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SIMPLIFIED AFFINE PHASE STRUCTURES

A b s t r a c t. Phase models for affine linear logic were independently devised by Lafont [10] and Piazza [15], although foreshadowed by Ono [14]. However, the existing semantics either contain no explicit directions for the construction of models in the general case, or else are forced to resort to additional conditions extending Girard's semantics (preordering of the monoids, the condition that the set of antiphases be an ideal). We dispense with these extra postulates - at least for the subexponential fragment of this logic - considering structures where the set of antiphases is concretely constructed. Moreover, we show the equivalence of our phase models and the algebraic models of affine linear logic by means of a representation theorem.

1. Introduction

Three different research tradition merge into affine linear logic, i.e. linear logic plus unrestricted weakening.

In the first place, there is the "French" tradition of mainstream linear logicians. Viewed as an extension of linear logic, affine logic is particularly

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interesting because of the decidability of its full propositional fragment (Kopylov [9]). Lafont [10] and Piazza [15] provided phase semantics for affine logic using, respectively, phase structures where the set of antiphases is an ideal, and a special kind of preordered phase structures.

In the second place, we have the "Russian" tradition of contraction-free logics. Grishin [8] studied the proof theory and algebra of classical logic minus contraction, which is equivalent to affine logic in the $\{\neg, \&, \rightarrow\}$-vocabulary. Ono [13, 14] thoroughly investigated its relational semantics.

Lastly, there is the "Italian" tradition of comparative logicians. Ettore Casari (see e.g. [1–3]) introduced comparative logics in the early 1980s in order to account for the Aristotelian logic of comparative propositions ("x is more P than y", "x is more P than Q", etc.). Later, Casari's investigations focused on the algebraic structures underlying such logics - he suggested an interesting common abstraction of the notions of Boolean algebra and Abelian l-group, viz. the concept of Abelian l-pregroup (since in what follows we shall only be concerned with commutative structures, the specification "Abelian" will often be omitted). Bounded comparative logic is the logic of l-zeroids, i.e. bounded l-pregroups. Casari [3] has recently shown that bounded comparative logic is equivalent to Grishin's contraction-free logic and that l-zeroids amount to Grishin's L_0-algebras. Moreover, Minari [12] represented l-zeroids as subdirect products of minimum irreducible l-zeroids.

In the following, we shall consider a different, phase-semantical representation of l-zeroids and exploit it as a device to construct relational models for subexponential affine logic, much in the same way as the relational semantics for orthologic (Dalla Chiara [4]) rests upon a phase-semantical representation of ortholattices.

2. Syntactic and algebraic preliminaries

The following section is aimed both at fixing up terminology and notation and at making the paper self-contained. No new result will be presented herewith.

Subexponential affine logic (AL) is completely axiomatized by the following postulates expressed in a propositional language containing the connectives $\neg$ (affine linear negation), & (additive conjunction), $\to$ (affine linear implication):

A1. $A \to (B \to A)$

A2. $(A \to B) \to ((B \to C) \to (A \to C))$

A3. $(A \to (B \to C)) \to (B \to (A \to C))$

A4. $\neg\neg A \to A$

A5. $(A \to B) \to (\neg B \to \neg A)$

A6. $A\&B \to A$

A7. $A\&B \to B$

A8. $(A \to B)\&(A \to C) \to (A \to B\&C)$

R1. $A, A \to B \Rightarrow B$

Additive disjunction and multiplicative conjuntion and disjunction can be defined in the usual way. The notions of proof, theorem, etc. are introduced in the standard fashion.

An *l-pregroup* is a structure $\mathcal{G} = \langle G, +, -, 0, \leq\rangle$ s.t.

(i) $\langle G, +\rangle$ is an Abelian semigroup;

(ii) $\langle G, \leq\rangle$ is a lattice;

(iii) $-$ is an involution on $\langle G, \leq\rangle$;

(iv) For every x, y in G:

(iv.i) $0 + 0 = 0$;

(iv.ii) $x + -x = 0$;

(iv.iii) $0 \leq -x + y$ iff $x \leq y$.

The product operation $x \cdot y$ is defined as $-(-x + -y)$.

An l-pregroup is an l-*group* iff $0 = -0$. An l-pregroup is an l-*zeroid* iff for every x it holds that $x + 0 = 0$. Remark that the only l-zeroid which is also an l-group is the trivial group $\{0\}$.

Lemma 1. *In every l-zeroid* $\mathcal{Z} = \langle Z, +, -, 0, \leq \rangle$*:*

(1.i) $-0 \leq x$;

(1.ii) $x \leq 0$;

(1.iii) $x \cdot y \leq x$;

(1.iv) $x \leq x + y$;

(1.v) $\langle Z, \cdot, 0 \rangle$ *is an Abelian monoid;*

(1.vi) $-0 \cdot -0 = -0$;

(1.vii) $x \cdot -0 = -0$;

(1.viii) $x \cdot -y = -0$ *iff* $x \leq y$;

(1.ix) *if* $x \leq y,\ x' \leq y'$ *then* $x + x' \leq y + y'$;

(1.x) *if* $x \leq y,\ x' \leq y'$ *then* $x \cdot x' \leq y \cdot y'$.

Proof. Scattered over several papers of Casari [1–3]. Remark that the above properties hold for all partially ordered zeroids, not only for the lattice-ordered ones. ■

Lest the reader should be misled by our unusual notation (aimed at stressing similarities with groups), let us emphasize that in an l-zeroid the elements referred to as "0" and "-0" are respectively the top and the bottom element, i.e. what in standard terminology of bounded algebras are usually denoted by "1" and "0".

A *Z-algebraic model* is a pair $\mathcal{M} = \langle \mathcal{Z}, h \rangle$ s.t. 1) $\mathcal{Z}$ is an l-zeroid and 2) the homomorphism $h : \text{FOR} \to \mathcal{Z}$ extends the arbitrary map $v : \text{VAR} \to Z$ in such a way that:

$h(p) = v(p)$;

$h(\neg A) = -h(A)$;

$h(A\&B) = h(A) \wedge h(B)$;

$h(A \to B) = -h(A) + h(B)$.

A is *true in* $\mathcal{M}$ ($\mathcal{M} \models_Z A$) iff $0 = h(A)$; A is *Z-logically valid* ($\models_Z A$) iff A is true in every Z-algebraic model $\mathcal{M}$.

Theorem 1. *$\vdash_{AL} A$ iff $\models_Z A$.*

Proof. See Casari [1]. ■

3. Constructing l-zeroids out of Abelian monoids

This section will be devoted to the exposition of a general, phase-semantical method for the construction of l-zeroids.

Let $\mathcal{M} = \langle P, \cdot, 1\rangle$ be an Abelian monoid. Of course, P contains at least one idempotent element (viz., the unit 1). Let i be a designated element of P having such a property. We now define:

$$\perp = \{i \cdot x : x \in P\}.$$

Since $\perp$ is nonempty (it contains at least i), $\mathcal{P} = \langle P, \cdot, 1, \perp\rangle$ is a *phase structure* (cf. Girard [7], Gallier [6], Troelstra [16]) with $\perp$ its set of antiphases. As usual, thus, we can define an orthogonality operation $X^\perp$ on nonempty subsets of P (we shall henceforth omit the specification "nonempty"), along with the corresponding closure operation $C(X) = X^{\perp\perp}$, a generalized product XY and a sum operation $X \oplus Y$. We set:

$X^\perp =_{\text{df}} \{y : \forall x(x \in X \Rightarrow x \cdot y \in \perp)\}$;

$XY =_{\text{df}} \{x \cdot y : x \in X \& y \in Y)\}$;

$X \oplus Y =_{\text{df}} (X^\perp Y^\perp)^\perp$.

By $\mathcal{C}(P)$ we denote the set of all C-closed subsets of P.

Now we can prove the following

Lemma 2.

(i) $i \in \bot$;

(ii) $\bot^\bot = P$;

(iii) $P^\bot = \bot$;

(iv) *if* $X \subseteq P$ *belongs to* $\mathcal{C}(P)$, *then* $\bot \subseteq X$;

(v) $i = 1$ *implies* $P = \bot$;

(vi) $i = 1$ *implies* $\mathcal{C}(P) = \{P\}$;

(vii) $i = 1$ *iff* $1 \in \bot$;

(viii) *every antiphase is representable as the product of two antiphases;*

(ix) *the product of two antiphases is itself an antiphase;*

(x) *if* $X \subseteq P$ *belongs to* $\mathcal{C}(P)$, *then* $P = (X^\bot X)^\bot$;

(xi) *if* $X \subseteq P$, *then* $\bot = \bot X^\bot$.

Proof. (i) Trivial: $i = i \cdot 1 = i \cdot i$.

(ii) $\bot^\bot = \{x : \forall y(y \in \bot \Rightarrow x \cdot y \in \bot)\}$. But $y = i \cdot z$ for some z, hence for every $x, x \cdot (i \cdot z) = i \cdot (x \cdot z) \in \bot$.

(iii) $P^\bot = \{y : \forall x(x \cdot y \in \bot)\}$. Now, $\bot \subseteq P^\bot$ since $(w \cdot i) \cdot z = i \cdot (z \cdot w) \in \bot$; but also $P^\bot \subseteq \bot$ since if $x \notin \bot$, then $1 \cdot x = x \notin \bot$.

(iv) If X is a C-closed subset of P, $X^{\bot\bot} = X$; but $X^{\bot\bot} = \{x : \forall y(y \in X^\bot \Rightarrow x \cdot y \in \bot)\}$, and, as we have seen in (ii), every antiphase z is such that, for every phase x, $z \cdot x \in \bot$.

(v) If $i = 1$, $\bot = \{1 \cdot x : x \in P\} = P$.

(vi) From (iv) and (v).

(vii) If $1 \in \bot$, then for some x we have $1 = i \cdot x$; but then $1 = i \cdot x = (i \cdot i) \cdot x = (i \cdot x) \cdot i = i$. If $i = 1$, then $1 = i \cdot 1 \in \bot$.

(viii) If x is an antiphase, for some y we have $x = i \cdot y = (i \cdot i) \cdot y = i \cdot (i \cdot y)$.

(ix) Indeed $(i \cdot x) \cdot (i \cdot y) = (i \cdot i) \cdot (x \cdot y) = i \cdot (x \cdot y)$.

(x) Since by (ii) $\bot^\bot = P$, it is sufficient to show that $X^\bot X = \bot$. Now, we have $X^\bot X \subseteq \bot$ as the product of a member of an X whatsoever by a phase which, if multiplied by any member of X, gives an antiphase is in any case an antiphase. However, if X belongs to $\mathcal{C}(P)$, it also holds $\bot \subseteq X^\bot X$.

This means, indeed, that every antiphase is representable as the product of an element of X by an element of its dual. But, for any X, we have $\bot \subseteq X^\bot$, since $x \cdot (i \cdot y) = i \cdot (x \cdot y) \in \bot$. Hence, given the antiphase $i \cdot x$, $i \cdot x = (i \cdot i) \cdot x = i \cdot (i \cdot x)$, viz. the product of a member of X (i.e. i) by a member of its dual (i.e. $i \cdot x$).

(xi) $\bot X^\bot \subseteq \bot$ because any phase, if multiplied by an antiphase, produces an antiphase (cf. the proof of (ii) above). Moreover $\bot \subseteq \bot X^\bot$, as $\bot \subseteq X^\bot$ (cf. the proof of (x)): hence a generic antiphase $i \cdot x = (i \cdot i) \cdot x = i \cdot (i \cdot x)$ is representable as the product of a member of $\bot$ (that is, i) by a member of $X^\bot$ (viz., $i \cdot x$). ■

Now, let $S = \langle \mathcal{C}(P), \oplus, ^\bot, P, \subseteq \rangle$, and let Π be a subset of $\mathcal{C}(P)$ containing $\bot$ and P and closed w.r.t. duals, sums and binary intersections (by ordinary phase semantics, $\mathcal{C}(P)$ is itself an appropriate example). Let now $S^{\mathcal{Z}} = \langle \Pi, \oplus, ^\bot, P, \subseteq \rangle$. Obviously, $S^{\mathcal{Z}}$ is a substructure of S.

Theorem 2. *$S^{\mathcal{Z}}$ is an l-zeroid; moreover, if $i = 1$, then $S^{\mathcal{Z}}$ is the trivial l-group $\{P\}$.*

Proof. Π is a nonempty set, since it contains $\bot$ and P. Moreover, by our stipulations and ordinary phase semantics, $\langle \Pi, \subseteq \rangle$ is a lattice, and $X = X^{\bot\bot}$ holds as Π is a subset of $\mathcal{C}(P)$. Trivially, $X \subseteq Y$ implies $Y^\bot \subseteq X^\bot$. By ordinary phase semantics, $\langle \Pi, \oplus, \bot \rangle$ is an Abelian monoid and $X \subseteq Y$ holds iff $P \subseteq X^\bot \oplus Y$. Let us now show that a (redundant) set of conditions holds whence the theorem follows.

(A) $P \oplus P = P$, i.e. $(P^\bot P^\bot)^\bot = P$. By Lemma 2(iii) it is sufficient to prove that $(\bot\bot)^\bot = P$ and by Lemma 2(ii) it is enough to show that $\bot\bot = \bot$. But this follows from Lemma 2(viii)–(ix).

(B) $X \oplus X^\bot = P$, i.e. $(X^\bot X^{\bot\bot})^\bot = P$, i.e. ($X$ is a C-closed subset of P) $(X^\bot X)^\bot = P$. Since X is C-closed, this follows from Lemma 2(x).

(C) $X \subseteq P$. Trivial.

(D) $\bot \subseteq X$. As X is a C-closed subset of P, this follows from Lemma 2(iv).

(E) $X \oplus P = P$, i.e. $(X^{\perp}P^{\perp})^{\perp} = P$, i.e. $(X^{\perp}\perp)^{\perp} = P$. In virtue of Lemma 2(ii) it is enough to prove that $X^{\perp}\perp = \perp$; this, however, follows from the commutativity of product and from Lemma 2(xi).

(F) If i = 1, then $S^{\mathcal{Z}}$ is the trivial l-group $\{P\}$. This is a consequence of Lemma 2(v)-(vii).

Remark. If, given a phase structure $\langle P, \cdot, 1, \perp\rangle$ constructed as above, we define $x \leq y =_{\mathrm{df}} \exists z(x = z \cdot y)$ (i.e. the converse of the divisibility relation), then $\langle P, \cdot, 1, \perp, \leq\rangle$ is a preordered affine phase space as in Piazza [15], where $\perp = \{x : x \leq i\}$.

Example. The following l-zeroid, constructed by the method just expounded, is neither an MV-algebra nor, all the more so, a Boolean algebra. Let us consider the Abelian monoid $\mathcal{M} = \langle\{-1, a, b, c, d, 1\}, \cdot, 1\rangle$ (cf. the monoid P_2 of Minari [11]). The following matrix defines the operation of product:

	-1	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1
a	-1	-1	-1	-1	-1	a
b	-1	-1	a	-1	a	b
c	-1	-1	-1	c	c	c
d	-1	-1	a	c	c	d
1	-1	a	b	c	d	1

Let $i = c$. According to our definitions, we have:

$\perp = \{-1, c\}$;

$\Pi = \mathcal{C}(\{-1, a, b, c, d, 1\}) = \{\{-1, c\}, \{-1, a, c\}, \{-1, a, c, d\}, \{-1, a, b, c, d\}, \{-1, a, b, c, d, 1\}\}$ (this l-zeroid is linearly ordered by set-theoretical inclusion);

$\{-1, c\}^{\perp} = \{-1, a, b, c, d, 1\}$;

$\{-1, a, c\}^{\perp} = \{-1, a, b, c, d\}$;

$\{-1, a, c, d\}^{\perp} = \{-1, a, c, d\}$;

$\{-1, a, b, c, d\}^{\perp} = \{-1, a, c\}$;

$\{-1, a, b, c, d, 1\}^{\perp} = \{-1, c\}$.

Now, if $X = \{-1, a, c, d\}$, then $X \oplus X \neq X$. In fact,

$X \oplus X = (X^{\perp}X^{\perp})^{\perp} = \{-1, a, b, c, d, 1\} \neq \{-1, a, c, d\}$.

Moreover, if $X = \{-1, a, c, d\}$ and $Y = \{-1, a, c\}$, then $(X^{\perp} \oplus Y)^{\perp} \oplus Y \neq (Y^{\perp} \oplus X)^{\perp} \oplus X$. In fact, $(X^{\perp} \oplus Y)^{\perp} \oplus Y = ((XY^{\perp})^{\perp}Y^{\perp})^{\perp} = \{-1, a, b, c, d\}$, whereas $(Y^{\perp} \oplus X)^{\perp} \oplus X = ((YX^{\perp})^{\perp}X^{\perp})^{\perp} = \{-1, a, c, d\}$.

4. Representing l-zeroids

We are now in a position to provide a set-theoretical representation of l-zeroids. As previously remarked, however, there already exists a different representation theorem for the class of l-zeroids (Minari [12]).

Theorem 3. *Every l-zeroid is isomorphic to an l-zeroid arising out of an Abelian monoid.*

Proof. Let $\mathcal{Z} = \langle Z, +, -, 0, \leq \rangle$ be an l-zeroid, and let $x \cdot y$ be defined as usual as $-(-x + -y)$. Then, in virtue of Lemma 1.(v), $\langle Z, \cdot, 0 \rangle$ is an Abelian monoid. Therefore, we can consider the phase structure $\mathcal{P} = \langle Z, \cdot, 0, \perp \rangle$, where $\perp =_{\mathrm{df}} \{-0\} = \{-0 \cdot x : x \in Z\}$. That is, we select -0 as the idempotent element (cf. Lemma 1.(vi)) generating the set of antiphases.

By the way, notice that in every l-zeroid $\{-0\}$ coincides with the set of all products of mutually orthogonal elements. Indeed, $-0 = -0 \cdot -0$ is such that $z \leq -0$ and $z \leq -0$ implies $z = -0$; conversely, suppose $\forall z(z \leq x \;\&\; z \leq y \Rightarrow z = -0)$; by Lemma 1,(iii)-(iv) $x \cdot y \leq x$ and $x \cdot y \leq y$, hence $x \cdot y = -0$. Then, our definition of $\perp$ is consistent with the one used in most phase-semantical representations of algebraic structures.

Now, for X, $Y \subseteq Z$ define as usual $X^{\perp} =_{\mathrm{df}} \{y : \forall x(x \in X \Rightarrow x \cdot y \in \perp)\}$; $XY =_{\mathrm{df}} \{x \cdot y : x \in X \;\&\; y \in Y)\}$; $X \oplus Y =_{\mathrm{df}} (X^{\perp}Y^{\perp})^{\perp}$; $C(X) =_{\mathrm{df}} X^{\perp\perp}$; $\mathcal{C}(Z) = \{X : C(X) = X\}$. Notice that $X^{\perp} = \{y : \forall x(x \in X \Rightarrow x \cdot y = -0)\}$. The latter set, by Lemma 1.(viii), is tantamount to $\{y : \forall x(x \in X \Rightarrow x \leq -y)\}$. In other words, the dual of X is the set of the members of Z which are orthogonal to every x in X.

Moreover, let Π be the set of all principal lattice ideals of Z. By $I(x)$ we shall hereafter refer to the principal lattice ideal of Z generated by x. Given what has been proved in Theorem 2 above, we can conclude that $\mathcal{A} = \langle \Pi, \oplus, ^{\perp}, Z, \subseteq \rangle$ is an l-zeroid if we manage to show that Π:

1) is a subset of $\mathcal{C}(Z)$;

2) contains $\{-0\}$ and Z;

3) is closed w.r.t. duals, sums, and binary intersections.

Ad 1). We have to prove that for every x in Z, $I(x) = \{y : \forall z(\forall w(w \leq x \Rightarrow w \leq -z) \Rightarrow y \leq -z)\}$. Indeed, if $y \leq x$, then for every z, if $w \leq x$ implies $w \leq -z$, it follows that $y \leq -z$. Conversely, if $\forall z(\forall w(w \leq x \Rightarrow w \leq -z) \Rightarrow y \leq -z)$, then $y \leq x$. Suppose indeed, for the sake of argument, that it is not the case that $y \leq x$; then for $z = 0$ we should have that $\forall w(w \leq x \Rightarrow w \leq -0) \Rightarrow y \leq -0$, i.e. $x \leq -0 \Rightarrow y \leq -0$. But then $y \leq x$, against the hypothesis.

Ad 2). $\{-0\} \in \Pi$ because $\{-0\} = I(-0)$; $Z \in \Pi$ because $Z = I(0)$.

Ad 3). We shall show that: a) $I(x)^{\perp} = I(-x)$; b) $I(x) \oplus I(y) = I(x+y)$; c) $I(x) \cap I(y) = I(x \wedge y)$.

a) Remember that $I(x)^{\perp} = \{y : \forall z(z \leq x \Rightarrow z \leq -y)\}$. If $y \leq -x$ and $z \leq x$, i.e. $-x \leq -z$, then $y \leq -z$, i.e. $z \leq -y$. Conversely, if $\forall z(z \leq x \Rightarrow z \leq -y)$, for $z = x$ we get $x \leq x \Rightarrow x \leq -y$. But the antecedent is certainly true: hence $x \leq -y$, i.e. $y \leq -x$.

b) By a), it is enough to show that $I(x+y) = (I(-x)I(-y))^{\perp}$. Let us suppose $z \leq x+y$, $z' \leq -x$, $z'' \leq -y$. We have to show $z' \cdot z'' \leq -z$. From $z' \leq -x$ and $z'' \leq -y$, by Lemma 1.(x), we conclude $z' \cdot z'' \leq -x \cdot -y$, i.e. $x + y \leq -(z' \cdot z'')$. By transitivity, then, $z \leq -(z' \cdot z'')$, which is the same as $z' \cdot z'' \leq -z$. Conversely, suppose $\forall w(w = w' \cdot w'' \& \forall u'u''((u' \leq x \Rightarrow u' \leq -w') \& (u'' \leq y \Rightarrow u'' \leq -w')) \Rightarrow w \leq -z)$. Let $w = -x \cdot -y$. Surely if $u' \leq x$ then $u' \leq --x$; likewise, if $u'' \leq y$ then $u'' \leq --y$. So $-x \cdot -y \leq -z$, that is $z \leq x + y$.

c) From standard lattice theory we know that $I(x) \cap I(y) = I(x \wedge y)$. Now, let $f : Z \to \Pi$ be such that $f(x) = I(x)$. We have to show that f is an isomorphism. Clearly, it is one-one. Moreover, it is just as evident that $x \leq y$ implies $I(x) \subseteq I(y)$. Lastly, as we have already proved:

1) $f(0) = I(0) = Z$;
2) $f(-x) = I(-x) = I(x)^{\perp} = f(x)^{\perp}$;
3) $f(x + y) = I(x + y) = I(x) \oplus I(y) = f(x) \oplus f(y)$.

This concludes the proof of our theorem. ■

Remark that we did not use anywhere the hypothesis that $\mathcal{Z}$ is a *lattice*-ordered zeroid (even principal lattice ideals in $\mathcal{Z}$ might have been characterized otherwise). Thus, what we proved is that every *partially* ordered zeroid is isomorphic to a *po*-zeroid of sets - an algebraically stronger (although logically weaker!) result.

5. From algebraic models to relational models

The methods and techniques employed throughout this section are mainly borrowed from Dalla Chiara [4].

A *Z-relational model* $\mathcal{R}$ is a pair $\langle \mathcal{F}, \rho \rangle$ such that:

(1) $\mathcal{F} = \langle W, \cdot, 1, i, \Pi \rangle$, where $\langle W, \cdot, 1 \rangle$ is an Abelian monoid, i is a designated idempotent element of W, $\Pi \subseteq \mathcal{C}(W)$ is closed w.r.t. ${}^{\perp}$, $\oplus$, $\cap$, $\perp$, W, defined as above;

(2) ρ is a map assigning to every variable of the language of AL an element of Π, extended to a homomorphism by the following clauses:
$\rho(\neg A) = \rho(A)^{\perp}$;
$\rho(A \& B) = \rho(A) \cap \rho(B)$;
$\rho(A \rightarrow B) = \rho(A)^{\perp} \oplus \rho(B)$.

We could just as well define a binary accessibility relation on W by setting:

$Rxy =_{\mathrm{df}} x \cdot y \notin \perp$.

Hence,

$x \in \rho(\neg A)$ iff $x \in \{y : \forall z(z \in \rho(A) \Rightarrow y \cdot z \in \perp\}$ iff $x \in \{y : \forall z(y \cdot z \notin \perp \Rightarrow z \notin \rho(A)\}$.

If $x \models A$ is short for $x \in \rho(A)$, then we may reformulate the above equality as $x \models \neg A$ iff $\forall z(Rxz \Rightarrow z \not\models A)$.

We stipulate that A *is true in* $\mathcal{R}$ ($\mathcal{R} \models_R A$) iff $\rho(A) = W$, and that A is *logically valid* ($\models_R A$) iff, for every $\mathcal{R}$, $\mathcal{R} \models_R A$.

We now proceed to show how to convert algebraic models into relational models, and vice versa.

Theorem 4. *For every Z-algebraic model $\mathcal{M}$ there is a Z-relational model $\mathcal{R}^{\mathcal{M}}$ s.t. for every A, $\mathcal{M} \models_Z A$ iff $\mathcal{R}^{\mathcal{M}} \models_R A$; likewise, for every Z-relational model $\mathcal{R}$ there is a Z-algebraic model $\mathcal{M}^{\mathcal{R}}$ s.t. for every A, $\mathcal{R} \models_R A$ iff $\mathcal{M}^{\mathcal{R}} \models_Z A$.*

Proof. As to the first claim, let $\mathcal{M} = \langle \mathcal{Z}, h \rangle$ be a Z-algebraic model, where $\mathcal{Z} = \langle Z, -, +, 0, \leq \rangle$. If Π is the set of all the principal lattice ideals of Z, $\mathcal{R}^{\mathcal{M}}$ may be identified with $\langle \mathcal{F}, \rho \rangle$, where $\mathcal{F} = \langle Z, \cdot, 0, -0, \Pi \rangle$ and $\rho(A) = \{x \in Z : x \leq h(A)\} = I(h(A))$. By the results of Section 4, $\mathcal{F}$ is a frame. Moreover, we have checked earlier that $I(x)^{\perp} = I(-x)$, $I(x) \oplus I(y) = I(x+y)$, $I(x) \cap I(y) = I(x \wedge y)$. Hence:

$\rho(\neg A) = I(h(\neg A)) = I(-h(A)) = I(h(A))^{\perp} = \rho(A)^{\perp}$;

$\rho(A \to B) = I(h(A \to B)) = I(-h(A) + h(B)) = I(-h(A)) \oplus I(h(B)) = I(h(A))^{\perp} \oplus I(h(B)) = \rho(A)^{\perp} \oplus \rho(B)$;

$\rho(A \& B) = I(h(A \& B)) = I(h(A) \wedge h(B)) = I(h(A)) \cap I(h(B)) = \rho(A) \cap \rho(B)$.

Thus, ρ is perfectly well-defined. Moreover, $\mathcal{M} \models_Z A$ iff $h(A) = 0$ iff $\rho(A) = Z$ iff $\mathcal{R}^{\mathcal{M}} \models_R A$.

As regards our second claim, let $\mathcal{R} = \langle \mathcal{F}, \rho \rangle$ be a Z-relational model where $\mathcal{F} = \langle W, \cdot, 1, i, \Pi \rangle$. $\mathcal{M}^{\mathcal{R}}$ can be identified with $\langle \mathcal{Z}, h \rangle$, where $\mathcal{Z} = \langle \Pi, ^{\perp}, \oplus, W, \subseteq \rangle$ and $h(A) = \rho(A)$. By our results of Section 3, Z is an l-zeroid, and h is trivially (according to our definitions) a well-defined algebraic valuation. Moreover, $\mathcal{R} \models_R A$ iff $\rho(A) = W$ iff $h(A) = W$ iff $\mathcal{M}^{\mathcal{R}} \models_Z A$. ■

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