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ON THE LOGIC DETERMINED BY SOME FINITE RELATIONAL SYSTEMS

A b s t r a c t. Let $\mathbf{N}_n = \langle \{1, 2, \ldots, n\}, =, 1, 2, \ldots, n \rangle$ be the relational system of cardinality n with identity and with designated elements $1, 2, \ldots, n$. In the paper we consider the predicate logic $L(\mathbf{N}_n)$ determined by the relational system $\mathbf{N}_n$ (cf. [1]).

Let $\mathcal{L}_n$ be the first-order language for $\mathbf{N}_n$. Thus, $\mathcal{L}_n$ contains the symbols $=, \overline{1}, \overline{2}, \ldots, \overline{n}$ interpreted in $\mathbf{N}_n$ as the identity and the natural numbers $1, 2, \ldots, n$. By F_n we denote the set of all formulas of $\mathcal{L}_n$. Hence, F_n is the smallest set such that:

a. $t_i = t_j \in F_n$,

b. if $A, B \in F_n$, then: $\neg A$, $A \to B$, $A \wedge B$, $A \vee B$, $A \equiv B$, $\bigvee_{x_k} A$, $\bigwedge_{x_k} A \in F_n$.

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The symbol $\mathbf{N}_n \models A$ will be the abbreviation of the expression: the formula A is true in $\mathbf{N}_n$. By $Vf(A)$ we denote the set of all free variables occurring in A.

We shall use the symbols: $\Rightarrow$, $\Leftrightarrow$, $\forall x$, $\exists x$ as the well-known propositional connectives and quantifiers from metalanguage.

Let us now define some further notions. We say that the formulas $A, B \in F_n$ are similar, in symbols $A \approx B$ if they differ at most on bound variables and all occurrences of distinct variables in A remain so in B and *vice versa*. By $[A]_m$ we denote an operation which increases the indices of bound variables: $[\bigwedge_{x_k} B]_m = \bigwedge_{x_{k+m+r}} [B(x_k/x_{k+m+r})]$, where r is the greatest index among all the indices of all the individual variables occurring in B. By x_k/x_m we denote the substitution for individual variables. The operation $[A]_m$ preserves logical connectives and atomic formulas.

The set of all logical schemes SL is the smallest set such that:

a. $P_k^m(x_{i_1}, \ldots, x_{i_m}) \in SL$, for $k, m = 1, 2, \ldots$,

b. if $\alpha, \beta \in SL$ then: $\neg\alpha$, $\alpha \to \beta$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \equiv \beta$, $\bigwedge_{x_k} \alpha$, $\bigvee_{x_k} \alpha \in SL$, where P_k^m is n-ary predicate letter.

The set of all atomic schemes $P_k^m(x_{i_1}, \ldots, x_{i_m})$, with $k, m = 1, 2, \ldots$ will be denoted by At.

We define now the set V_n of functions as follows:

$$v \in V_n \iff \begin{array}{l} \text{a. } v : At \longrightarrow F_n \\ \text{b. } Vf(v(\alpha)) = Vf(\alpha) \\ \text{c. } v(\alpha(x_k/x_n)) \approx [v(\alpha)]_{k+n}(x_k/x_n) \end{array}$$

for every $\alpha \in At \subseteq SL$.

Every function $v \in V_{\mathbf{N}}$ can be extended to a homomorphism $h^v : SL \longrightarrow F_n$, with respect to the algebraic structure imposed in a natural way on SL and F_n by the operations $\neg, \to, \wedge, \vee, \equiv, \bigvee_{x_k}, \bigwedge_{x_k}$.

Definition 1. (cf.[1]). A scheme $\alpha \in SL$ is a tautology of the relational system $\mathbf{N}_n$ if and only if for every $v \in V_n : \mathbf{N}_n \models h^v(\alpha)$.

The set of all predicate tautologies of $\mathbf{N}_n$ is denoted by $L(\mathbf{N}_n)$ and called the predicate logic of the system $\mathbf{N}_n$.

Let now K^n be the function from SL into SL defined as follows:

Definition 2.

a. $K^n(\alpha) = \alpha$, if $\alpha \in At$,

b. $K^n(\neg\alpha) = \neg K^n(\alpha)$,

c. $K^n(\alpha \circ \beta) = K^n(\alpha) \circ K^n(\beta)$, where $\circ \in \{\wedge, \vee, \rightarrow, \equiv\}$,

d. $K^n(\bigwedge_{x_t} \alpha) = K^n([\alpha]_{t+1}(x_t/x_1)) \wedge \ldots \wedge K^n([\alpha]_{t+n}(x_t/x_n))$,

e. $K^n(\bigvee_{x_t} \alpha) = K^n([\alpha]_{t+1}(x_t/x_1)) \vee \ldots \vee K^n([\alpha]_{t+n}(x_t/x_n))$,

for $n = 1, 2, \ldots$

The aim of the present paper is to prove the theorem announced below:

For every natural number $n \geq 1$ and any closed logical scheme α we have: $\alpha \in L(\mathbf{N}_n) \iff K^n(\alpha) \in L_2$, where L_2 is the set of all theorems of the classical first-order logic.

Let $Q_t(x_1, \ldots, x_{t+1})$ denote the following scheme:

$$\begin{aligned}
&\bigwedge_y [P_2^2(x_1, y) \equiv P_2^2(x_2, y)] \vee \bigwedge_y [P_2^2(x_1, y) \equiv P_2^2(x_3, y)] \vee \ldots \\
&\qquad \ldots \vee \bigwedge_y [P_2^2(x_1, y) \equiv P_2^2(x_{t+1}, y)] \\
&\vee \bigwedge_y [P_2^2(x_2, y) \equiv P_2^2(x_3, y)] \vee \ldots \vee \bigwedge_y [P_2^2(x_2, y) \equiv P_2^2(x_{t+1}, y)] \\
&\qquad \vdots \\
&\vee \bigwedge_y [P_2^2(x_t, y) \equiv P_2^2(x_{t+1}, y)].
\end{aligned}$$

Then we have:

Lemma 1. *For every $\alpha \in SL$:*
$\neg Q_{n-1}(x_1, \ldots, x_n) \rightarrow \big(([\alpha]_{t+1}(x_t/x_1) \wedge \ldots \wedge [\alpha]_{t+n}(x_t/x_n) \equiv \bigwedge_{x_t} \alpha\big) \in L(\mathbf{N}_n)$, *where* $n = 2, 3, \ldots$

Its easy proof, based on Definition 1, is left to the reader.

Lemma 2. *For every* $\alpha \in SL$*:*
$\neg Q_{n-1}(x_1,\ldots,x_n) \to (\alpha \equiv K^n(\alpha)) \in L(\mathbf{N}_n)$, *for* $n = 2,3,\ldots n$

Proof. We prove this by induction on the number of logical constants occurring in α. For $\alpha \in At$, the lemma holds trivially.

Let $\alpha = \bigwedge_{x_t} \beta$. By the inductive assumption, we get:

(1) $\neg Q_{n-1}(x_1,\ldots,x_n) \to \big([\beta]_{t+i}(x_t/x_i) \equiv K^n([\beta]_{t+i}(x_t/x_i))\big) \in L(\mathbf{N}_n)$, for $i = 1,2,\ldots$

Hence,

(2) $\neg Q_{n-1}(x_1,\ldots,x_n) \to \big([\beta]_{t+1}(x_t/x_1) \wedge \ldots \wedge [\beta]_{t+n}(x_t/x_n) \equiv K^n(\bigwedge_{x_t})\big) \in L(\mathbf{N}_n)$, for $i = 1,2,\ldots$

Then, by Lemma 1, we have:

(3) $\neg Q_{n-1}(x_1,\ldots,x_n) \to \big(\bigwedge_{x_t}\beta \equiv K^n(\bigwedge_{x_t}\beta)\big) \in L(\mathbf{N}_n)$.

If $\alpha = \beta_1 \circ \beta_2$, with $\circ \in \{\to,\wedge,\vee,\equiv\}$, or $\alpha = \neg\beta$, then the lemma obviously holds. Thus, Lemma 2 is proved. ■

Let now $N_r = \{1,2,\ldots,r\}$ and $\overline{At} = \{\neg\alpha : \alpha \in At\}$. Thus, N_n is the universe of $\mathbf{N}_n$. We then have,

Lemma 3. *For all functions* $f : At \longrightarrow At \cup \overline{At}$ *and* $g_i : N_k \longrightarrow N_n$ *such that* $\forall\alpha \in At : f(\alpha) \in \{\alpha,\neg\alpha\}$ *and* $\forall i,j \in N_r : i \neq j \Rightarrow g_i \neq g_j$*, the following holds:*

$$\underbrace{\neg Q_{n-1}(x_1,\ldots,x_n) \to \sum_{i\in N_r} f(P_t^k(x_{g_i(1)},\ldots,x_{g_i(k)}))}_{\gamma_n} \notin L(\mathbf{N}_n),$$

where $\sum_{i\in N_r}\alpha_i = \alpha_1 \vee \ldots \vee \alpha_r$.

Proof. Let,

(1) $f : At \longrightarrow At \cup \overline{At}$ and $\forall\alpha \in At : f(\alpha) \in \{\alpha,\neg\alpha\}$,

(2) $g_i : N_k \longrightarrow N_n$ and $\forall i,j \in N_r : i \neq j \Rightarrow g_i \neq g_j$.

We define a set $I \subseteq N_r$ as follows:

(3) $I = \{i \in N_r : f\big(P_t^k(x_{g_i(1)}, \ldots, x_{g_i(k)})\big) = \neg P_t^k(x_{g_i(1)}, \ldots, x_{g_i(k)})\}$.

Assume $I \neq \emptyset$ and let $v : At \longrightarrow F_n$ be a function from V_n such that:

(4) $v(P_t^k(x_1, \ldots, x_k)) = \sum_{i \in I}\big(x_1 = \overline{g_i(1)} \wedge x_2 = \overline{g_i(2)} \wedge \ldots \wedge x_k = \overline{g_i(k)}\big)$, and

(5) $v(P_2^2(x_1, x_2)) = (x_1 = x_2)$.

Hence,

(6) $v(P_t^k(x_{i_1}, \ldots, x_{i_k})) = \sum_{i \in I}\big(x_{i_1} = \overline{g_i(1)} \wedge \ldots \wedge x_{i_k} = \overline{g_i(k)}\big)$, and

(7) $v(P_2^2(x_i, y)) = (x_i = y)$.

Thus,

(8) $$\mathbf{N}_n \models h^v\big(Q_{n-1}(x_1, \ldots, x_n) \equiv (x_1 = x_2 \vee x_1 = x_3 \vee \ldots \vee x_1 = x_n \vee \vee x_2 = x_3 \vee \ldots \vee x_2 = x_n \vee \vdots \vee x_{n-1} = x_n)\big).$$

It follows from (8), (6) and (3) that for the sequence $\Sigma : N \longrightarrow N_n$ such that,

(9) $\Sigma(i) = \begin{cases} i \text{ if } i \leq n \\ 1 \text{ if } i > n \end{cases}$

we have:

(10) $\mathbf{N}_n \models h^v\big(\neg Q_{n-1}(x_1, \ldots, x_n)\big)[\Sigma]$, and

(11) $\mathbf{N}_n \not\models h^v\big(f(P_t^k(x_{g_j(1)}, \ldots, x_{g_j(k)}))\big)[\Sigma]$, for every $j \in N_r$.

Therefore, $\mathbf{N}_n \not\models h^v(\gamma_n)$.

If $I \neq \emptyset$, then for $v \in V_n$ such that $v\big(P_t^k(x_1, \ldots, x_k)\big) = (x_1 \neq x_1 \wedge \ldots \wedge x_k \neq x_k)$ and $v\big(P_2^2(x_i, y)\big) = (x_i = y)$ we also have $\mathbf{N}_n \not\models h^v(\gamma_n)$. Thus, Lemma 3 is proved. ■

Theorem. *For every natural number $n \geq 1$ and any closed logical scheme α:*

$$\alpha \in L(\mathbf{N}_n) \iff K^n(\alpha) \in L_2.$$

Proof. Assume that $n > 1$, and

(1) $Vf(\alpha) = \emptyset$,

(2) $P_2^2 \notin Pr(\alpha)$, where $Pr(\alpha)$ is the set of all predicate letters occurring in α.

Firstly, we will show that:

(∗) $K^n(\alpha) \in L_2 \Rightarrow \alpha \in L(\mathbf{N}_n)$.

Let us suppose that:

(1.1) $K^n(\alpha) \in L_2$, and

(1.2) $\alpha \notin L(\mathbf{N}_n)$.

Then, there exists a function $v_1 \in V_n$ such that

(1.3) $\mathbf{N}_n \not\models h^{v_1}(\alpha)$.

We now define a function $v_0 \in V_n$ as follows:

(1.4) $v_0\big(P_t^k(x_1,\ldots,x_k)\big) = \begin{cases} v_0\big(P_t^k(x_1,\ldots,x_k)\big) \text{ if } k \neq 2 \text{ or } t \neq 2 \\ x_1 = x_2 \text{ if } k = 2 \text{ and } t = 2 \end{cases}$

From Lemma 2 and (1.1) it follows that:

(1.5) $\neg Q_{n-1}(x_1,\ldots,x_n) \to \alpha \in L(\mathbf{N}_n)$.

Thus,

(1.6) $\mathbf{N}_n \models h^{v_0}(\neg Q_{n-1}(x_1,\ldots,x_n)) \to h^{v_0}(\alpha)$.

Hence, by (2) and (1.4) we obtain:

(1.7) $\mathbf{N}_n \models h^{v_0}(\neg Q_{n-1}(x_1,\ldots,x_n)) \to h^{v_1}(\alpha)$.

Therefore, by (1) and (1.4), we get:

(1.8) $\mathbf{N}_n \models \bigvee_{x_1} \ldots \bigvee_{x_n} \to h^{v_1}(\alpha)$.

But then, $\mathbf{N}_n \models h^{v_1}(\alpha)$, which contradicts (1.3). Thus, the implication (∗) is proved.

Now we are going to prove that:

(∗∗) $\alpha \in L(\mathbf{N}_n) \Rightarrow K^n(\alpha) \in L_2$.

Assume that:

(2.1) $\alpha \in L(\mathbf{N}_n)$, and

(2.2) $K^n(\alpha) \notin L_2$.

Then, there are $\gamma_1^1,\ldots,\gamma_1^{k_1},\ldots,\gamma_s^1,\ldots,\gamma_s^{k_s} \in At \cup \overline{At}$ such that:

(2.3) $K^n(\alpha) \equiv (\gamma_1^1 \vee \ldots \vee \gamma_1^{k_1}) \wedge \ldots \wedge (\gamma_s^1 \vee \ldots \vee \gamma_s^{k_s}) \in L_2$ and, for some $i \leq s$, we have:

(2.4) $\gamma_i^1 \vee \ldots \vee \gamma_i^{k_i} \notin L_2$.

It follows from (2.1), (2.3) and from Lemma 2 that:

(2.5) $\neg Q_{n-1}(x_1, \ldots, x_n) \to \gamma_i^1 \vee \ldots \vee \gamma_i^{k_i} \in L(\mathbf{N}_n)$.

However, by Lemma 3 and (2.4) we get:

(2.6) $\neg Q_{n-1}(x_1, \ldots, x_n) \to \gamma_i^1 \vee \ldots \vee \gamma_i^{k_i} \notin L(\mathbf{N}_n)$,

which contradicts (2.5). Thus, the implication (∗∗) is proved. ■

It is easily verified that if $n = 1$, then $\alpha \equiv \bigwedge_{x_t} \alpha \in L(\mathbf{N}_n)$, $K^n(\alpha) \equiv \alpha \in L(\mathbf{N}_n)$, and $\sum_{i \in N_r} f\big(P_t^k(x_{g_i(1)}, \ldots, x_{g_i(k)})\big) \notin L(\mathbf{N}_n)$, for $f : At \longrightarrow At \cup \overline{At}$ and $g_i : N_k \longrightarrow N_n$ such that $\forall \alpha \in At : f(\alpha) \in \{\alpha, \neg\alpha\}$ and $\forall i, j \in N_r : i \neq j \Rightarrow g_i \neq g_j$. Thus, our Theorem holds also for $n = 1$.

It is easily seen that $K^s(\alpha) \equiv K^{s+1}(\alpha)(x_{s+1}/x_s) \in L_2$. Hence, by the above theorem, we get:

Corollary. *If $n \leq s$, then $L(\mathbf{N}_s) \subseteq L(\mathbf{N}_n)$.*

References

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