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FORMALIZATIONS FOR THE CONSEQUENCE RELATION OF VISSER'S PROPOSITIONAL LOGIC

A b s t r a c t. Visser's propositional logic (**VPL**) was first considered in Visser [5] as the propositional logic embedded into the modal logic **K4** by Gödel's translation. He gave a natural deduction system $\vdash_V$ for the consequence relation of **VPL**. An essential difference from the consequence relation $\vdash_I$ of intuitionistic propositional logic is $\{a, a \supset b\} \nvdash_V b$ while $\{a, a \supset b\} \vdash_I b$. In other words, in $\vdash_V$, modus ponens does not hold in general. So, we may well say that systems for the consequence relation of **VPL** are obtained from the systems for $\vdash_I$ by replacing the inference rule, which corresponds to modus ponens, by other inference rules. For instance, Visser's system $\vdash_V$ was obtained from Gentzen's natural deduction system $\vdash_{NJ}$ by replacing implication elimination rule ($\supset E$) by three other inference rules. Ardeshir [1] and [2] gave sequent style systems for $\vdash_V$ by replacing the inference rule ($\supset\rightarrow$) in Gentzen's sequent system **LJ** by other inference rules. However, it seems difficult to give a finite Hilbert style system for the consequence relation of **VPL** because modus pones in Hilbert style system for $\vdash_I$ has the role of translating axioms into inference rules, and so, we have to find another inference rule having the same role in $\vdash_V$. This problem was raised in Suzuki, Wolter and Zakharyaschev [4]. Here we introduce a restricted modus

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ponens, which mostly has the same role as modus ponens in Hilbert style system for $\vdash_I$ has. Using the restricted modus ponens and adjunction, we give a formalization for $\vdash_V$, and at the same time we show that $\vdash_V$ can not be formalized by any systems with a restricted modus ponens as only one inference rule.

1. Preliminary

We use lower case Latin letters for propositional variables. Formulas are defined, as usual, from the propositional variables and the logical constant $\bot$ (contradiction) by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\supset$ (implication). We assume $\wedge$ and $\vee$ to connect stronger than $\supset$ and omit those brackets that can be recovered according to this priority of connectives. We use upper case Latin letters, possibly with suffixes, for formulas. We use Greek letters Γ and Σ for finite sets of formulas. The expression $\top$ is an abbreviation for $\bot \supset \bot$.

The natural deduction system $\vdash_V$ introduced in [5] consists of the following inference rules.

$$(\bot E)\frac{\bot}{A} \quad (\wedge I)\frac{A \quad B}{A \wedge B} \quad (\wedge E_1)\frac{A \wedge B}{A} \quad (\wedge E_2)\frac{A \wedge B}{B}$$

$$(\vee I_1)\frac{A}{A \vee B} \quad (\vee I_2)\frac{B}{A \vee B} \quad (\vee E)\frac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C} \quad (\supset I)\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}$$

$$(\wedge I_f)\frac{A \supset B \quad A \supset C}{A \supset B \wedge C} \quad (\vee E_f)\frac{A \supset C \quad B \supset C}{A \vee B \supset C} \quad (Tr)\frac{A \supset B \quad B \supset C}{A \supset C}$$

The consequence relation $\vdash_V$ is defined by the axiom

(1) $\Gamma \vdash_V A$ if $A \in \Gamma$

and the above inference rules inductively. We put $\mathbf{VPL} = \{X | \emptyset \vdash_V X\}$. A Hilbert style formalization for **VPL** has been given in Suzuki and Ono [3] as follows.

Lemma 1.1. *The closure under modus ponens and substitution of the set of following 12 axioms coincides with* **VPL***:*

$(\supset_1)$	$a \supset a$,
$(\supset_2)$	$a \supset (b \supset a)$,
$(\supset_3)$	$(b \supset c) \wedge (a \supset b) \supset (a \supset c)$,
$(\wedge_1)$	$a \wedge b \supset a$,
$(\wedge_2)$	$a \wedge b \supset b$,
$(\wedge_3)$	$(c \supset a) \wedge (c \supset b) \supset (c \supset a \wedge b)$,
$(\wedge_4)$	$a \supset (b \supset a \wedge b)$,
$(\vee_1)$	$a \supset a \vee b$,
$(\vee_2)$	$b \supset a \vee b$,
$(\vee_3)$	$(a \supset c) \wedge (b \supset c) \supset (a \vee b \supset c)$,
$(\vee_4)$	$a \wedge (b \vee c) \supset (a \wedge b) \vee (a \wedge c)$,
$(\bot)$	$\bot \supset a$.

[5] also showed the completeness theorem for $\vdash_V$ using Kripke models defined below. A Kripke model is a triple $M = \langle W, R, P\rangle$, where R is a transitive binary relation on a set $W \neq \emptyset$ and P is a mapping from the set of all propositional variables to the set

$$\{S \in 2^W \mid \text{if } \alpha R \beta \text{ and } \alpha \in S, \text{ then } \beta \in S\}.$$

The truth valuation $\models$ is defined in the following way:

(1) $(M, \alpha) \models a$ if and only if $\alpha \in P(a)$,

(2) $(M, \alpha) \not\models \bot$,

(3) $(M, \alpha) \models A \wedge B$ if and only if $(M, \alpha) \models A$ and $(M, \alpha) \models B$,

(4) $(M, \alpha) \models A \vee B$ if and only if $(M, \alpha) \models A$ or $(M, \alpha) \models B$,

(5) $(M, \alpha) \models A \supset B$ if and only if for any $\beta \in \{\gamma \in W | \alpha R \gamma\}$, $(M, \beta) \models A$ implies $(M, \beta) \models B$.

The expression $M \models A$ denotes $(M, \alpha) \models A$ for every $\alpha \in W$. We write $(M, \alpha) \models \Gamma$ if $(M, \alpha) \models A$ for every $A \in \Gamma$.

Lemma 1.2([5]). *(1)* **VPL** $= \{A|$ *for every Kripke model* $M, M \models A\}$. *(2)* $\Gamma \vdash_V A$ *if and only if for every* M *and every* $\alpha \in W$,

$$(M, \alpha) \models \Gamma \text{ implies } (M, \alpha) \models A.$$

From Lemma 1.2, we can see that $\{a, a \supset b\} \not\vdash_V b$ by a Kripke model $\langle\{\alpha\}, \emptyset, P\rangle$, where $P(a) = \{\alpha\}, P(b) = \emptyset$. So, in $\vdash_V$, modus ponens does not hold in general.

In the next section, we give a formalization for the consequence relation of **VPL** with axioms described in Lemma 1.1 and with two inference rules, adjunction and a restricted modus ponens. In section 3, we prove that there is no formalization for $\vdash_V$ with a restricted modus ponens as only one inference rule.

2. A formalization for the consequence relation of VPL

In this section, we give a formalization for $\vdash_V$. By **A**, we mean the set of all substitution instances of axioms described in Lemma 1.1 except $(\wedge_4)$.

Definition 2.1. We define the consequence relation $\vdash_{V^*}$ inductively as follows:

(axi) if $A \in \mathbf{A}$, then $\Gamma \vdash_{V^*} A$,

(asp) if $A \in \Gamma$, then $\Gamma \vdash_{V^*} A$,

(rmp) if $\Gamma \vdash_{V^*} A$ and $\emptyset \vdash_{V^*} A \supset B$, then $\Gamma \vdash_{V^*} B$,

(adj) if $\Gamma \vdash_{V^*} A$ and $\Gamma \vdash_{V^*} B$, then $\Gamma \vdash_{V^*} A \wedge B$.

Our main theorem in this section is

Theorem 2.2. $\Gamma \vdash_{V^*} A$ *if and only if* $\Gamma \vdash_V A$.

In order to prove the above theorem, we show some lemmas.

Lemma 2.3.

(1) *if* $\Sigma \subseteq \Gamma$ *and* $\Sigma \vdash_{V^*} A$*, then* $\Gamma \vdash_{V^*} A$*,*

(2) *if* $\Gamma \vdash_{V^*} A$ *and* $\Sigma \cup \{A\} \vdash_{V^*} B$*, then* $\Gamma \cup \Sigma \vdash_{V^*} B$*.*

Proof. (1) is trivial. We show only (2). We use an induction on the number of inference rules used in the proof of $\Sigma \cup \{A\} \vdash_{V^*} B$.

If $B \in \mathbf{A} \cup \Sigma$, then (2) is obvious.

If $B = A$, then we have $\Gamma \vdash_{V^*} B$. So, $\Gamma \cup \Sigma \vdash_{V^*} B$.

If $\Sigma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Sigma \cup \{A\} \vdash_{V^*} C \text{ and } \emptyset \vdash_{V^*} C \supset B$$

for some C by (rmp) then, by the induction hypothesis, we have $\Gamma \cup \Sigma \vdash_{V^*} C$. Using (rmp), we obtain the lemma.

If $\Sigma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Sigma \cup \{A\} \vdash_{V^*} C \text{ and } \Sigma \cup \{A\} \vdash_{V^*} D$$

for some C and D such that $B = C \wedge D$ by (adj) then, by the induction hypothesis, we have $\Gamma \cup \Sigma \vdash_{V^*} C$ and $\Gamma \cup \Sigma \vdash_{V^*} D$. Using (adj), we obtain the lemma. ■

Lemma 2.4. $\Gamma \vdash_{V^*} A$ *implies* $\Gamma \vdash_V A$*.*

Proof. From Lemma 1.1, $\mathbf{A} \subseteq \mathbf{VPL}$ and we can easily check that (rmp) and (adj) hold in every Kripke model. Using Lemma 1.2, we obtain the lemma. ■

We put $(\wedge\emptyset) = \top$ and by $(\wedge\Gamma)$, we mean a conjunction of all the formulas in Γ if $\Gamma \neq \emptyset$.

Lemma 2.5. $\emptyset \vdash_{V^*} (\wedge\Gamma) \supset B$ *implies* $\Gamma \vdash_{V^*} B$*.*

Proof. If $\Gamma = \emptyset$, then we have $\Gamma \vdash_{V^*} (\wedge\Gamma)$ from the axiom $(\supset_1)$. If not, we also have $\Gamma \vdash_{V^*} (\wedge\Gamma)$ using (adj), possibly several times. Using $\emptyset \vdash_{V^*} (\wedge\Gamma) \supset B$ and (rmp), we obtain the Lemma. ■

By the above Lemma, Lemma 2.3.(2) and the axioms $(\supset_2)$, $(\supset_3)$, $(\wedge_3)$ and $(\vee_3)$, we have

Corollary 2.6. *The following rules hold in* $\vdash_{V^*}$*:*

$(R\supset_2)$ *if* $\Gamma \vdash_{V^*} A$*, then* $\Gamma \vdash_{V^*} B \supset A$*,*

$(R\supset_3)$ *if* $\Gamma \vdash_{V^*} B \supset C$ *and* $\Gamma \vdash_{V^*} A \supset B$*, then* $\Gamma \vdash_{V^*} A \supset C$*,*

$(R\wedge_3)$ *if* $\Gamma \vdash_{V^*} C \supset A$ *and* $\Gamma \vdash_{V^*} C \supset B$*, then* $\Gamma \vdash_{V^*} C \supset A \wedge B$*,*

$(R\vee_3)$ *if* $\Gamma \vdash_{V^*} A \supset C$ *and* $\Gamma \vdash_{V^*} B \supset C$*, then* $\Gamma \vdash_{V^*} A \vee B \supset C$*.*

Lemma 2.7. *If* $\Gamma \cup \{A\} \vdash_{V^*} B$*, then* $\Gamma \vdash_{V^*} A \supset B$*.*

Proof. We use an induction on the number of inference rules used in the proof of $\Gamma \cup \{A\} \vdash_{V^*} B$.

If $B = A$, then $A \supset B \in \mathbf{A}$. So, we have $\Gamma \vdash_{V^*} A \supset B$.

If $B \in \mathbf{A} \cup \Gamma$, then we have $\Gamma \vdash_{V^*} B$. So, using $(R\supset_2)$, we obtain the lemma.

If $\Gamma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Gamma \cup \{A\} \vdash_{V^*} C \text{ and } \emptyset \vdash_{V^*} C \supset B$$

for some C by (rmp) then, by the induction hypothesis and Lemma 2.3(1), we have

$$\Gamma \vdash_{V^*} A \supset C \text{ and } \Gamma \vdash_{V^*} C \supset B.$$

Using $(R\supset_3)$, we obtain the lemma.

If $\Gamma \cup \{A\} \vdash_{V^*} B$ is derived from

$$\Gamma \cup \{A\} \vdash_{V^*} C \text{ and } \Gamma \cup \{A\} \vdash_{V^*} D$$

for some C and D such that $B = C \wedge D$ by (adj) then, by the induction hypothesis, we have

$$\Gamma \vdash_{V^*} A \supset C \text{ and } \Gamma \vdash_{V^*} A \supset D.$$

Using $(R\wedge_3)$, we obtain the lemma. ■

Here we can see $\emptyset \vdash_{V^*} a \supset (b \supset a \wedge b)$ by (adj) and the above Lemma. Hence we confirm that $(\wedge_4)$ does not necessarily belong to $\mathbf{A}$.

Lemma 2.8. $A \in \Gamma$ *implies* $(\wedge\Gamma) \vdash_{V^*} A$.

Proof. Using the axioms $(\wedge_1)$ and $(\wedge_2)$ and (rmp), possibly several times, we obtain the Lemma. ∎

Lemma 2.9. $\Gamma \vdash_V A$ *implies* $\Gamma \vdash_{V^*} A$.

Proof. We use an induction on the number of inference rules used in the proof of $\Gamma \vdash_V A$.

If $A \in \Gamma$, then the lemma is trivial.

Suppose that $\Gamma \vdash_V A$ is proved using at least one inference rule. Let I be the inference rule that introduces $\Gamma \vdash_V A$. If I is either one of the inference rules

$$(\bot E), (\wedge E_1), (\wedge E_2), (\vee I_1) \text{ and } (\vee I_2),$$

then we obtain the lemma by the induction hypothesis, (rmp) and the corresponding axioms

$$(\bot), (\wedge_1), (\wedge_2), (\vee_1) \text{ and } (\vee_2),$$

respectively. If I is $(\wedge I)$, then the lemma follows from (adj) and the induction hypothesis. If I is $(\supset I)$, then the lemma follows from Lemma 2.7 and the induction hypothesis. If I is either one of the inference rules $(\wedge I_f), (\vee E_f)$ and (Tr), then we obtain the lemma by the induction hypothesis and inference rules in Corollary 2.6.

The remaining case is that I is $(\vee E)$. I is of the form $\dfrac{B \vee C \quad \begin{matrix}[B] \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[C] \\ \vdots \\ A\end{matrix}}{A}$. By the induction hypothesis, we have

$$\Gamma \cup \{B\} \vdash_{V^*} A.$$

On the other hand, from Lemma 2.8, we have

$$\{(\wedge\Gamma) \wedge B\} \vdash_{V^*} D$$

for any $D \in \Gamma \cup \{B\}$. Using Lemma 2.3(2),

$$\{(\wedge\Gamma) \wedge B\} \vdash_{V^*} A.$$

Using Lemma 2.7,

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge B \supset A.$$

Similarly, we have

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge C \supset A.$$

Using $(R\vee_3)$,

$$\emptyset \vdash_{V^*} ((\wedge\Gamma) \wedge B) \vee ((\wedge\Gamma) \wedge C) \supset A.$$

By $(\vee_4)$, we also have

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge (B \vee C) \supset ((\wedge\Gamma) \wedge B) \vee ((\wedge\Gamma) \wedge C).$$

Using $(R \supset_3)$,

$$\emptyset \vdash_{V^*} (\wedge\Gamma) \wedge (B \vee C) \supset A.$$

Using Lemma 2.5,

$$\Gamma \cup \{B \vee C\} \vdash_{V^*} A.$$

By the induction hypothesis, we also have

$$\Gamma \vdash_{V^*} B \vee C.$$

Hence, using Lemma 2.3(2),

$$\Gamma \vdash_{V^*} A.$$

■

Now, Theorem 2.2 follows from Lemma 2.4 and Lemma 2.9.

Corollary 2.10. $\{A | \emptyset \vdash_V A\} = \{A | \emptyset \vdash_{V^*} A\}$.

By modifying the system $\vdash_{V^*}$, we can easily define a system $\vdash_{V^{**}}$ for the consequence relation of **VPL** with only one inference rule.

Definition 2.11. We define the consequence relation $\vdash_{V^{**}}$ inductively as follows:

(axi) if $A \in \mathbf{A}$, then $\Gamma \vdash_{V^{**}} A$,

(asp) if $A \in \Gamma$, then $\Gamma \vdash_{V^{**}} A$,

(rmp*) if $\Gamma \vdash_{V^{**}} A_1, \Gamma \vdash_{V^{**}} A_2$ and $\emptyset \vdash_{V^{**}} A_1 \wedge A_2 \supset B$, then $\Gamma \vdash_{V^{**}} B$.

Lemma 2.12. $\Gamma \vdash_{V^{**}} A$ *if and only if* $\Gamma \vdash_{V^{*}} A$.

Proof. We show "if" part. It is sufficient to show that (rmp) and (adj) hold in $\vdash_{V^{**}}$. By $\emptyset \vdash_{V^{**}} A \wedge B \supset A \wedge B$ and (rmp*), we can see that (adj) holds in $\vdash_{V^{**}}$. From the following proof, we can also see that (rmp) holds in $\vdash_{V^{**}}$.

(1) $\Gamma \vdash_{V^{**}} A$ assumption,

(2) $\emptyset \vdash_{V^{**}} A \supset B$ assumption,

(3) $\emptyset \vdash_{V^{**}} A \wedge A \supset A$ $(\wedge_1)$,

(4) $\emptyset \vdash_{V^{**}} (A \supset B) \wedge (A \wedge A \supset A) \supset (A \wedge A \supset B)$ $(\supset_3)$,

(5) $\emptyset \vdash_{V^{**}} A \wedge A \supset B$ (2),(3),(4), (rmp*),

(6) $\Gamma \vdash_{V^{**}} B$ (1),(5) (rmp*). ■

3. Restricted modus ponens and $\vdash_V$

In the previous section, we show that $\vdash_V$ can be formalized by the inference rules (rmp), a restricted modus ponens, and (adj). Also by (rmp*) alone. Here we prove that $\vdash_V$ can not be formalized by any restricted modus ponens as only one inference rule. As a corollary, we find that (adj) is not redundant in $\vdash_{V^*}$.

First of all, we have to make the meaning of "restricted modus ponens" clear. By a restricted modus ponens, we mean an inference rule obtained from modus ponens, i.e.,

(mp) for any pair (A, B) of formulas,

$$\text{if } \Gamma \vdash A \text{ and } \Gamma \vdash A \supset B, \text{ then } \Gamma \vdash B$$

by restricting the domain of the pair (A, B) of variables. For instance, the inference rule

(rmp′) for any pair $(A, B) \in \{(X, Y) | X \supset Y \in \mathbf{VPL}\}$,

$$\text{if } \Gamma \vdash A \text{ and } \Gamma \vdash A \supset B, \text{ then } \Gamma \vdash B$$

is a restricted modus ponens. Since the inference rule (rmp) in Definition 2.1 is equivalent to the above inference rule, we might as well say that (rmp) is a restricted modus ponens.

Definition 3.1. Let $\mathbf{S}$ be a set of formulas and let $\mathbf{MP}$ be a set of pairs of formulas. We define the consequence relation $\vdash_{\mathbf{S},\mathbf{MP}}$ inductively as follows:

(AXI) if $A \in \mathbf{S}$, then $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} A$,
(ASP) if $A \in \Gamma$, then $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} A$,
(RMP) for any pair $(A, B) \in \mathbf{MP}$,

$$\text{if } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A \text{ and } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A \supset B, \text{ then } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} B.$$

Our main theorem in this section is

Theorem 3.2. *There exists no pair* $(\mathbf{S}, \mathbf{MP})$ *satisfying that for any* Γ *and any* A,

$$\Gamma \vdash_V A \textit{ if and only if } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A.$$

To prove the above theorem, we provide some preparations. It is easily seen that if $\Sigma \subseteq \Gamma$ and $\Sigma \vdash_{\mathbf{S},\mathbf{MP}} A$, then $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} A$.

Lemma 3.3. *Let* $\mathbf{MP}_1 \subseteq \mathbf{MP}_2$ *and* $\mathbf{S}_1 \subseteq \mathbf{S}_2$. *Then*

$$\Gamma \vdash_{\mathbf{S}_1,\mathbf{MP}_1} A \textit{ implies } \Gamma \vdash_{\mathbf{S}_2,\mathbf{MP}_2} A.$$

Proof. Every axiom in $\vdash_{\mathbf{S}_1,\mathbf{MP}_1}$ is also an axiom in $\vdash_{\mathbf{S}_2,\mathbf{MP}_2}$. And the inference rule in $\vdash_{\mathbf{S}_1,\mathbf{MP}_1}$ holds in $\vdash_{\mathbf{S}_2,\mathbf{MP}_2}$. ■

Let us consider the consequence relation $\vdash_{\mathbf{VPL},\mathbf{MP}_V}$, where

$$\mathbf{MP}_V = \{(X,Y) | \text{ for any } \Gamma \text{ if } \Gamma \vdash_V X \text{ and } \Gamma \vdash_V X \supset Y, \text{ then } \Gamma \vdash_V Y\}.$$

The following lemma is almost immediate.

Lemma 3.4. $\Gamma \vdash_{\mathbf{VPL},\mathbf{MP}_V} A$ *implies* $\Gamma \vdash_V A$.

Lemma 3.5. $\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} A \supset B$.

Proof. It is easily seen that $\emptyset \vdash_V A \supset \top$ and $\emptyset \vdash_V \bot \supset B$. Using (Tr) twice, we have

(1) $\{\top \supset \bot\} \vdash_V A \supset B$.

Using Lemma 2.3(2), $\Gamma \vdash_V \top \supset \bot$ implies $\Gamma \vdash_V A \supset B$, and so, we have $(\top \supset \bot, A \supset B) \in \mathbf{MP}_V$. On the other hand, by (1) and $(\supset I)$, we have $(\top \supset \bot) \supset (A \supset B) \in \mathbf{VPL}$. So, we have

$$\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} (\top \supset \bot) \supset (A \supset B).$$

We also have $\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} \top \supset \bot$. Using (RMP), we have $\{\top \supset \bot\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} A \supset B$. ■

Lemma 3.6. *If* $\{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C$*, then either*

$$\{\top \supset \bot, A\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C \text{or } \{\top \supset \bot, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C.$$

Proof. We use an induction on the number of inference rules used in the proof for $\{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C$.

If $C \in \mathbf{VPL} \cup \{A, B, \top \supset \bot\}$, then the lemma is trivial.

Suppose that there exists a formula D such that $(D, C) \in \mathbf{MP}_V$,

$$\{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \text{ and } \{\top \supset \bot, A, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \supset C.$$

By the induction hypothesis, we have either

$$\{\top \supset \bot, A\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \text{ or } \{\top \supset \bot, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D.$$

On the other hand, by Lemma 3.5, we have

$$\{\top \supset \bot, E\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} D \supset C, \text{ for any } E \in \{A, B\}.$$

Since $(D, C) \in \mathbf{MP}_V$, we can use (RMP). Hence, we have either

$$\{\top \supset \bot, A\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C \text{ or } \{\top \supset \bot, B\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} C.$$

■

Lemma 3.7. $\{a, b\} \nvdash_{\mathbf{VPL},\mathbf{MP}_V} a \wedge b$.

Proof. Suppose that $\{a, b\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} a \wedge b$. Then

$$\{\top \supset \bot, a, b\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} a \wedge b.$$

By Lemma 3.6, we have either

$$\{\top \supset \bot, a\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} a \wedge b \text{ or } \{\top \supset \bot, b\} \vdash_{\mathbf{VPL},\mathbf{MP}_V} a \wedge b.$$

Using Lemma 3.4, we have either

$$\{\top \supset \bot, a\} \vdash_V a \wedge b \text{ or } \{\top \supset \bot, b\} \vdash_V a \wedge b.$$

However, using a Kripke model, we can easily show

$$\{\top \supset \bot, a\} \nvdash_V a \wedge b \text{ and } \{\top \supset \bot, b\} \nvdash_V a \wedge b.$$

This is a contradiction. ■

Proof of Theorem 3.2. Suppose that there exists a pair $(\mathbf{S}, \mathbf{MP})$ satisfying that for any Γ and any A,

$$\Gamma \vdash_V A \text{ if and only if } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} A.$$

If $\mathbf{S} \not\subseteq \mathbf{VPL}$, then there exists a formula $B \in \mathbf{S} - \mathbf{VPL}$. So, we have $\emptyset \vdash_{\mathbf{S},\mathbf{MP}} B$ and $\emptyset \nvdash_V B$. This is a contradiction.

If $\mathbf{MP} \not\subseteq \mathbf{MP}_V$, then there exists a pair $(B, C) \in \mathbf{MP} - \mathbf{MP}_V$. By $(B, C) \notin \mathbf{MP}_V$, there exists a set Σ of formulas such that $\Sigma \vdash_V B$, $\Sigma \vdash_V B \supset C$ and $\Sigma \nvdash_V C$. Using Lemma 2.3(2), $\Sigma \cup \{B, B \supset C\} \nvdash_V C$. On the other hand, we have $(B, C) \in \mathbf{MP}$. So, for any Γ,

$$\text{if } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} B \text{ and } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} B \supset C, \text{ then } \Gamma \vdash_{\mathbf{S},\mathbf{MP}} C.$$

By replacing Γ by $\Sigma \cup \{B, B \supset C\}$, we have $\Sigma \cup \{B, B \supset C\} \vdash_{\mathbf{S},\mathbf{MP}} C$. This is a contradiction.

So, we assume that $\mathbf{S} \subseteq \mathbf{VPL}$ and $\mathbf{MP} \subseteq \mathbf{MP}_V$. By Lemma 3.3, $\Gamma \vdash_{\mathbf{S},\mathbf{MP}} B$ implies $\Gamma \vdash_{\mathbf{VPL},\mathbf{MP}_V} B$. Using Lemma 3.7, we have

$$\{a, b\} \nvdash_{\mathbf{S},\mathbf{MP}} a \wedge b.$$

But, by $(\wedge I)$, we have $\{a, b\} \vdash_V a \wedge b$. This is a contradiction.

Hence, we obtain the theorem. ∎

From the above proof, we have

Corollary 3.8. *If* $\vdash_{\mathbf{S},\mathbf{MP}} \subseteq \vdash_V$, *then* $\{a, b\} \nvdash_{\mathbf{S},\mathbf{MP}} a \wedge b$.

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