x_n)$.

Hence, from (1.2) and (11) by means of Theorem II (4) and Theorem III we have

(15) $C\hat{\phi}\psi \in Z_3$ .

Let

(1.3) $\psi = A\phi_1\phi_2$

and assume inductively that

$(b_1)$ $C\hat{\phi}\phi_1 \in Z_3$ or $(b_2)$ $C\hat{\phi}\phi_2 \in Z_3$.

Hence, from (1) and (3) in case $(b_1)$ and $(b_2)$ by means of Theorem II (15) and (16) we obtain

(16) $C\hat{\phi}\psi \in Z_3$ .

Let

(1.4) $\psi = K\phi_1\phi_2$

and assume inductively that

(17) $C\hat{\phi}\phi_1 \in Z_3$ and $C\hat{\phi}\phi_2 \in Z_3$.

By Theorem II (3) we get

(18) $C\hat{\phi}\psi \in Z_3$.

Let

(1.5) $\psi = Q_n\psi'$, where $Q_n \in \{\bigwedge x_n, \bigvee x_n\}$

and assume inductively that

(19) $C\hat{\phi}\psi' \in Z_3$.

Therefore by means of Theorem II (5) and (3) we have

(20) $C\hat{\phi}\psi \in Z_3$,

which ends the proof in case $(a_2)$. Thus the proof of Lemma 3.2 is completed. ■

**Lemma 3.3.** *Let* $\phi, \psi \in S_1$, $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)h^v(i(\phi)) = 1$, $Cn(R_{0+}, L_2 \cup X) = Z_3$ *and* $(\forall e \in \mathcal{E}_*)[h^e(\phi) \in Z_3 \Rightarrow h^e(\psi) \in Z_3]$. *Then* $C\hat{\phi}\psi \in Z_3$.

**Proof.** By Lemma 3.2 and by the well-known Theorem of Replacement (see [2] p.128). ■

**Lemma 3.4.** *Let* $\alpha_1, \beta_1 \in S_1$, $(\forall e \in \mathcal{E}_*)[h^e(\alpha_1) \in Z_3 \Rightarrow h^e(\beta_1) \in Z_3]$, $Cn(R_{0+}, L_2 \cup X) = Z_3$ *and* $(\forall \alpha \in \overline{Z}_2)[\alpha \in Z_3 \lor N\alpha \in Z_3]$. *Then* $C\hat{\alpha}\beta_1 \in Z_3$.

**Proof.** Assume that

(1) $\alpha_1, \beta_1 \in S_1$ and $Cn(R_{0+}, L_2 \cup X) = Z_3$,

(2) $(\forall \alpha \in \overline{Z}_2)[\alpha \in Z_3 \lor N\alpha \in Z_3]$,

(3) $(\forall e \in \mathcal{E}_*)[h^e(\alpha_1) \in Z_3 \Rightarrow h^e(\beta_1) \in Z_3]$

We have two cases:

$(a_1)$ $\alpha_1 \in Z_3$ or $(a_2)$ $\alpha_1 \notin Z_3$.

In case $(a_2)$ we have the following two possibilities:

$(b_1)$ $(\forall e \in \mathcal{E}_*)[h^e(CC\check{\beta}_1\check{\alpha}_1\check{\alpha}_1) \in Z_3 \Rightarrow h^e(\check{\alpha}_1) \in Z_3]$

or

$(b_2)$ $(\exists e_1 \in \mathcal{E}_*)[h^{e_1}(CC\check{\beta}_1\check{\alpha}_1\check{\alpha}_1) \in Z_3 \land h^{e_1}(\check{\alpha}_1) \notin Z_3]$.

In $(b_1)$ from (1) and (3) by means of Lemma 3.3. we obtain

(5) $C\hat{\alpha}_1\beta_1 \in Z_3$ or $CCC\check{\beta}_1\check{\alpha}_1\check{\alpha}_1\check{\alpha}_1 \in Z_3$.

So in $(a_2)(b_1)$ by means of Theorem II: (1), (2), (9), (17) and Theorem III we have

(6) $C\hat{\alpha}_1\beta_1 \in Z_3$.

Next, in $(a_2)(b_2)$ we have the following possibilities:

$(c_1)$ $Nh^{e_1}(\check{\alpha}_1) \in Z_2^*$ or

$(c_2)$ $h^{e_1}(\check{\alpha}_1) \in Z_2^*$ or

$(c_3)$ $Nh^{e_1}(\check{\alpha}_1) \notin Z_2^* \land h^{e_1}(\check{\alpha}_1) \notin Z_2^*$.

From (2), $(b_2)$ and $(c_1)$ it follows that

(7) $Nh^{e_1}(\check{\alpha}_1) \in Z_3$.

Hence by Theorem II: (1), (2), (7) we get

(8) $h^{e_1}(\alpha_1) \in Z_3$.

Hence and from (3) it follows that

(9) $h^{e_1}(\beta_1) \in Z_3$.

Next, in $(b_2)$ by (1), (7) and Theorem II (9) we get

(10) $Nh^{e_1}(C\check{\beta}_1\check{\alpha}_1) \in Z_3$.

Hence by (1), Theorem II: (18), (20), (22) we have

(11) $Nh^{e_1}(\hat{\beta}_1) \in Z_3$.

From (1), (9) it follows that

(12) $h^{e_1}(\hat{\beta}_1) \in Z_3$.

Hence, in case $(a_2)(b_2)(c_1)$ from (1) and (11) it follows that

(13) $C\hat{\alpha}_1\beta_1 \in Z_3$.

In case $(a_2)(b_2)(c_2)$ by (2) we have

(14) $Nh^{e_1}(\check{\alpha}_1) \in Z_3$.

Hence, analogously as in case $(c_1)$ we get

(15) $C\hat{\alpha}_1\beta_1 \in Z_3$.

In case $(a_2)(b_2)(c_3)$ we obtain

(16) $(\exists v : At_0 \longrightarrow |\mathfrak{M}_2|)h^v(i(\alpha_1)) = 1$.

Hence, from (1) and (3) by Lemma 3.3 we get

(17) $C\hat{\alpha}_1\beta_1 \in Z_3$,

which completes the proof of Lemma 3.4. ■

Finally

**Theorem 3.5.** *Let* $X \subseteq S_1$ *and* $Cn(R_{0+}, L_2 \cup X) = Z_3$. *Then* $(\forall\alpha \in \overline{Z}_2)[\alpha \in Z_3 \ \vee \ N\alpha \in Z_3] \ \Rightarrow \ \langle R_{0+}, L_2 \cup X\rangle \in SCpl$.

**Proof.** By Lemma 3.4. and Theorem IV. ■

The following corollary is a generalisation of a result from [1].

**Corollary 3.6.** $X \subseteq S_1^+ \ \Rightarrow \ \langle R_{0+}, L_2^+ \cup X\rangle \in SCpl$.

**Proof.** By Lemma 3.3., Theorem I and Theorem IV. ■

## Section 4

At first we prove

**Lemma 4.1.** *Let* $X \subseteq S_1$ *and* $Cn(R_{o+}, L_2 \cup X) = Z_3 \neq S_1$. *Then*

$\langle R_{o+}, L_2 \cup X\rangle \in SCpl \;\Rightarrow\; (\forall \alpha \in \overline{Z}_2)[\alpha \in Z_3 \vee N\alpha \in Z_3]$.

**Proof.** Assume that

(1) $X \subseteq S_1$,

(2) $Cn(R_{o+}, L_2 \cup X) = Z_3$,

(3) $Z_3 \neq S_1$,

(4) $\langle R_{o+}, L_2 \cup X\rangle \in SCpl$.

Suppose that

(5) $(\exists \alpha \in \overline{Z}_2)[\alpha \notin Z_3 \wedge N\alpha \notin Z_3]$.

Hence, let

(6) $A_1 = \{\alpha_1, N\alpha_1\}$, where $\alpha_1$ is an arbitrary but fixed formula such that $\alpha_1 \in \overline{Z}_2 - Z_3$ and $N\alpha_1 \notin Z_3$.

Hence

(7) $A_1 = \{\alpha_1, \alpha_2\}$, where $\alpha_2 = N\alpha_1$.

Suppose now that

$$(8)\ \neg(\forall e \in \mathcal{E}_*)(\forall \alpha_j \in A_1)(\exists \alpha_i \in A_1)[(h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \;\Rightarrow\; \Rightarrow\; h^e(N\alpha_j) \in Z_3) \;\Rightarrow\; h^e(N\alpha_i) \in Z_3]$$

Hence

$$(9)\ (\exists e \in \mathcal{E}_*)(\exists \alpha_j \in A_1)(\forall \alpha_i \in A_1)[(h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \;\Rightarrow\; \Rightarrow\; h^e(N\alpha_j) \in Z_3) \wedge h^e(N\alpha_i) \notin Z_3]$$

Hence, and from (5) – (7) we get

$(a_1)$ $\alpha_j = \alpha_1$

or

$(a_2)$ $\alpha_j = N\alpha_1$.

In the case $(a_1)$ from (5) – (7) and (9) we get that there exists $e_1 \in \mathcal{E}_*$ such that

(10) $[h^{e_1}(EN\alpha_1 CN\alpha_1 N\alpha_1) \in Z_3 \;\Rightarrow\; h^{e_1}(N\alpha_1) \in Z_3] \;\wedge\; h^{e_1}(N\alpha_1) \notin Z_3$

and

(11) $[h^{e_1}(E\alpha_1 CN\alpha_1\alpha_1) \in Z_3 \;\Rightarrow\; h^{e_1}(N\alpha_1) \in Z_3] \;\wedge\; h^{e_1}(\alpha_1) \notin Z_3.$

From (2) and (11)

(12) $h^{e_1}(N\alpha_1) \in Z_3.$

From (10)

(13) $h^{e_1}(N\alpha_1) \notin Z_3,$

what contradics (12).

In the case $(a_2)$ from (5) – (7) and (9) we get that there exists $e_1 \in \mathcal{E}_*$ such that

(14) $[h^{e_1}(EN\alpha_1 C\alpha_1 N\alpha_1) \in Z_3 \;\Rightarrow\; h^{e_1}(\alpha_1) \in Z_3] \;\wedge\; h^{e_1}(N\alpha_1) \notin Z_3$

and

(15) $[h^{e_1}(E\alpha_1 C\alpha_1\alpha_1) \in Z_3 \;\Rightarrow\; h^{e_1}(\alpha_1) \in Z_3] \;\wedge\; h^{e_1}(\alpha_1) \notin Z_3.$

From (2) and (14)

(16) $h^{e_1}(\alpha_1) \in Z_3.$

From (15)

(17) $h^{e_1}(\alpha_1) \notin Z_3,$

what contradics (16).

In consequence

(18) $(\forall e \in \mathcal{E}_*)(\forall \alpha_j \in A_1)(\exists \alpha_i \in A_1)[(h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \;\Rightarrow$
$\Rightarrow\; h^e(N\alpha_j) \in Z_3) \;\Rightarrow\; h^e(N\alpha_i) \in Z_3]$

Suppose now that

(19) $\neg(\forall e \in \mathcal{E}_*)(\forall \alpha_i \in A_1)(\exists \alpha_j \in A_1)[((h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \;\Rightarrow$

$\Rightarrow\; h^e(N\alpha_j) \in Z_3) \;\Rightarrow\; h^e(N\alpha_i) \in Z_3) \Longleftrightarrow h^e(E\alpha_i\alpha_j) \notin Z_3]$

Hence

(20) $(\exists e \in \mathcal{E}_*)(\exists \alpha_i \in A_1)(\forall \alpha_j \in A_1)[((h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \;\Rightarrow$
$\Rightarrow\; h^e(N\alpha_j) \in Z_3) \;\Rightarrow\; h^e(N\alpha_i) \in Z_3) \Longleftrightarrow h^e(E\alpha_i\alpha_j) \in Z_3]$

Hence, and from (5) – (7) it follows that

$(a_1)\ \alpha_i = \alpha_1$

or

$(a_2)\ \alpha_i = N\alpha_1$.

In $(a_1)$ from (2) and (20) it follows that there exists $e_1 \in \mathcal{E}_*$ such that

(21) $h^{e_1}(N\alpha_1) \in Z_3$

and

(22) $(h^{e_1}(\alpha_1) \in Z_3 \ \Rightarrow\ h^{e_1}(N\alpha_1) \in Z_3) \ \Rightarrow\ h^{e_1}(\alpha_1) \in Z_3$.

From (2), (21) and (22) it follows that

(23) $h^{e_1}(\alpha_1) \in Z_3$.

From (2), (21) and (23) it follows that

(24) $Z_3 = S_1$,

what contradicts (3).

In $(a_2)$ from (2) and (20) it follows that there exists $e_1 \in \mathcal{E}_*$ such that

(25) $h^{e_1}(\alpha_1) \in Z_3$

and

(26) $(h^{e_1}(N\alpha_1) \in Z_3 \ \Rightarrow\ h^{e_1}(\alpha_1) \in Z_3) \ \Rightarrow\ h^{e_1}(N\alpha_1) \in Z_3$.

From (2), (25) and (26) it follows that

(27) $h^{e_1}(N\alpha_1) \in Z_3$.

From (2), (25) and (27) it follows that

(28) $Z_3 = S_1$,

what contradicts (3).

Thus,

(29) $$(\forall e \in \mathcal{E}_*)(\forall \alpha_i \in A_1)(\exists \alpha_j \in A_1)[((h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \ \Rightarrow \\ \Rightarrow\ h^e(N\alpha_j) \in Z_3) \ \Rightarrow\ h^e(N\alpha_i) \in Z_3) \Longleftrightarrow h^e(E\alpha_i\alpha_j) \notin Z_3]$$

From (18) and (29)

(30) $$(\forall e \in \mathcal{E}_*)(\forall \alpha_i \in A_1)(\exists \alpha_j \in A_1)(\exists \alpha_i' \in A_1) \\ [(((h^e(EN\alpha_i CN\alpha_j N\alpha_i) \in Z_3 \ \Rightarrow\ h^e(N\alpha_j) \in Z_3) \ \Rightarrow\ h^e(N\alpha_i) \in Z_3) \Longleftrightarrow \\ \Longleftrightarrow h^e(E\alpha_i\alpha_j) \notin Z_3) \ \wedge\ ((h^e(EN\alpha_i' CN\alpha_i N\alpha_i') \in Z_3 \ \Rightarrow \\ \Rightarrow\ h^e(N\alpha_i) \in Z_3) \ \Rightarrow\ h^e(N\alpha_i') \in Z_3)]$$

Hence and from (5) – (7) it follows that

(I) $\alpha_i = \alpha_1$

and

(II) $\alpha_i = N\alpha_1$.

In the situation (I) from (30)

(31) $(\forall e \in \mathcal{E}_*)(\exists \alpha_j \in A_1)(\exists \alpha_i' \in A_1)[(((h^e(EN\alpha_1 CN\alpha_j N\alpha_1) \in Z_3 \;\Rightarrow$

$$\Rightarrow\; h^e(N\alpha_j) \in Z_3) \;\Rightarrow\; h^e(N\alpha_1) \in Z_3) \Longleftrightarrow h^e(E\alpha_1\alpha_j) \notin Z_3) \wedge$$
$$\wedge\; ((h^e(EN\alpha_i' CN\alpha_1 N\alpha_i') \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3) \;\Rightarrow\; h^e(N\alpha_i') \in Z_3)]$$

From (2), (6), (7) and (31)

(32) $(\forall e \in \mathcal{E}_*^0)[h^e(N\alpha_1) \in Z_3 \;\Rightarrow\; h^e(\alpha_1) \in Z_3]$, when $\alpha_j = \alpha_1$,

(33) $(\forall e \in \mathcal{E}_*^1)[h^e(\alpha_1) \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3]$, when $\alpha_j = N\alpha_1$.

From (31) – (33)

(34) $\mathcal{E}_*^0 \cup \mathcal{E}_*^1 = \mathcal{E}_*$

and

(35) $\alpha_1 \in A_1$.

From (34)

$(a_1)$ $\mathcal{E}_*^0 = \emptyset$

or

$(a_2)$ $\mathcal{E}_*^0 \neq \emptyset$.

In $(a_1)$ from (32) – (35)

(36) $(\forall e \in \mathcal{E}_*)[h^e(\alpha_1) \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3]$.

From (1), (2), (4) – (7) and (36) we obtain by means of Theorem I and Theorem IV that

(37) $N\alpha_1 \in Z_3$.

what contradicts (5) – (7).

Thus

(38) $\mathcal{E}_*^0 \neq \emptyset$.

Next from (34)

$(b_1)\ \mathcal{E}^1_* = \emptyset$

or

$(b_2)\ \mathcal{E}^1_* \neq \emptyset.$

The case $(b_1)$ is excluded by the analogous reasoning as in the case $(a_1)$. Thus,

(39) $\mathcal{E}^1_* \neq \emptyset.$

Hence, from (5) – (7), (31) and (33) if follows that

$(c_1)\ \alpha_j = N\alpha_1 \ \ \wedge \ \ \alpha'_i = N\alpha_1$

or

$(c_2)\ \alpha_j = N\alpha_1 \ \ \wedge \ \ \alpha'_i = \alpha_1.$

In $(c_1)$ from (2) and (31)

(40) $(\forall e \in \mathcal{E}^2_*)[h^e(N\alpha_1) \in Z_3 \ \Rightarrow \ h^e(\alpha_1) \in Z_3].$

In $(c_2)$ from (2) and (31)

(41) $(\forall e \in \mathcal{E}^3_*)[h^e(N\alpha_1) \in Z_3].$

From (31), (33), (39), (40), (41), $(c_1)$ and $(c_2)$ it follows that

(42) $\mathcal{E}^2_* \cup \mathcal{E}^3_* = \mathcal{E}^1_* \neq \emptyset.$

Next in $(c_1)$ from (32) and (40)

(43) $(\forall e \in \mathcal{E}^0_* \cup \mathcal{E}^2_*)[h^e(N\alpha_1) \in Z_3 \ \Rightarrow \ h^e(\alpha_1) \in Z_3].$

From (34) and (42)

$(d_1)\ \mathcal{E}^3_* = \emptyset$

or

$(d_2)\ \mathcal{E}^3_* \neq \emptyset.$

In $(d_1)$ from (42) it follows that

(44) $\mathcal{E}^2_* = \mathcal{E}^1_*.$

Hence in $(d_1)$ from (32), (34), (42) and (43) it follows that

(45) $(\forall e \in \mathcal{E}_*)[h^e(N\alpha_1) \in Z_3 \ \Rightarrow \ h^e(\alpha_1) \in Z_3].$

From (1), (2), (4) – (7) and (45) we obtain by means of Theorem I and Theorem IV that

(46) $\alpha_1 \in Z_3.$

what contradicts (5) – (7).

Thus,

(47) $\mathcal{E}_*^3 \neq \emptyset$.

In the case $(d_2)$ from (34), (42) and (43) it follows that

(48) $(\forall e \in \mathcal{E}_*^4)[h^e(N\alpha_1) \in Z_3 \;\Rightarrow\; h^e(\alpha_1) \in Z_3]$.

and

(49) $\mathcal{E}_*^0 \cup \mathcal{E}_*^2 = \mathcal{E}_*^4$

and

(50) $\mathcal{E}_*^4 \cup \mathcal{E}_*^3 = \mathcal{E}_*$.

In the situation (II) from (30)

(51) $(\forall e \in \mathcal{E}_*)(\exists \alpha_j \in A_1)(\exists \alpha_i' \in A_1)[(((h^e(E\alpha_1 CN\alpha_j \alpha_1) \in Z_3 \;\Rightarrow$
$\Rightarrow\; h^e(N\alpha_j) \in Z_3) \;\Rightarrow\; h^e(\alpha_1) \in Z_3) \Longleftrightarrow h^e(EN\alpha_1\alpha_j) \notin Z_3) \;\wedge$
$\wedge\; ((h^e(EN\alpha_i' C\alpha_1 N\alpha_i') \in Z_3 \;\Rightarrow\; h^e(\alpha_1) \in Z_3) \;\Rightarrow\; h^e(N\alpha_i') \in Z_3)]$

From (2), (6), (7) and (51)

(52) $(\forall e \in \mathcal{E}_*^5)[h^e(\alpha_1) \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3]$, when $\alpha_j = N\alpha_1$,

(53) $(\forall e \in \mathcal{E}_*^6)[h^e(N\alpha_1) \in Z_3 \;\Rightarrow\; h^e(\alpha_1) \in Z_3]$, when $\alpha_j = \alpha_1$.

From (51) – (53)

(54) $\mathcal{E}_*^5 \cup \mathcal{E}_*^6 = \mathcal{E}_*$

and

(55) $\alpha_1 \in A_1$.

From (54)

$(e_1)$ $\mathcal{E}_*^5 = \emptyset$

or

$(e_2)$ $\mathcal{E}_*^5 \neq \emptyset$.

In $(e_1)$ from (51) – (55)

(56) $(\forall e \in \mathcal{E}_*)[h^e(N\alpha_1) \in Z_3 \;\Rightarrow\; h^e(\alpha_1) \in Z_3]$.

From (1), (2), (4) – (7) and (56) we obtain by means of Theorem I and Theorem IV that

(57) $\alpha_1 \in Z_3$.

what contradicts (5) – (7).

Thus

(58) $\mathcal{E}_*^5 \neq \emptyset$.

Next from (54)

($f_1$) $\mathcal{E}_*^6 = \emptyset$

or

($f_2$) $\mathcal{E}_*^6 \neq \emptyset$.

The case ($f_1$) is excluded by the analogous reasoning as in the case ($e_1$). Thus,

(59) $\mathcal{E}_*^6 \neq \emptyset$.

Hence, from (5) – (7), (51) and (53) if follows that

($g_1$) $\alpha_j = \alpha_1 \quad \wedge \quad \alpha'_i = \alpha_1$

or

($g_2$) $\alpha_j = \alpha_1 \quad \wedge \quad \alpha'_i = N\alpha_1$.

In ($g_1$) from (51)

(60) $(\forall e \in \mathcal{E}_*^7)[h^e(\alpha_1) \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3]$.

In ($g_2$) from (51)

(61) $(\forall e \in \mathcal{E}_*^8)[h^e(\alpha_1) \in Z_3]$.

From (51), (53), (60), (61), ($g_1$) and ($g_2$) it follows that

(62) $\mathcal{E}_*^7 \cup \mathcal{E}_*^8 = \mathcal{E}_*^6$.

From (54), (59) and (62) it follows that

($h_1$) $\mathcal{E}_*^8 = \emptyset$

or

($h_2$) $\mathcal{E}_*^8 \neq \emptyset$.

In the case ($h_1$) from (62)

(63) $\mathcal{E}_*^7 = \mathcal{E}_*^6$.

Hence, in the case ($h_1$) from (51), (52), (54) and (60) it follows that

(64) $(\forall e \in \mathcal{E}_*)[h^e(\alpha_1) \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3]$.

From (1), (2), (4) – (7) and (64) we obtain by means of Theorem I and Theorem IV that

(65) $N\alpha_1 \in Z_3$.

what contradicts (5) – (7).

Thus

(66) $\mathcal{E}_*^8 \neq \emptyset$.

Hence in the case ($h_2$) from (51), (52), (58) and (60) it follows that

(67) $(\forall e \in \mathcal{E}_*^9)[h^e(\alpha_1) \in Z_3 \;\Rightarrow\; h^e(N\alpha_1) \in Z_3]$.

and

(68) $\mathcal{E}_*^5 \cup \mathcal{E}_*^7 = \mathcal{E}_*^9 \neq \emptyset$.

From (54), (62) and (68)

(69) $\mathcal{E}_*^9 \cup \mathcal{E}_*^8 = \mathcal{E}_*$.

From (2), (3), (32), (38), (41), (47), (48) and (49) it follows that

(70) $\mathcal{E}_*^4 \cap \mathcal{E}_*^3 = \emptyset$.

From (2), (3), (41), (47), (61) and (66) it follows that

(71) $\mathcal{E}_*^3 \cap \mathcal{E}_*^8 = \emptyset$.

From (2), (3), (61), (66), (67) and (68) it follows that

(72) $\mathcal{E}_*^9 \cap \mathcal{E}_*^8 = \emptyset$.

From (50) and (69)

(73) $\mathcal{E}_*^4 \cup \mathcal{E}_*^3 = \mathcal{E}_*^9 \cup \mathcal{E}_*^8$.

Hence

(74) $\mathcal{E}_*^8 = (\mathcal{E}_*^4 \cup \mathcal{E}_*^3) - \mathcal{E}_*^9$.

Hence, from (71) and (72)

(75) $\mathcal{E}_*^8 \subseteq \mathcal{E}_*^4$.

Hence, from (50) and (69)

(76) $\mathcal{E}_*^4 \cup \mathcal{E}_*^9 = \mathcal{E}_*$.

From (50) and (76)

(77) $\mathcal{E}_*^4 \cup \mathcal{E}_*^3 = \mathcal{E}_*^4 \cup \mathcal{E}_*^9$.

Hence

(78) $\mathcal{E}_*^3 = \mathcal{E}_*^9$.

Hence, from (69)

(79) $\mathcal{E}_*^8 \cup \mathcal{E}_*^3 = \mathcal{E}_*$.

Hence, from (41) and (61)

(80) $\alpha_1 \in Z_3$

or

(81) $N\alpha_1 \in Z_3$,

what contradicts (5) – (7). This ends the proof. ■

Finally,

**Theorem 4.2.** *Let $X \subseteq S_1$ and $Cn(R_{o+}, L_2 \cup X) = Z_3$. Then*

$$\langle R_{o+}, L_2 \cup X\rangle \in SCpl \quad \text{iff} \quad (\forall \alpha \in \overline{Z}_2)[\alpha \in Z_3 \;\; \vee \;\; N\alpha \in Z_3].$$

**Proof.** By Theorem 3.5 and Lemma 4.1. ■

**Theorem 4.3.** *Let $X \subseteq S_1$. Then $\langle R_{o+}, L_2 \cup X\rangle \in SCpl$ iff*

$$(\forall Y \subseteq S_1)[\langle R_{0+}, L_2 \cup X \cup Y\rangle \in SCpl].$$

**Proof.** By Theorem 4.2. ■

**Corollary 4.4.** $Cn(R_{o+}, L_2 \cup X) \subset Z_2^* \;\Rightarrow\; \langle R_{o+}, L_2 \cup X\rangle \notin SCpl$.

**Proof.** By Theorem 4.2. ■

**Corollary 4.5.** *There exists only one system $\langle R, X\rangle \; > \; \langle R_{0+}, L_2\rangle$ such that $\langle R, X\rangle \in SCpl \cap Inw$ and $Cn(R, X) \subset S_1$ i.e.*

$$\langle R, X\rangle = \langle R_{0+}, Z_2^*\rangle.$$

**Proof.** By Theorem 4.2. and Corollary 4.4. ■

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Pedagogical College
ul. Podchorążych 2
30-084 Kraków
Poland