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Diderik BATENS, Kristof De CLERCQ
and Natasha KURTONINA

EMBEDDING AND INTERPOLATION FOR SOME PARALOGICS. THE PROPOSITIONAL CASE.

A b s t r a c t. We consider the very weak paracomplete and paraconsistent logics that are obtained by a straightforward weakening of Classical Logic, as well as some of their maximal extensions that are a fragment of Classical Logic. We prove (for the propositional case) that these logics may be faithfully embedded in Classical Logic (as well as in each other), and that the interpolation theorem obtains for them.

1. Aim of this paper

In the semantics of Classical Logic—henceforth **CL**—negation is characterized by the consistency requirement—if $v_M(A) = 1$ then $v_M(\sim A) = 0$—and the completeness requirement—if $v_M(A) = 0$ then $v_M(\sim A) = 1$. By dropping the consistency requirement, the completeness requirement,

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or both, we obtain respectively the logics **CLuN**, **CLaN**, and **CLoN** (allowing respectively for gl*u*ts, for g*a*ps, and for b*o*th gluts and gaps with respect to negation). These very weak logics are in a sense basic with respect to **CL**—see [3]. They contain the full positive part of **CL**, give up consistency, completeness or both, and do not compensate for this by reintroducing double negation, de Morgan properties, etc. [1]

Unlike in previous papers on paraconsistent logics, we shall take the language schema $\mathcal{L}$ to contain the constant *bottom*, written as "$\bot$" and defined as usual (viz. semantically as $v_M(\bot) = 0$). Given this convention, classical negation is definable in the above systems ($\neg A =_{df} A \supset \bot$), whence **CLuN**, **CLaN**, and **CLoN** may be seen as **CL** extended with a negation ($\sim$) that is respectively paraconsistent, paracomplete, and both. The presence of bottom greatly simplifies the meta-theoretic proofs and many results may be transferred directly to the systems obtained by removing bottom from the language schema.

The importance of **CLuN**, **CLaN**, and **CLoN** is double. First, many paralogics are extensions of them. Studying properties of the basic logics is often a good starting point for studying properties of their extensions. Next, the very weak basic logics themselves have interesting applications as the lower limit logic of adaptive logics; for many applications, such adaptive logics are preferable to adaptive logics based on richer lower limit logics (this position is defended in [4], [5] and [6] with respect to inconsistency-adaptive logics—see also [7]).

In the present paper we shall show that the three paralogics may be faithfully embedded in **CL** (as well as in each other) and that interpolation holds for them. The results are presented together because the proof methods are closely related. We restrict ourselves to the propositional level. The predicative case involves complications of a rather different nature and

[1] As a result, these logics do not spread abnormalities: if an abnormality (inconsistency or incompleteness) obtains in a model, for example in that the model verifies both A and $\sim A$, this does not entail that any subformula or superformula of A behaves abnormally in the model.

is studied in a forthcoming paper. We do however extend the results to several paralogics that are maximal—see section 5.

2. The basic paralogics

Let $\mathcal{L}$ (including "$\bot$") be the language schema of **CL**. Let $\mathcal{S}$ be the set of sentential letters, $\mathcal{W}$ the set of wffs and ${}^{\sim}\mathcal{W} = \{\sim A \mid A \in \mathcal{W}\}$. The paraconsistent and paracomplete **CLoN** is simply full positive **CL**, characterized syntactically by, *e.g.*,

MP	From A and $A \supset B$ to derive B
A⊃1	$A \supset (B \supset A)$
A⊃2	$((A \supset B) \supset A) \supset A$
A⊃3	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A⊥	$\bot \supset A$
A&1	$(A \& B) \supset A$
A&2	$(A \& B) \supset B$
A&3	$A \supset (B \supset (A \& B))$
A∨1	$A \supset (A \vee B)$
A∨2	$B \supset (A \vee B)$
A∨3	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A≡1	$(A \equiv B) \supset (A \supset B)$
A≡2	$(A \equiv B) \supset (B \supset A)$
A≡3	$(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$

The paraconsistent **CLuN** is obtained by adding the axiom

$$(A \supset\sim A) \supset\sim A \text{ (alternatively: } A \vee \sim A),$$

the paracomplete **CLaN** by adding the axiom

$$A \supset (\sim A \supset B).$$

We first present a decent semantic characterization for **CLoN** in which $v_M(A)$ is fully determined by the assignment values of subformulas of A, and that agrees with the semantics for the predicative extension. A **CLoN**-model M is a singleton $\langle v \rangle$ in which v is an assignment function defined (in a fully classical meta-language) by:

C1.1 $v : \mathcal{S} \rightarrow \{0, 1\}$
C1.2 $v : {}^{\sim}\mathcal{W} \rightarrow \{0, 1\}$

The valuation function v_M determined by the model M is defined as follows:

C2.1 $v_M : \mathcal{W} \rightarrow \{0, 1\}$
C2.2 where $A \in \mathcal{S}$, $v_M(A) = v(A)$; $v_M(\bot) = 0$
C2.3^o $v_M(\sim A) = 1$ iff $v(\sim A) = 1$
C2.4 $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C2.5 $v_M(A \& B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
C2.6 $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$
C2.7 $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$

Truth in a model, semantic consequence and validity are defined as usual.

The semantic characterization for **CLuN** is obtained by replacing C2.3^o by

C2.3^u $v_M(\sim A) = 1$ iff $v_M(A) = 0$ or $v(\sim A) = 1$,

that for **CLaN** by replacing C2.3^o by

C2.3^a $v_M(\sim A) = 1$ iff $v_M(A) = 0$ and $v(\sim A) = 1$.

It goes without saying that the **CL**-semantics is obtained by adding "$v_M(\sim A) = 1$ iff $v_M(A) = 0$" in which case C1.2 becomes pointless. Remark that we take "$\sim$" to be the original negation symbol of **CL**, not to be confused with "$\neg$" that is definable in all logics considered.

3. Embedding the paralogics in CL

In order to show that our paralogics can be faithfully embedded in **CL**, we first extend $\mathcal{L}$ into $\mathcal{L}^{\#}$ by extending $\mathcal{S}$ with a denumerable set of new sentential letters: $\mathcal{S}^{\#} = \mathcal{S} \cup \{p_{\sim A} \mid A \in \mathcal{W}\}$. Let $\mathcal{W}^{\#}$ be the set of wffs of $\mathcal{L}^{\#}$ and let the **CL**-semantics be extended to $\mathcal{L}^{\#}$. Next, we define a translation function $Tr : \mathcal{W} \rightarrow \mathcal{W}^{\#}$. For **CLoN**, it is defined by:

(i) where $A \in \mathcal{S}$, $Tr(A) = A$; $Tr(\bot) = \bot$
(ii^o) $Tr(\sim A) = p_{\sim A}$
(iii) $Tr(A \supset B) = Tr(A) \supset Tr(B)$
(iv) $Tr(A \vee B) = Tr(A) \vee Tr(B)$
(v) $Tr(A\&B) = Tr(A)\&Tr(B)$

We extend the translation function to sets: $Tr(\Gamma) = \{Tr(A) \mid A \in \Gamma\}$. Replacing (ii^o) by

(ii^u) $Tr(\sim A) = \sim Tr(A) \vee p_{\sim A}$

we obtain the translation function for **CLuN**; replacing (ii^o) by

(ii^a) $Tr(\sim A) = \sim Tr(A)\&p_{\sim A}$

we obtain the translation function for **CLaN**. The connection with clause C2.3 of the semantic characterizations is immediate.

Where $M = \langle v \rangle$ is a **CLoN**-model for $\mathcal{L}$ and $M' = \langle v' \rangle$ a **CL**-model for $\mathcal{L}^{\#}$, let RMM' iff, for all $A \in \mathcal{S}$, $v'(A) = v(A)$, and, for all $\sim A \in {}^{\sim}\mathcal{W}$, $v'(p_{\sim A}) = v(\sim A)$.

Lemma 1. *R is a bijection that connects each* **CLoN**-*model for $\mathcal{L}$ to a* **CL**-*model for $\mathcal{L}^{\#}$ and vice versa.*

Proof. Where $M = \langle v \rangle$ is a **CLoN**-model for $\mathcal{L}$, the **CL**-model $M' = \langle v' \rangle$ for $\mathcal{L}^{\#}$ such that RMM' is defined by: for all $A \in \mathcal{S}$, $v'(A) = v(A)$; for all $\sim A \in {}^{\sim}\mathcal{W}$, $v'(p_{\sim A}) = v(\sim A)$. As the assignment function is completely defined, so is M'.

Where $M' = \langle v' \rangle$ is a **CL**-model for $\mathcal{L}^{\#}$, define the **CLoN**-model $M = \langle v \rangle$ for $\mathcal{L}$ by: for all $A \in \mathcal{S}$, $v(A) = v'(A)$; for all $\sim A \in {}^{\sim}\mathcal{W}$, $v(\sim A) = v'(p_{\sim A})$. As the assignment function is completely defined, so is M, and RMM'. ■

Lemma 2. *Where* $M = \langle v \rangle$ *is a* **CLoN**-*model for* $\mathcal{L}$, $M' = \langle v' \rangle$ *is a* **CL**-*model for* $\mathcal{L}^{\#}$, *and* RMM', $v_M(A) = v_{M'}(Tr(A))$ *for all* $A \in \mathcal{W}$.

Proof. We proceed by induction on the complexity of formulas (the number of connectives occurring in them). Where A has complexity 0, $v_M(A) = v_{M'}(Tr(A))$ in view of Tr and R. For the induction step, the only non-obvious case concerns negation. It is established as follows. We first recall C2.3^o:

$$v_M(\sim A) = 1 \text{ iff } v(\sim A) = 1. \tag{1}$$

By the induction hypothesis,

$$v_M(A) = v_{M'}(Tr(A)) \tag{2}$$

and by the definition of R and C2.2,

$$v(\sim A) = v'(p_{\sim A}) = v_{M'}(p_{\sim A}). \tag{3}$$

From (1)–(3):

$$v_M(\sim A) = 1 \text{ iff } v_{M'}(p_{\sim A}) = 1. \tag{4}$$

By the definition of Tr,

$$v_M(\sim A) = 1 \text{ iff } v_{M'}(Tr(\sim A)) = 1. \tag{5}$$

■

Theorem 1. $\Gamma \models_{\mathbf{CLoN}} A$ *iff* $Tr(\Gamma) \models_{\mathbf{CL}} Tr(A)$. *(Translatability).*

Proof. Immediate in view of Lemmas 1 and 2. ■

Corollary 1. **CLoN** *can be faithfully embedded in* **CL**.

For **CLuN**, we first replace "**CLoN**" by "**CLuN**" in the definition of R. The proof of Lemma 3 is identical to that of Lemma 1.

Lemma 3. *R is a bijection that connects each* **CLuN**-*model for $\mathcal{L}$ to a* **CL**-*model for $\mathcal{L}^{\#}$ and vice versa.*

Lemma 4. *Where $M = \langle v \rangle$ is a* **CLuN**-*model for $\mathcal{L}$, $M' = \langle v' \rangle$ is a* **CL**-*model for $\mathcal{L}^{\#}$, and RMM', $v_M(A) = v_{M'}(Tr(A))$ for all $A \in \mathcal{W}$.*

Proof. As for Lemma 2, except in that

(1) $$v_M(\sim A) = 1 \text{ iff } v_M(A) = 0 \text{ or } v(\sim A) = 1$$

and hence

(4) $$v_M(\sim A) = 1 \text{ iff } v_{M'}(Tr(A)) = 0 \text{ or } v_{M'}(p_{\sim A}) = 1.$$

As M' is a **CL**-model

(5) $$v_M(\sim A) = 1 \text{ iff } v_{M'}(\sim Tr(A) \vee p_{\sim A}) = 1.$$

By the definition of Tr,

(6) $$v_M(\sim A) = 1 \text{ iff } v_{M'}(Tr(\sim A)) = 1.$$

■

Theorem 2. $\Gamma \models_{\mathbf{CLuN}} A$ *iff* $Tr(\Gamma) \models_{\mathbf{CL}} Tr(A)$. *(Translatability)*

Proof. Immediate in view of Lemmas 3 and 4. ■

Corollary 2. **CLuN** *can be faithfully embedded in* **CL**.

Proceeding in the same way for **ClaN**, delivers:

Lemma 5. *R is a bijection that connects each* **CLaN**-*model for $\mathcal{L}$ to a* **CL**-*model for $\mathcal{L}^{\#}$ and vice versa.*

Lemma 6. *Where $M = \langle v \rangle$ is a* **CLaN**-*model for $\mathcal{L}$, $M' = \langle v' \rangle$ is a* **CL**-*model for $\mathcal{L}^{\#}$, and RMM', $v_M(A) = v_{M'}(Tr(A))$ for all $A \in \mathcal{W}$.*

Proof. As for Lemma 2, except in that

(1) $$v_M(\sim A) = 1 \text{ iff } v_M(A) = 0 \text{ and } v(\sim A) = 1$$

and hence

(4) $$v_M(\sim A) = 1 \text{ iff } v_{M'}(Tr(A)) = 0 \text{ and } v_{M'}(p_{\sim A}) = 1.$$

As M' is a **CL**-model

(5) $$v_M(\sim A) = 1 \text{ iff } v_{M'}(\sim Tr(A) \& p_{\sim A}) = 1.$$

By the definition of Tr,

(6) $$v_M(\sim A) = 1 \text{ iff } v_{M'}(Tr(\sim A)) = 1.$$

■

Theorem 3. $\Gamma \models_{\mathbf{CLaN}} A$ *iff* $Tr(\Gamma) \models_{\mathbf{CL}} Tr(A)$. *(Translatability)*

Proof. Immediate in view of Lemmas 5 and 6. ■

Corollary 3. **CLaN** *can be faithfully embedded in* **CL**.

The upshot is that decidability in **CLoN**, **CLuN**, and **CLaN** is reduced to decidability in **CL**, and that all proof search techniques for the latter are available for the former three. The idea underlying the proofs derives directly from the semantics. In the paralogics, $v_M(\sim A)$ depends in part on $v_M(A)$ and in part on an independent element, viz. $v(\sim A)$—in **CLoN** the former part reduces to nil. The embeddings are realized by assigning the role of $v(\sim A)$ to a new schematic letter, viz. $p_{\sim A}$, in the **CL**-models. Given this convention, the assignment of a paralogic model corresponds to the assignment of a **CL**-model and *vice versa*. As, in our semantic style, the valuation function is completely determined by the assignment function, the result is easily extended to all three paralogics (only the translation function has to be accommodated to the specific clause for negation).

Our proof of the theorems is wholly independent of the presence of bottom in $\mathcal{L}$, and hence holds also for the three systems if bottom is removed from the language. If bottom is present, the reverse embedding is immediate. If bottom is absent, no finite set of formulas will do as the translation of a **CL**-formula $\sim A$.

4. Interpolation

The proof of the interpolation theorem for our three paralogics follows the plot of the proof of the interpolation theorem for **CL**, as it is presented by Boolos & Jeffrey's [8], pp. 235–242. The required changes are due to the abnormal behaviour of negation. As negation behaves abnormally, we cannot just apply the 'replacement procedure' of Boolos & Jeffrey.

We may assume that A is satisfiable and that C is not valid. For if A is unsatisfiable, $\bot$ will do as B; if C is valid, $\top =_{df} \bot \supset \bot$ will do.

For **CLuN** and **ClaN**, A and C have to be 'prepared' in a specific way. **CLoN** does not require such preparation. However, even for **CLoN** the proof is different from the one in Boolos & Jeffrey. The cause is the role of the independent element mentioned in the next to last paragraph of the previous section. We shall start with the proof for **CLoN**, and then introduce the further complication required by **CLuN** and **ClaN**.

Theorem 4. *If $A \vdash_{\mathbf{CLoN}} C$, there is a B, such that $A \vdash_{\mathbf{CLoN}} B$ and $B \vdash_{\mathbf{CLoN}} C$, and B contains only non-logical symbols that are contained in both A and C.*

Proof. Our hypothesis is that A is satisfiable, C is not valid, $A \vdash_{\mathbf{CLoN}} C$, and each of A and C is a truth-functional compound [2] of members of $\mathcal{S} \cup {}^{\sim}\mathcal{W}$—let us call these *atoms*.

We may assume that at least one atom Q is contained in A but not C (if not, we may take $B = A$). Let D_1 be the result of everywhere replacing occurrences of Q (outside the scope of a "$\sim$") in A by $\top$; and D_2 the result

[2] Negation ($\sim$) is *not* a truth-function in any of our paralogics.

of everywhere replacing occurrences of Q (outside the scope of a "$\sim$"), by $\bot$. Let $D = D_1 \vee D_2$. We have to show that $A \vdash_{\mathbf{CLoN}} D$, and $D \vdash_{\mathbf{CLoN}} C$.

Fact 1. $A \vdash_{\mathbf{CLoN}} D$. For suppose $v_M(A) = 1$. Then if $v_M(Q) = 1$, it follows that $v_M(A) = v_M(D_1)$, whence $v_M(D) = 1$. And if $v_M(Q) = 0$, then $v_M(A) = v_M(D_2)$, whence $v_M(D) = 1$.

Fact 2. $D \vdash_{\mathbf{CLoN}} C$. For suppose that $v_M(D_1) = 1$ but $v_M(C) = 0$. Let $v_{M'}$ differ from v_M (if at all) only in that $v_{M'}(Q) = 1$. Then $v_{M'}(A) = 1$ and, since C does not contain Q, $v_{M'}(C) = v_M(C) = 0$, contradicting the assumption that $A \vdash_{\mathbf{CLoN}} C$. By the same reasoning, $D_2 \vdash_{\mathbf{CLoN}} C$. Therefore $D = D_1 \vee D_2 \vdash_{\mathbf{CLoN}} C$.

So, $A \vdash_{\mathbf{CLoN}} D$, $D \vdash_{\mathbf{CLoN}} C$, and every atom in D is in A. Moreover A contains one more atom foreign to C than D does. So repeating the construction sufficiently often, using the D of one stage as the A of the next, eventually yields a sentence B, such that $A \vdash_{\mathbf{CLoN}} B$ and $B \vdash_{\mathbf{CLoN}} C$, and B contains no atoms not contained in both A and C. ■

Let us now turn to the preparation of A and C that is required by **CLuN**. In both A and C we replace every (sub)formula of the form $\sim E$ which itself does not occur within the scope of a "$\sim$", by $(\sim E^{\dagger} \vee \neg E)$.[3] Eventually we reach the stage where every atom of the form $\sim E$ that occurs itself outside the scope of a "$\sim$", is marked with a "$\dagger$". Let the result of these transformations be $A^{\dagger}$ and $C^{\dagger}$. Where, for example, $A = \sim\sim (p\& \sim q)$, the successive transformations proceed as follows:

$$\sim\sim (p\& \sim q)^{\dagger} \vee \neg \sim (p\& \sim q)$$
$$\sim\sim (p\& \sim q)^{\dagger} \vee \neg(\sim (p\& \sim q)^{\dagger} \vee \neg(p\& \sim q))$$
$$\sim\sim (p\& \sim q)^{\dagger} \vee \neg(\sim (p\& \sim q)^{\dagger} \vee \neg(p\&(\sim q^{\dagger} \vee \neg q))) = A^{\dagger}$$

Once the replacement is finished, all occurrences of "$\dagger$" are removed. As $\vdash_{\mathbf{CLuN}} \sim E \equiv (\sim E \vee \neg E)$ and the rule of replacement of equivalents

[3] The "$\dagger$" in "$E^{\dagger}$" is merely a mark indicating that the atom $\sim E$ has been replaced by the equivalent wff $\sim E \vee \neg E$. The mark prevents us to perform the same replacement over and over again.

outside the scope of a negation holds in **CLuN**, $A \vdash_{\mathbf{CLuN}} A^\dagger$ (and vice versa) and $C \vdash_{\mathbf{CLuN}} C^\dagger$ (and vice versa), and thus also $A^\dagger \vdash_{\mathbf{CLuN}} C^\dagger$. Moreover, no new non-logical symbols are introduced in the transition from A and C to $A^\dagger$ and $C^\dagger$ respectively. So if B is an interpolant for $A^\dagger \vdash_{\mathbf{CLuN}} C^\dagger$, B is also an interpolant for $A \vdash_{\mathbf{CLuN}} C$.

An important heuristic remark is at hand here. For **CLoN** we might have reasoned as follows: $A \vdash_{\mathbf{CLoN}} C$; hence $Tr(A) \vdash_{\mathbf{CL}} Tr(C)$; hence there is an interpolant B such that $Tr(A) \vdash_{\mathbf{CL}} B$ and $B \vdash_{\mathbf{CL}} Tr(C)$; as the converse of Tr is a function, there is a D such that $B = Tr(D)$; hence D is an interpolant for A and C. However, for **CLuN** (and **ClaN**) the converse of Tr is not a function, and hence the presence of a **CL**-interpolant for $Tr(A)$ and $Tr(C)$ does not warrant that there is a **CLuN**-interpolant for A and C. This is why we need the preparation of A and C. The preparation does not introduce any symbols beyond $\mathcal{L}$, but still has the same effect as the translation function: any truth-functional effect of a formula on its subformulas is made explicit when the preparation comes to an end. If the resulting formula has a subformula of the form $\sim E^\dagger$ this subformula functions as an 'independent element'. However, the problem connected to back-translation is avoided: the occurrences of "$\dagger$" are removed and $\sim E$ is **CLuN**-equivalent to $\sim E \vee \neg E$.

The previous paragraph is not only important from a heuristic point of view. It also clarifies why, *after* the transformation is performed, formulas of the form $\sim E$ may be considered as atomic.

Theorem 5. *If $A \vdash_{\mathbf{CLuN}} C$, there is a B, such that $A \vdash_{\mathbf{CLuN}} B$, $B \vdash_{\mathbf{CLuN}} C$, and B contains only non-logical symbols that are contained in both A and C.*

The proof proceeds exactly as that of Theorem 4, except that **CLoN**, A, and C are replaced by **CLuN**, $A^\dagger$, and $C^\dagger$ respectively. As pointed out before, the interpolant B for $A^\dagger$ and $C^\dagger$ is an interpolant for A and C.

For **CLaN**, the matter is as for **CLuN**, except that A and C have to be prepared differently, viz. by replacing every (sub)formula of the form

$\sim E$ which itself does not occur within the scope of a "$\sim$", by $(\sim E^\dagger \& \neg E)$. By a proof wholly analogous to that of Theorem 5, one establishes:

Theorem 6. *If $A \vdash_{\mathbf{CLaN}} C$, there is a B, such that $A \vdash_{\mathbf{CLaN}} B$ and $B \vdash_{\mathbf{CLaN}} C$, and B contains only non-logical symbols that are contained in both A and C.*

Our interpolation results are not independent of the presence of bottom in $\mathcal{L}$, mainly because, if bottom were absent, D_2 could not be defined. It remains an open problem whether the Interpolation Theorems are provable in the absence of bottom.

5. Extending the results to some maximal paralogics

We now extend our results to six paralogics that are *maximal* in that their only proper extensions are **CL** and richer systems—at the propositional level these reduce to the trivial system. The logics **CLoNs**, **CLuNs**, and **CLaNs** (the "s" refers to Schütte who devised the propositional fragments in [9]) are obtained from respectively **CLoN**, **CLuN**, and **CLaN** by adding all properties that drive negations inwards. For the syntactic characterization, the following axioms are added to the weak systems:

A$\sim\sim$	$\sim\sim A \equiv A$
A$\sim\supset$	$\sim (A \supset B) \equiv (A \& \sim B)$
A$\sim \&$	$\sim (A \& B) \equiv (\sim A \vee \sim B)$
A$\sim \vee$	$\sim (A \vee B) \equiv (\sim A \& \sim B)$
A$\sim\equiv$	$\sim (A \equiv B) \equiv ((A \vee B) \& (\sim A \vee \sim B))$

For the semantic characterization, we first weaken clause C2.3 for the respective systems as follows:[4]

C2.3^{os}	where $A \in \mathcal{S}$, $v_M(\sim A) = 1$ iff $v(\sim A) = 1$
C2.3^{us}	where $A \in \mathcal{S}$, $v_M(\sim A) = 1$ iff $v_M(A) = 0$ or $v(\sim A) = 1$
C2.3^{as}	where $A \in \mathcal{S}$, $v_M(\sim A) = 1$ iff $v_M(A) = 0$ and $v(\sim A) = 1$

[4] Clause C1.2 *may* obviously be restricted to "$v : {}^{\sim}\mathcal{S} \rightarrow \{0, 1\}$" (where ${}^{\sim}\mathcal{S} = \{\sim A | A \in \mathcal{S}\}$). The same obtains for the Vasil'ev systems discussed below in the text.

and then add the clauses:

C2.3.1	$v_M(\sim\sim A) = v_M(A)$
C2.3.2	$v_M(\sim (A \supset B)) = v_M(A\& \sim B)$
C2.3.3	$v_M(\sim (A\&B)) = v_M(\sim A\vee \sim B)$
C2.3.4	$v_M(\sim (A \vee B)) = v_M(\sim A\& \sim B)$
C2.3.5	$v_M(\sim (A \equiv B)) = v_M((A \vee B)\&(\sim A\vee \sim B))$

Alternatively, **CLuNs** and **CLaNs** are characterized by three-valued matrices (with one value designated for **CLaNs** and two values designated for **CLuNs**) and **CLoNs** is characterized by four-valued matrices. These systems are among the most popular many-valued logics. The $\sim$-$\vee$-&-fragment of **CLaNs** is Kleene's $\mathbf{SK}_3$, that of **CLuNs** is Priest's **LP**, and the first degree fragment of **CLoNs** characterizes tautological entailments (see [1]).[5]

The logics **CLoNv**, **CLuNv**, and **CLaNv** (the "v" refers to Arruda's Vasil'ev system from [2], which is the propositional fragment of **CLuNv**) are obtained from respectively **CLoN**, **CLuN**, and **CLaN** by making negation behave classically in front of complex formulas. For the syntactic characterization, the addition of the two following axiom schemes is sufficient:[6]

$\mathrm{A}\sim^{uv}$	where $A \in \mathcal{W} - \mathcal{S}$, $\sim A \supset (A \supset B)$
$\mathrm{A}\sim^{av}$	where $A \in \mathcal{W} - \mathcal{S}$, $(A \supset\sim A) \supset\sim A$

For the semantics, the negation clause is again restricted to $\mathrm{C2.3}^{os}$, $\mathrm{C2.3}^{us}$, and $\mathrm{C2.3}^{as}$ respectively, and the following clause is added:

$\mathrm{C2.3}^{v}$ where $A \in \mathcal{W} - \mathcal{S}$, $v_M(\sim A) = 1$ iff $v_M(A) = 0$

In order to extend our embedding result to these six logics, we modify the translation function as follows. For all systems, we restrict clause (ii) to

[5] 'Material implication' is often defined by $\sim A \vee B$, and hence is not detachable. The detachable material implication of the systems **CLoNs**, **CLuNs**, and **CLaNs** is somewhat tricky: although $\sim (A \supset B)$ is equivalent to $A\& \sim B$ and hence also to $\sim (\sim A \vee B)$, $\sim A \vee B$ is derivable from $A \supset B$ but not conversely.

[6] $\mathrm{A}\sim^{uv}$ is redundant in **CLaNv**, $A \sim^{av}$ in **CLuNv**.

(ii^{op}) where $A \in \mathcal{S}$, $Tr(\sim A) = p_{\sim A}$

for **CLoNs** and **CLoNv** and similarly for the other systems. Next one paraphrases the semantic clauses. For the Schütte systems:

(ii^{s}) $Tr(\sim\sim A) = Tr(A)$; $Tr(\sim (A \supset B)) = Tr(A\& \sim B)$; etc.

and for the Vasil'ev systems;

(ii^{v}) where $A \in \mathcal{W} - \mathcal{S}$, $Tr(\sim A) =\sim Tr(A)$.

Given these modifications, the proofs from section 3 are easily adapted to show that **CLoNs**, **CLuNs**, **CLaNs**, **CLoNv**, **CLuNv**, and **CLaNv** can be faithfully embedded in **CL**.

Some more embedding results are easily obtained. Replacing (ii^{op}), (ii^{up}), and (ii^{ap}) by

(ii^{p}) where $A \in \mathcal{S}$, $Tr(\sim A) =\sim A$

the proofs from section 3 are easily adapted to show that **CLoNs** and **CLoNv** can be faithfully embedded in **CLoN**, **CLuNs** and **CLuNv** in **CLuN**, and **CLaNs** and **CLaNv** in **CLaN**. Weakening (ii^{o}), (ii^{u}), and (ii^{a}) in the translation functions from section 3 to

(ii^{oc}) where $A \in \mathcal{W} - \mathcal{S}$, $Tr(\sim A) = p_{\sim A}$

etc., and adding, for all systems,

(ii^{p}) where $A \in \mathcal{S}$, $Tr(\sim A) =\sim A$

one easily adapts the proofs from section 3 to show that **CLoN** can be faithfully embedded in **CLoNs** as well as in **CLoNv**, etc. We leave it to the reader to adapt the translation functions to show that **CLoNs** and **CLoNv** can be embedded in each other. So, the upshot is that all systems can be embedded in **CL**, that **CLoN**, **CLoNs** and **CLoNv** can be embedded in each other, and similarly for the other two groups. Remark that classical negation can be defined in the Vasil'ev systems without relying on bottom, *e.g.*, by $\neg A =_{df} \sim (A\&A)$; hence all systems can be embedded in the Vasil'ev systems. All these results are independent of the presence of bottom in $\mathcal{L}$.

In the presence of bottom, all systems can obviously be embedded in each other.

As the reader might expect, also our interpolation result is easily extended to the six maximally paraconsistent logics. Three kinds of modifications are required; the first two concern the preparation, the third the interpretation of "atom" in the proofs. (i) For the Schütte systems, a first preparatory step consists in driving negations inwards: for all subformulas D of A and C: if $D = \sim\sim E$, replace it by E; if $D = \sim (A \supset B)$, replace it by $A \& \sim B$; etc. Nothing should be done at this stage for the Vasil'ev systems. (ii) The preparation required for **CLuN**, respectively **CLaN**, in section 4, is restricted (for **CLuNs** and **CLuNv**, respectively **CLaNs** and **CLaNv**) to members of ${}^{\sim}\mathcal{S}$. (iii) In the proofs, only schematic letters and negations of schematic letters (instead of negations of wffs) are treated as *atoms*. Given these modifications, the proofs of the Interpolation Theorem for the six logics can be left as an easy exercise for the reader.

6. In conclusion

The embedding results obviously extend to fragments of the logics (even in the absence of bottom), such as $\mathbf{SK}_3$ and **LP**. They also easily extend to some other maximal paralogics. We have some results on the application of our method to systems (such as da Costa's $\mathbf{C}_1$) between the basic logics (such as **CLuN**) and the maximal paralogics (such as **CLuNv**), but many open problems remain.

Other open problems concern the Interpolation Theorem for related logics, partly because our method requires the presence of bottom, partly because it is not obvious how the method should be adapted if, for example, $A \supset\sim\sim A$ (without its converse) is added as an axiom to **CLuN**.

References

[1] Anderson A. R. and Belnap N. D., Jr., Entailment. The Logic of Relevance and Necessity, volume 1. Princeton University Press, 1975.

[2] Arruda A. I., *On the imaginary logic of N.A. Vasil'ev.* In Ayda I. Arruda, Newton C.A. da Costa, and R. Chuaqui, editors, Non-classical Logics, Model Theory and Computability, pages 3–24. North-Holland, Amsterdam, 1977.

[3] Batens D., *Paraconsistent extensional propositional logics.* Logique et Analyse, 90–91:195–234, 1980.

[4] Batens D., *Dynamic dialectical logics.* In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.

[5] Batens D., *Inconsistency-adaptive logics.* In Ewa Orłowska, editor, Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa, pages 445–472. Physica Verlag (Springer), Heidelberg, New York, 1998.

[6] Batens D., *Rich inconsistency-adaptive logics. The clash between heuristic efficiency and realistic reconstruction.* In print.

[7] Batens D., *A survey of inconsistency-adaptive logics.* In Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, editors, Frontiers of Paraconsistent Logic, pages 59–83. Research Studies Press, Baldock, UK, in print.

[8] Boolos G. S. and Jeffrey R. J., Computability and Logic. Cambridge University Press, 1989. (Third edition).

[9] Schütte K., Beweistheorie. Springer, Berlin, 1960.

Centre for Logic and Philosophy of Science
Department of Philosophy
Wijsbegeerte
Universiteit Gent
Blandijnberg 2
B-9000 Gent (Belgium)
e-mail: `Diderik.Batens@rug.ac.be`