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VARIETIES OF TENSE ALGEBRAS

A b s t r a c t. The paper has two parts preceded by quite comprehensive preliminaries.

In the first part it is shown that a subvariety of the variety $\mathcal{T}$ of all tense algebras is discriminator if and only if it is semisimple. The variety $\mathcal{T}$ turns out to be the join of an increasing chain of varieties $\mathcal{D}_n$, which are discriminator varieties. The argument carries over to all finite type varieties of boolean algebras with operators satisfying some term conditions. In the case of tense algebras, the varieties $\mathcal{D}_n$ can be further characterised by certain natural conditions on Kripke frames.

In the second part it is shown that the lattice of subvarieties of $\mathcal{D}_0$ has two atoms, the lattice of subvarieties of $\mathcal{D}_1$ has countably many atoms, and for $n > 1$, the lattice of subvarieties of $\mathcal{D}_n$ has continuum atoms. The proof of the second of the above statements involves a rather detailed description of zero-generated simple algebras in $\mathcal{D}_1$.

Almost all the arguments are cast in algebraic form, but both parts begin with an outline describing their contents from the dual point of view of tense logics.

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Introduction and acknowledgements

The departure point of this investigation can be traced down to J. Perzanowski's "topography of modal logics" as presented in [17]. What began as an attempt at following the guidelines provided by this book, soon proved to live a life of its own, driving the author more and more into the algebraic framework which the work has ultimately acquired.

Yet, it would not have acquired any form whatsoever, had it not been for the help, friendliness, and endurance of many people. I am especially grateful to Jerzy Perzanowski—for the initial idea of this investigation, constant help during its preparation, and putting up with my various delays; to Andrzej Wroński—for opening up the world of algebra to me, and being such a wonderful boss; and to Frank Wolter—for helping me avoid one particularly spectacular disaster which was about to happen in Aix-en-Provence in August 1997. Thanks are also due to the anonymous referee for pointing out a number of errors, eliminating which has led not as much to improving the paper as to shortening it considerably (given its present length, this might well be considered a merit in itself).

To help in distinguishing the work of the present author from the old-established theorems and elements of folklore, a labelling convention has been adopted: anything to which the author claims some rights (not necessarily of priority, but certainly of independence) has been labelled Theorem, Lemma, Corollary, or, just once, Example. This, admittedly, leads at places to calling a trivial observation "Lemma", and an important theorem "Fact", but—the author hopes—it clarifies something, at the very least the state of his own mind.

Preliminaries

1. Algebraic generalities

Our algebraic terminology and notation will follow that of [13]. For reader's convenience we repeat here some of the most frequently used notions and conventions.

As usual, boldface capitals will stand for algebras, and italicised capitals for their universes. **Con A** stands for the *congruence lattice* of an algebra **A**, whereas Con **A** denotes the bare set of all congruences of **A**. Further, $\mathbf{Cg}^{\mathbf{A}}(X)$ and $\mathbf{Sg}^{\mathbf{A}}(X)$ stand for the congruence generated by the set $X \subseteq A$ and the subuniverse generated by $X \subseteq A$, respectively. We will write $\mathbf{Cg}^{\mathbf{A}}(a, b)$ instead of $\mathbf{Cg}^{\mathbf{A}}(\{a, b\})$ in the case of *principal* congruences, i.e., congruences generated by a single pair of elements. A congruence is called *compact,* or finitely generated iff it is a join of finitely many principal congruences.

If **Con A** is a distributive lattice, the algebra **A** is called *congruence distributive;* if for $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ we have $\alpha \circ \beta = \beta \circ \alpha$, **A** is called *congruence permutable;* when this is the case, we also have $\alpha \vee \beta = \alpha \circ \beta$. An algebra is *arithmetical* iff it is both congruence distributive and permutable. If for any $\alpha, \beta \in \mathrm{Con}\,\mathbf{A}$ and any $x \in A$ we have: $[x]_\alpha = [x]_\beta$ implies $\alpha = \beta$, then **Con A** is called *regular.*

If any of the above properties is true of all algebras in a class $\mathcal{K}$, the appropriate notion is applied to the whole class.

A class $\mathcal{K}$ of algebras is said to have the *congruence extension property* (CEP, for short) iff for any $\mathbf{A} \subseteq \mathbf{B} \in \mathcal{K}$ and any $\Theta \in \mathrm{Con}\,\mathbf{A}$, there is a $\Phi \in \mathrm{Con}\,\mathbf{B}$ such that Θ is the restriction of Φ to $A \times A$.

For a class $\mathcal{K}$ of algebras, $I(\mathcal{K})$, $H(\mathcal{K})$, $S(\mathcal{K})$, $P(\mathcal{K})$, $P_S(\mathcal{K})$, $P_U(\mathcal{K})$, denote the classes of: all isomorphic images, homomorphic images, subalgebras, direct products (which always contains the trivial algebra), subdirect products, and ultraproducts of the algebras from $\mathcal{K}$.

Algebras are classified into varieties. A class $\mathcal{K}$ of algebras is a variety iff $\mathcal{K}$ is closed under homomorphic images, subalgebras and direct products ($\mathcal{K} = HSP(\mathcal{K})$) iff $\mathcal{K}$ is an equational class (in other words, $\mathcal{K}$ consists of precisely these algebras that satisfy all equations from a given set Γ, which is often said to *axiomatise* the variety $\mathcal{K}$).

For a class of algebras $\mathcal{K}$ of the same type, $\mathcal{V}(\mathcal{K})$ and $\mathcal{Q}(\mathcal{K})$ will stand, respectively, for the variety and the quasivariety generated by $\mathcal{K}$.

Certain algebras serve as "building-blocks" for others. The most important of these are *subdirectly irreducible* (from now on: si) algebras, i.e., these whose congruence lattices have a unique atom (called the *monolith*). By the famous Birkhoff Theorem, every algebra can be built from (more precisely: is a subdirect product of) si algebras. Two other important types of "building-blocks" are: *simple* algebras, i.e., these whose congruence lattices are two-element; and *directly indecomposable,* i.e., these that cannot be nontrivially represented as Cartesian products.

Occasionally, $\mathcal{V}_{SI}(\mathcal{K})$ will denote the class of all si algebras from $\mathcal{V}(\mathcal{K})$, and $\mathcal{V}_S(\mathcal{K})$ — the class of all simple algebras from $\mathcal{V}(\mathcal{K})$.

Finally, let us recall yet another celebrated theorem which will be of use in the sequel, namely, Jónsson Theorem, stating that, for a congruence distributive variety $\mathcal{V}$, generated by a class $\mathcal{K}$, we have: $\mathcal{V}_{SI}(\mathcal{K}) \subseteq HSP_U(\mathcal{K})$.

2. Tense logics and tense algebras

The language of propositional tense logics consists of the countable set $\mathrm{Var} = \{x_i : i \in \omega\}$ of propositional variables; the set of logical connectives Λ_t, comprising: (i) the standard connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, of which conjunction and negation can be viewed as primitive and the others as defined, (ii) the constants $\bot =_{\mathrm{df}} x_0 \wedge \neg x_0$ and $\top =_{\mathrm{df}} \neg\bot$, (iii) two unary connectives h, g, whose intuitive reading is *it has always been the case that* and *it is always going to be the case that*, respectively. The set of formulas For is defined in the standard recursive way.

A *tense logic* is any set of formulas closed under substitution and under some consequence operation given by:

- A set of axioms, containing:
  (A0) All classical tautologies,
  (A1) $h(\phi \to \psi) \to (h\phi \to h\phi)$,
  (A2) $g(\phi \to \psi) \to (g\phi \to g\phi)$,
  (A3) $\phi \to h\neg g\neg\phi$,
  (A4) $\phi \to g\neg h\neg\phi$.
- A set of rules, containing:
  (R0) from $\phi, \phi \to \psi$ infer ψ (Detachment),
  (R1) from ϕ infer $h\phi$ (Gödel rule for h),
  (R2) from ϕ infer $g\phi$ (Gödel rule for g).

The *minimal tense logic* $\mathbf{K}_t$ is the logic whose axioms are just A0–A4, and inference rules just R0–R2. Thus, any tense logic is an extension of the minimal one. By the lattice of tense logics we mean the lattice of all extensions of $\mathbf{K}_t$. The following are examples of tense logics:

(i) $\mathbf{K}_t$, the minimal tense logic.
(ii) $\mathbf{K4}_t = \mathbf{K}_t$ + the axiom:
(4) $hx \to hhx$ (or: $gx \to ggx$).
(iii) $\mathbf{Lin}_t = \mathbf{K4}_t$ + the axiom:
(Lin) $hx \wedge x \wedge gx \to hgx \wedge ghx$.
(iv) $\mathbf{Dens}_t = \mathbf{Lin}_t$ + the axiom:
(Dens) $hhx \to hx$ (or: $ggx \to gx$).
(v) $\mathbf{Disc}_t = \mathbf{Lin}_t$ + the axioms:
(Disc) $x \wedge hx \to g\bot \vee \neg g\neg hx$, $x \wedge gx \to h\bot \vee \neg h\neg gx$.
(vi) $\mathbf{Ver}_t = \mathbf{K}_t$ + the axioms:

(Ver) hx, gx.

(vii) $\mathbf{Tr}_t = \mathbf{K}_t$ + the axioms:

(Tr) $hx \leftrightarrow x$, $gx \leftrightarrow x$.

Of the above, the logics $\mathbf{K4}_t$ and $\mathbf{Lin}_t$, as well as lattices of their extensions, have been so extensively studied that any set of references would probably be incomplete: [10], and especially [21], [22], their own importance notwithstanding, can serve as comprehensive sources of references. As for theoretical significance of the logics on the list above, the items (i), (ii), (iii), and (vi), deserve attention: (i) is the minimal logic, (ii) and (iii) have rich theories of lattices of their extensions, and (vi) is the only one that splits the lattice of all tense logics. The others have been selected more or less at random (cf. e.g., [6]), only as examples.

The variety $\mathcal{T}$ of *tense algebras* is defined as the variety of algebras $\mathbf{A} = \langle A; \wedge, \neg, h, g, 0, 1\rangle$, of the type $\tau = \langle 2, 1, 1, 1, 0, 0\rangle$ such that $\langle A; \wedge, \neg, 0, 1\rangle$ is a boolean algebra and the unary functions h, g, satisfy the following identities:

(i) $g1 = 1, \quad h1 = 1;$

(ii) $g(x \wedge y) = gx \wedge gy, \quad h(x \wedge y) = hx \wedge hy;$

(iii) $\neg x \vee g\neg h\neg x = 1, \quad \neg x \vee h\neg g\neg x = 1.$

Let us take the absolutely free algebra $\mathbf{For} = \langle \mathrm{For}; \wedge, \neg, h, g, \bot, \top\rangle$, in the type τ over a countable set of generators Var. Functions $\vee$, $\rightarrow$, and $\leftrightarrow$ are introduced as terms, by $x \vee y =_{\mathrm{df}} \neg(\neg x \wedge \neg y)$, $x \rightarrow y =_{\mathrm{df}} \neg x \vee y$, $x \leftrightarrow y =_{\mathrm{df}} (x \rightarrow y) \wedge (y \rightarrow x)$. Endomorphisms $e : \mathbf{For} \longrightarrow \mathbf{For}$ are (counterparts of) substitutions. This "algebraization" of syntax will enable us to move freely from *formulas* of tense logics to *terms* of tense algebras and *vice versa* without the need of any formal "translation" since in our framework the former are simply identical to the latter.

Consider a tense logics $\mathcal{L}$. By $\mathcal{V}(\mathcal{L})$ we will denote the variety corresponding to $\mathcal{L}$, by which we mean, as usual, the variety defined by the set of identities $\{\phi = 1 : \phi \in \mathcal{L}\}$. The relation $\equiv\ \subseteq \mathrm{For}^2$, defined by: $\phi \equiv \psi$ iff $\phi \leftrightarrow \psi \in \mathcal{L}$, is a congruence on **For**. The algebra $\mathbf{Alt}(\mathcal{L}) = \mathbf{For}/_{\equiv}$ is called the *Lindenbaum-Tarski algebra* of $\mathcal{L}$.

We have the following fundamental:

Fact 1.1. *For every tense logic $\mathcal{L}$, the following hold:*

(i) $\phi \in \mathcal{L}$ *iff* $\phi = 1$ *holds in* $\mathbf{Alt}(\mathcal{L})$,

(ii) $\mathbf{Alt}(\mathcal{L})$ *is the countably generated free algebra in* $\mathcal{V}(\mathcal{L})$,

(iii) $\mathcal{V}(\mathcal{L})$ *is a subvariety of* $\mathcal{T}$,

Let ν be the map from the lattice of tense logics into the lattice of subvarieties of $\mathcal{T}$, defined by $\nu(\mathcal{L}) = \mathcal{V}(\mathcal{L})$, and λ be the map from the lattice of subvarieties of $\mathcal{T}$ into the lattice of tense logics, defined by $\lambda(\mathcal{V}) = \mathcal{L}(\mathcal{V})$, where $\mathcal{L}(\mathcal{V}) = \{\phi : \phi = 1 \text{ holds throughout } \mathcal{V}\}$.

Fact 1.2. *The maps ν and λ are mutually inverse and establish an isomorphism between the lattice of subvarieties of $\mathcal{T}$ and the (dual of) the lattice of tense logics.*

3. Tense frames and tense algebras.

Semantical considerations on tense logic are most often concerned with various types of Kripke-style *frames,* two types of which are of primary importance. The first type are *Kripke frames,* sometimes called *second-order structures* (cf. [19]). The second, and more important for the theory, is the class of *general frames,* or simply *frames.*

Definition 1.3.

- A tense *Kripke frame* is a triple $\boldsymbol{F} = \langle W, R, R^{-1}\rangle$, where W is a set, R is a binary relation on W, and R^{-1} is the converse of R.
- A tense *frame* is a pair $\langle \boldsymbol{F}; A\rangle$, where $\boldsymbol{F}$ is a Kripke frame; and A is a subset of 2^W closed under intersections and complements, and such

that if $X \in A$, then both $\{w \in W : R(w) \subseteq X\}$ and $\{w \in W : R^{-1}(w) \subseteq X\}$ also belong to A. Members of A are called *internal sets* of the frame.

Satisfaction, truth, and completeness are defined in an entirely standard way, we will repeat the definitions only to keep the preliminaries self-contained.

Definition 1.4.

- An *assignment* is a function v: Var $\longrightarrow A$.
- A formula α is *satisfied* at a point m in a relational frame $\boldsymbol{F}$ under an assignment v (written $m \models_{\langle \boldsymbol{F},v\rangle} \alpha$) if the following recursive conditions are met:

$$\begin{array}{lll} m \models_{\langle \boldsymbol{F},v\rangle} x & \text{iff} & m \in v(x), \text{ for a variable } x \\ m \models_{\langle \boldsymbol{F},v\rangle} \neg\phi & \text{iff} & m \not\models_{\langle \boldsymbol{F},v\rangle} \phi \\ m \models_{\langle \boldsymbol{F},v\rangle} \phi \wedge \psi & \text{iff} & m \models_{\langle \boldsymbol{F},v\rangle} \phi \text{ and } m \models_{\langle \boldsymbol{F},v\rangle} \psi \\ m \models_{\langle \boldsymbol{F},v\rangle} g\phi & \text{iff} & \text{for every } n, \text{ if } mRn, \text{ then } n \models_{\langle \boldsymbol{F},v\rangle} \phi \\ m \models_{\langle \boldsymbol{F},v\rangle} h\phi & \text{iff} & \text{for every } n, \text{ if } mR^{-1}n, \text{ then } n \models_{\langle \boldsymbol{F},v\rangle} \phi. \end{array}$$

- A formula α is *true* at a point m in a relational frame $\boldsymbol{F}$ (written $m \models_{\boldsymbol{F}} \alpha$) iff $m \models_{\langle \boldsymbol{F},v\rangle} \alpha$ under every assignment v.
- A formula α is *true* in a relational frame $\boldsymbol{F}$ (written $\boldsymbol{F} \models \alpha$) iff α is true at every point $m \in W$.
- A formula α is *true* in a class of frames $\mathcal{F}$ (written $\mathcal{F} \models \alpha$) iff α is true in every frame $\boldsymbol{F} \in \mathcal{F}$.
- A logic $\mathcal{L}$ is *complete* with respect to $\mathcal{F}$ iff for every formula α: $\mathcal{F} \models \alpha \iff \alpha \in \mathcal{L}$.

Let $\mathbf{A}$ be a tense algebra, and $\boldsymbol{F} = \langle W, R\rangle$ a tense frame. By $\mathrm{Ult}(\mathbf{A})$ we denote the set of all ultrafilters of the boolean reduct of $\mathbf{A}$.

Definition 1.5. The algebra $\mathbf{Alg}(\boldsymbol{F}) = \langle 2^W; \cap, \cup, -, h, g, \emptyset, W\rangle$, where the functions h and g are defined by:

- $g(X) = \{w \in W : R(w) \subseteq X\}, \quad h(X) = \{w \in W : R^{-1}(w) \subseteq X\}$,
  is called a *canonical algebra* of $\boldsymbol{F}$.

Definition 1.6. Let $\mathbf{A}$ be nontrivial, i.e., $|A| > 1$. The structure $\boldsymbol{F}(\mathbf{A}) = \langle \mathrm{Ult}(\mathbf{A}), R\rangle$, where $R \subseteq \mathrm{Ult}(\mathbf{A}) \times \mathrm{Ult}(\mathbf{A})$ is defined by:

- uRw iff $ga \in u$ implies $a \in w$, for all $a \in A$,
  or, equivalently, by:
- $uR^{-1}w$ iff $ha \in u$ implies $a \in w$, for all $a \in A$,
  is called a *canonical frame* for $\mathbf{A}$.

The following theorem is a particular case of the celebrated Representation Theorem for boolean algebras with operators from [8], [9].

Fact 1.7. *Let* $\mathbf{A}$ *be any nontrivial tense algebra, and* $\boldsymbol{F}$ *any tense frame. Then, the following hold:*

(i) $\boldsymbol{F}(\mathbf{A})$ *is a tense frame;* $\mathbf{Alg}(\boldsymbol{F})$ *is a tense algebra,*

(ii) $\mathbf{A}$ *is embeddable into* $\mathbf{Alg}(\boldsymbol{F}(\mathbf{A}))$,

(iii) $\boldsymbol{F} \models \phi$ *iff* $\mathbf{Alg}(\boldsymbol{F}) \models \phi = 1$,

(iv) *if* $\mathbf{A} \not\models \phi = 1$, *then* $\boldsymbol{F}(\mathbf{A}) \not\models \phi$,

(v) *if* $\mathbf{A}$ *is finite, then* $\mathbf{A}$ *is isomorphic to* $\mathbf{Alg}(\boldsymbol{F}(\mathbf{A}))$.

The embedding from (ii) can always be taken to be the *canonical embedding* defined as $e : A \longrightarrow 2^{\mathrm{Ult}(\mathbf{A})}$ with $e(a) = \{U \in \mathrm{Ult}(\mathbf{A}) : a \in U\}$.

The above allows for a reformulation of the definition of (general) frames, according to which a frame is a pair $\langle \boldsymbol{F}, \mathbf{A}\rangle$, where $\boldsymbol{F}$ is a Kripke frame, and $\mathbf{A}$ is a subalgebra of $\mathbf{Alg}(\boldsymbol{F})$. In particular, $\langle \boldsymbol{F}(\mathbf{A}), \mathbf{A}\rangle$ is a frame, and thus all tense logics are complete with respect to frames, since they are complete with respect to algebras. Kripke frames in this setting are frames of the form $\langle \boldsymbol{F}, \mathbf{Alg}(\boldsymbol{F})\rangle$.

One particular feature of $\mathbf{Alg}(\boldsymbol{F})$ is that it is always complete and atomic as a boolean algebra. Moreover, from [8], [9] we know that in tense algebras both g and h are completely distributive over meet, i.e., whenever $l(\bigwedge X)$ exists, we have $l(\bigwedge X) = \bigwedge\{lx : x \in X\}$, for $l \in \{g, h\}$.

Let us also recall another property of frames that corresponds nicely to algebraic properties.

Definition 1.8. A frame $\boldsymbol{F}$ is *connected* iff $\forall a, b \in W \exists n \in \omega : aS^n b$, where: $S = R \cup R^{-1}$; $S^0 = \mathrm{id}$, and $S^n = \underbrace{S \circ \ldots \circ S}_{n-\text{times}}$.

Fact 1.9. *A tense algebra* $\mathbf{A}$ *is si iff* $\boldsymbol{F}(\mathbf{A})$ *is a connected frame. If* $\mathbf{A}$ *is finite, then* $\mathbf{A}$ *is simple iff* $\boldsymbol{F}(\mathbf{A})$ *is a connected frame.*

4. Basics on tense algebras

Several properties of tense algebras have been isolated and investigated, whether for their own sake or in connection to tense logics. Quite a few facts are known about the variety of tense algebras. In the remaining part of this chapter the most important of these will be gathered together.

Definition 1.10. A subset F of the universe of a tense algebra $\mathbf{A}$ is called a *congruence-filter* iff F is a filter on the boolean reduct of $\mathbf{A}$ and, moreover, F is closed under operations g and h.

Fact 1.11. *For an algebra* $\mathbf{A} \in \mathcal{T}$*, the lattice of all congruence-filters on* $\mathbf{A}$ *is isomorphic to* $\mathbf{Con\,A}$*. The variety* $\mathcal{T}$ *is arithmetical, regular, has CEP, and is generated by its finite algebras.*

Another feature of tense algebras is that each of the tense operators determines the other in a unique way.

Fact 1.12. *For any element x of an $\mathbf{A} \in \mathcal{T}$, the following holds:*

$$gx = \neg \bigwedge h^{-1}[\neg x),$$

and the same with g and h interchanged.

Here $\bigwedge h^{-1}[z)$ denotes the smallest element of the inverse image, under h, of the principal boolean filter $[z)$.

A number of properties of tense algebras can be expressed by means of a unary term d, defined below, which also can be employed to define a natural chain of subvarieties of $\mathcal{T}$.

Definition 1.13. Let dx be the term-operation $hx \wedge x \wedge gx$. We set $d^0x = x$, and $d^{n+1}x = d(d^nx)$.

Fact 1.14. *Let $\mathbf{A}$ be a tense algebra. Then, for any $n \in \omega$, the following hold:*

(i) $d^n1 = 1, \qquad d^n0 = 0, \qquad d^{n+1}x \leq d^nx$;

(ii) $d^n(x \wedge y) = d^nx \wedge d^ny$;

(iii) *if $x \leq y$, then $d^nx \leq d^ny$, i.e., d^n is monotonic;*

(iv) $x \leq d^n\neg d^n\neg x$;

(v) *if $dx = x$, then $d\neg x = \neg x$.*

(vi) *$F \subseteq A$ is a congruence-filter iff F is a boolean filter closed under d.*

Proof. (i) and (ii) are straightforward, (iii) follows from them immediately; to prove (iv) observe that for $n = 0$ it holds trivially and simple calculation shows that it holds for $n = 1$ as well. From this it follows that $\neg d^n\neg x \leq d\neg d^{n+1}\neg x$. By monotonicity of d we get $d^n\neg d^n\neg x \leq d^{n+1}\neg d^{n+1}\neg x$, and the inductive hypothesis gives $x \leq d^n\neg d^n\neg x$, which completes the proof. To show (v), suppose that for some x we have $dx = x$. By (iv), with $n = 1$, it follows that $\neg x \leq d\neg dx$, and thus $\neg x \leq d\neg x$. On the other hand, by the definition of d, we have for any a: $da \leq a$, whence $d\neg x \leq \neg x$. Thus, $d\neg x = \neg x$ as required. The remaining (vi) is clear from the definitions involved. ■

Fact 1.15. *For any tense algebra* $\mathbf{A}$, *the set* $D \subseteq A$ *of all the elements satisfying* $da = a$ *is closed under the boolean operations of* $\mathbf{A}$.

Proof. Immediate from Fact 1.12. ■

The following three facts sum up the basic characteristics of simple, si, and directly indecomposable tense algebras.

Fact 1.16. *For an* $\mathbf{A} \in \mathcal{T}$ *the following are equivalent:*

(i) $\mathbf{A}$ *is simple,*

(ii) $\mathbf{A}$ *has no proper congruence-filters,*

(iii) *for every* $a \in A \setminus \{1\}$, *there is an* $n \in \omega$ *with* $d^n a = 0$.

Proof. The equivalence of (i) and (ii) is immediate from Fact 1.11. For the remaining equivalence, notice that $\mathbf{A}$ has no proper congruence-filters iff for any $a \neq 1$, the congruence-filter generated by a contains 0, i.e., we have $d^n a = 0$, for some $n \in \omega$. ■

Fact 1.17. *For an* $\mathbf{A} \in \mathcal{T}$ *the following are equivalent:*

(i) $\mathbf{A}$ *is subdirectly irreducible,*

(ii) *there is the smallest non-trivial congruence-filter on* $\mathbf{A}$,

(iii) *there is a* $b \in A \setminus \{1\}$, *such that: for any* $a \in A \setminus \{1\}$, *there is a* $k \in \omega$ *with* $d^k a \leq b$.

Proof. The equivalence of (i) and (ii) is obvious. To proceed further, let $[\![b)$ stand for the congruence-filter generated by b. Observe that:

$$a \in [\![b) \text{ iff } \exists n < \omega : d^n b \leq a. \tag{$*$}$$

To conclude the proof consider the following series of equivalences: There is the smallest non-trivial principal congruence-filter on $\mathbf{A}$ iff there is an element $b \neq 1_{\mathbf{A}}$ which generates this congruence-filter iff there is an element $b \neq 1_{\mathbf{A}}$ such that $[\![b)$ contains any other non-trivial principal congruence-filter iff $\exists b \neq 1_{\mathbf{A}} \forall a \neq 1_{\mathbf{A}} : [\![b) \subseteq [\![a)$ iff $\exists b \neq 1_{\mathbf{A}} \forall a \neq 1_{\mathbf{A}} : b \in [\![a)$ iff, by $(*)$, $\exists b \neq 1_{\mathbf{A}} \forall a \neq 1_{\mathbf{A}} \exists k < \omega : d^k a \leq b$. ■

Fact 1.18. *For an* $\mathbf{A} \in \mathcal{T}$ *the following are equivalent:*

(i) $\mathbf{A}$ *is directly indecomposable,*

(ii) *no proper congruence-filter on* $\mathbf{A}$ *has the smallest element,*

(iii) *for every* $a \in A \setminus \{0, 1\}$*:* $da < a$.

Proof. To prove the contrapositive of '(ii) implies (iii)' assume that there is an $x \notin \{0, 1\}$ with $dx = x$. Then, by Fact 1.14.(iii), we obtain that $[\![x)$ has the smallest element, namely x, thus contradicting (ii).

To show that (i) implies (ii), assume, again arguing for the contrapositive, that some congruence-filter F has the smallest element a. Now, by Fact 1.14.(iii), (iv), we get that $da = a$ and $d\neg a = \neg a$. Hence, $[\![a) = [a)$ and $[\![\neg a) = [\neg a)$, and thus, the congruences defined by $[\![a)$, $[\![\neg a)$, respectively, are the same as the congruences induced on the boolean reduct of $\mathbf{A}$ by $[a)$ and $[\neg a)$. These factorise $\mathbf{A}$ in an obvious way.

For the last remaining implication, assume, contrary to (i), that $\mathbf{A}$ is (nontrivially) factorised into $\prod_{i\in I} \mathbf{A}_i$, with $|I| > 1$. Consider the element $a_k = \langle x_i : i \in I\rangle$ such that $x_i = 1_i$, if $i = k$, and $x_i = 0_i$ otherwise. Evidently, $a_k \notin \{0, 1\}$. Calculating $d(a_k)$ coordinatewise, we obtain $d(a_k) = \langle d(x_i) : i \in I\rangle$ such that $x_i = 1_i$, if $i = k$, and $x_i = 0_i$ otherwise. By the properties of d, however, in each case $d(x_i) = x_i$. Thus, $d(a_k) = a_k$, contradicting (iii). ■

Fact 1.19. *A finite tense algebra* $\mathbf{A}$ *is uniquely factorable into a product of a finite number of finite simple algebras.*

Proof. Take the set $D \subseteq A$, of Fact 1.15. It is finite, since A itself is. Each co-atom c of D (to be more precise: of the boolean algebra which D is the universe of) is a maximal non-unit element of A such that $dc = c$. Thus, the congruence-filter generated by c is minimal nontrivial, and, moreover, it coincides with the boolean principal filter $[c)$. Now, it is straightforward to see that congruence-filters generated by the co-atoms of D define the appropriate factorisation. ■

Part One. Discriminator varieties

1. Outline

We show that a subvariety of $\mathcal{T}$ is discriminator if and only if it is semisimple. The variety $\mathcal{T}$ turns out to be the varietal join of an increasing chain of varieties $\mathcal{D}_n$, which are discriminator varieties. Some important varieties of tense algebras, including the much investigated $\mathcal{V}(\mathbf{Lin}_t)$ and a pretabular $\mathcal{V}(\mathbf{Ga})$ (the variety whose frames are so-called garlands, quite consequential for splittings of $\mathcal{T}$, or rather for the non-existence thereof) are subvarieties of $\mathcal{D}_1$.

The arguments have been formulated in terms of tense algebras, but they applies to any variety of *boolean algebras with operators* that possesses a unary term with the properties of our d. In particular it applies to any finite type variety of boolean algebras with conjugate operators. An example shows that the above finiteness requirement cannot be relaxed.

Although the notions of a discriminator and a semisimple variety have no direct logical associates, the notion of an EDPC variety, which will, in our case, turn out to be equivalent to each of the former has been long recognised to be the algebraic counterpart of a certain general form of deduction theorem (cf. [3], [7]). This correspondence covers an area much wider then boolean algebras with operators. In the (poly-) modal case, EDPC corresponds to *weak transitivity* of appropriate logics.

For the logics $\mathbf{D}_n$, corresponding to the varieties $\mathcal{D}_n$, the deduction theorem takes the following form:

$$\Gamma \cup \{\psi\} \vdash_{\mathbf{D}_n} \phi \quad \text{iff} \quad \Gamma \vdash_{\mathbf{D}_n} d^n\psi \to \phi.$$

Deduction theorems in this vein have been first formulated for several important (classes of) modal logics in [14] and [15].

2. The varieties $\mathcal{D}_n$

We begin with a couple of notions followed by an easy observation: The *ternary discriminator* is a function $t(x, y, z)$, such that:

$$t(x, y, z) = \begin{cases} x, & \text{for } x \neq y \\ z, & \text{for } x = y \end{cases}.$$

A *discriminator variety* is a variety $\mathcal{V}$, such that the ternary discriminator is a term-operation on every subdirectly irreducible algebra in $\mathcal{V}$.

A variety $\mathcal{V}$ is *semisimple* iff all subdirectly irreducible members of $\mathcal{V}$ are simple.

A variety $\mathcal{V}$ has *definable principal congruences* (DPC, for short) if there exists a formula $\Phi(x, y, u, v)$ in the first-order language of $\mathcal{V}$ such that for all $a, b, c, d, \in \mathbf{A} \in \mathcal{V}$ we have:

$$\langle a, b \rangle \in \mathbf{Cg}^{\mathbf{A}}(c, d) \text{ iff } \mathbf{A} \models \Phi(a, b, c, d).$$

If $\Phi(x, y, u, v)$ can be taken to be a finite set of polynomial equations, then $\mathcal{V}$ is said to have *equationally definable principal congruences* (EDPC).

The argument that follows depends only on the fact that boolean algebras with operators that we deal with possess a unary term d with the properties specified in Preliminaries. It is straightforward to show that such a term always exists in finite type varieties of boolean algebras with *conjugate* operators (see e.g., [8], [9] for the definition of conjugates). Even more generally, we can simply require that the variety we work with had such a term. Thus, from now on until the penultimate section of this part we fix a finite type variety of boolean algebras with operators, such that it possesses a unary term d with the properties stated in Fact 1.14. For simplicity sake, we will call this variety "the variety of tense algebras" and reserve the symbol $\mathcal{T}$ for it. To restrict attention to tense algebras as they were defined in Preliminaries, we will use the phrase: "tense algebras *sensu stricto*."

Definition 2.1. For any $n \in \omega$, $\mathcal{D}_n$ is the subvariety of $\mathcal{T}$ satisfying:

$$d^{n+1}x = d^n x. \tag{δ_n}$$

Notice that, by Facts 1.19. and 1.16., each finite tense algebra belongs to $\mathcal{D}_n$, for some n.

Fact 2.2. *Let* $\mathbf{B}$ *be a boolean algebra with an additional unary function* f *satisfying:*

$$fx = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \neq 1 \end{cases}$$

then the term-operation $t(x, y, z) = (z \wedge f(x \leftrightarrow y)) \vee (x \wedge \neg f(x \leftrightarrow y))$, *is the discriminator on* $\mathbf{B}$.

Fact 2.3. *Let* $\mathbf{B}$ *be a congruence distributive algebra, and* $\alpha \in \operatorname{Con} \mathbf{B}$ *be compact. Then, for any* $\beta \in \operatorname{Con} \mathbf{B}$, *such that* α *covers* β, *the set* $\Gamma = \{\gamma \in \operatorname{Con} \mathbf{B} : \gamma \geq \beta, \gamma \not\geq \alpha\}$ *is either empty, or has the largest element.*

Proof. We show that $\bigvee \Gamma$ belongs to Γ, whenever $\Gamma \neq \emptyset$. Suppose the contrary, then, $\bigvee \Gamma \geq \alpha$. Since α is compact, i.e., finitely generated, we have that $\bigvee \Gamma_0 \geq \alpha$, for some finite $\Gamma_0 \subseteq \Gamma$. We put $\Gamma_0 = \{\gamma_j : j \in J\}$, for some finite J, and calculate: $\alpha = \alpha \wedge \bigvee \{\gamma_j : j \in J\}$. This, by congruence distributivity, and finiteness of J, equals $\bigvee \{\alpha \wedge \gamma_j : j \in J\} = \beta$, since each γ_j belongs to Γ. Thus, we have arrived at a contradiction. ∎

Theorem 2.4. *Let* $\mathcal{V} \subseteq \mathcal{T}$. *Then the following are equivalent:*

(i) $\mathcal{V}$ *is a discriminator variety,*

(ii) $\mathcal{V}$ *has EDPC,*

(iii) $\mathcal{V}$ *has DPC,*

(iv) $\mathcal{V} \subseteq \mathcal{D}_n$, *for some* n,

(v) $\mathcal{V}$ *is semisimple.*

Proof. To show that (i) $\Longrightarrow$ (ii), it suffices to notice that if $t(x, y, z)$ is a discriminator term for $\mathcal{V}$, then for any $a, b, c, d \in \mathbf{A} \in \mathcal{V}$ we have:

$(c,d) \in \mathbf{Cg}^{\mathbf{A}}(a,b)$ iff $t(a,b,c) = t(a,b,d)$. The implication (ii) $\Longrightarrow$ (iii) is immediate, and a proof of (i) $\Longrightarrow$ (v) can be found in [20]. To see that (iv) $\Longrightarrow$ (i) take any si algebra $\mathbf{A} \in \mathbf{V}$. We will show that

$$d^n x = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \neq 1 \end{cases}.$$

That $d^n 1 = 1$ is obvious. Take any $x \neq 1$ and denote $d^n x$ by a. Since $\mathbf{A} \models \delta_n$, we have $d^{n+1}x = d^n x$, i.e., $da = a$. This implies, by Lemma 1.14., that $d\neg a = \neg a$ as well. If $a \neq 0$, then $\neg a \neq 1$ and $[a)$, $[\neg a)$ are both closed under d. By Fact 1.16. then, they are congruence-filters defining a pair of complementary factor congruences. Hence $\mathbf{A}$ is not si, contradicting the assumption. Therefore a must be 0, i.e., $d^n x = 0$ for any $x \neq 1$.

To show that (v) $\Longrightarrow$ (iv) assume $\mathcal{V} \subseteq \mathcal{T}$ is semisimple, yet for every $n \in \omega$, $\mathcal{V} \nsubseteq \mathcal{D}_n$. Take a sequence of simple algebras $\mathbf{A}_i$ $(i \in \omega)$ witnessing the above fact, i.e., $\mathbf{A}_i \not\models \delta_i$ for every i. Therefore, in each of the $\mathbf{A}_i$'s we can find an element a_i with $d^i a_i \neq 0$, $d^{i+1}a_i = 0$. Notice that $a_i \neq 1$, as otherwise we could not have $d^{i+1}a_i = 0$.

A little calculation shows that, for each $k \leq i$, $d^{k+1}\neg d^k \neg d^i a_i = 0$. For, $d^k \neg d^i a = d^k \neg d^k (d^{i-k} a) \geq \neg d^{i-k} a$. Thus, $\neg d^k \neg d^i a \leq d^{i-k} a$, and hence $d^k \neg d^k \neg d^i a \leq d^i a$. Thus, $d^{k+1} \neg d^k \neg d^i a \leq d^{i+1} a = 0$.

Let $p_i = \neg d^i a_i$, observe that $d^i p_i = d^i \neg d^i a_i \geq \neg a_i \neq 0$.

Take now an ultraproduct $\mathbf{A} = \prod_{i \in \omega} \mathbf{A}_i / U$ by a non-principal ultrafilter U on ω. Consider $p = \langle p_i : i \in \omega \rangle / U$, and $\alpha = \mathbf{Cg}^{\mathbf{A}}(p, 1)$. Since α is principal, there is a $\beta \in \mathrm{Con}(\mathbf{A})$ such that α covers β.

Consider $\Gamma = \{\gamma \in \mathrm{Con}(\mathbf{A}) : \gamma \geq \beta, \gamma \ngeq \alpha\}$. If $\Gamma = \emptyset$ then $\mathbf{A}/\beta$ is si, yet, by its definition, α is not the full congruence on $\mathbf{A}$, and thus, neither is it full on $\mathbf{A}/\beta$. This contradicts the assumption of $\mathcal{V}$ being semisimple. We may, therefore, assume, that Γ is nonempty.

It follows from Fact 2.3., that $\bigvee \Gamma \in \Gamma$. Let $\varphi = \bigvee \Gamma$. Consider $\mathbf{A}/\varphi$. This must be si, for whenever θ is a nontrivial congruence on $\mathbf{A}/\varphi$, we have $\theta > \varphi$ in $\mathbf{Con}(\mathbf{A})$, and, thus, $\theta > \alpha$ as well, which in turn shows that $\alpha \vee \varphi$ is the smallest nontrivial congruence on $\mathbf{A}/\varphi$.

We will now proceed to show that $\mathbf{A}/\varphi$ is not simple. In fact, we will argue that were $\alpha \vee \varphi$ full in $\mathrm{Con}(\mathbf{A})$, so would have to be φ. The latter, however, would contradict Fact 2.3.

Suppose thus, that $\alpha \vee \varphi$, which by congruence permutability equals $\varphi \circ \alpha$, is full in $\mathrm{Con}(\mathbf{A})$. Then, in particular, $(0, 1) \in \varphi \circ \alpha$. So, there is a $c \in A$ with $(0, c) \in \varphi$ and $(c, 1) \in \alpha$, or, equivalently, $(\neg c, 1) \in \varphi$, and $(c, 1) \in \alpha$.

Now, α is generated by $(p, 1)$, so, $c \geq d^k p$, for some fixed k. Recall, that we have chosen the elements p_i so that for every $k \leq i$, $d^{k+1}\neg d^k p_i = 0_i$ holds. As k is fixed, we obtain: for every coordinate i, with $i \geq k$, $d^{k+1}\neg d^k p_i = 0_i$. Hence, $d^{k+1}\neg d^k p = 0$ holds in the ultraproduct. Then we have: $c \geq d^k p$, thus, $\neg c \leq \neg d^k p$, and $d^{k+1}\neg c \leq d^{k+1}\neg d^k p = 0$. We obtain, thus, $d^{k+1}\neg c = 0$. This implies that $(0, 1) \in \mathbf{Cg}^{\mathbf{A}}(\neg c, 1)$, *a fortiori*, $(0, 1) \in \varphi$, i.e., φ is the full congruence on $\mathbf{A}$. Thus, $\varphi \geq \alpha$. But this, given the definition of φ, yields the announced contradiction with Fact 2.3.

To round off the proof we will show that (iii) $\Longrightarrow$ (iv). Suppose $\mathcal{V}$ has DPC and yet $\mathcal{V} \nsubseteq \mathcal{D}_n$, for any n. Thus, $\mathcal{V}$ must contain si algebras falsifying δ_n, for any n. Like in the previous part of the proof, let $\mathbf{A}_i$ be an si algebra witnessing the above for i, and $\mathbf{A}$ be the ultraproduct $\prod_{i \in \omega} \mathbf{A}_i/U$ for some non-principal ultrafilter U on ω.

As $\mathcal{V}$ has DPC, subdirect irreducibility becomes a first-order property and carries over to ultraproducts. Thus, $\mathbf{A}$ is si. Let μ be its monolith; $1/_{\mu}$ is then the smallest non-trivial congruence-filter on $\mathbf{A}$. Fact 1.17.(iii) implies that each $b \in 1/_{\mu}$ has the following property:

$$\forall a \neq 1_{\mathbf{A}} \exists k < \omega : d^k a \leq b. \tag{$*$}$$

Consider the sequence $\langle b_n : n < \omega \rangle$ such that, for each n, $b_n \neq 1_{\mathbf{A}_n}$, $b_n \in 1/_{\mu_n}$; where μ_n is the monolith of $\mathbf{A}_n$. By DPC, and the properties of ultraproducts, we may conclude that the element $\overline{b} = \langle b_n : n < \omega \rangle/U$ belongs to the monolithic congruence-filter $1/_{\mu}$ of $\mathbf{A}$, hence $\overline{b}$ should satisfy $(*)$. We proceed to exhibit an $\overline{a} \in \mathbf{A}$ for which it is not the case.

First, let's observe that for the coordinate $2t$ there is an element $a \in A_{2t}$, $a \neq 1$ with $d^t a \not\leq b_{2t}$. For suppose otherwise, then

$$\forall a \in A_{2t}, a \neq 1 : d^t a \leq b_{2t}. \qquad (**)$$

Thus, if $b_{2t} = 0$, then $\forall a \neq 1 : d^t a = 0$, whence $d^{t+1}x = d^t x$, i.e., $\mathbf{A}_{2t} \models \delta_t$. But then, $\mathbf{A}_{2t}$ would satisfy δ_{2t} as well, which contradicts the assumption that $\mathbf{A}_n \not\models \delta_n$.

We must then have $b_{2t} \neq 0$. Thus, $\neg b_{2t} \neq 1$, and so, by $(**)$ we have, in particular, $d^t \neg b_{2t} \leq b_{2t}$. On the other hand, by the properties of d, we have $d^t \neg b_{2t} \leq \neg b_{2t}$. Hence, $d^t \neg b_{2t} = 0$.

By the properties of d, we have: (i) $d^t \neg d^t b_{2t} \geq \neg b_{2t}$. Now, if $d^t b_{2t} \neq 0$, then $\neg d^t b_{2t} \neq 1$. Hence, by $(**)$, we get: (ii) $d^t \neg d^t b_{2t} \leq b_{2t}$. However, as $b_{2t} \neq 1$, (i) and (ii) contradict each other, so we must have $d^t b_{2t} = 0$ as well. Moreover, by monotonicity of d, $\forall x \leq b_{2t} : d^t x = 0$, which together with $(**)$ yield:

$$\forall a \in A_{2t}, a \neq 1 : d^{2t} a = 0.$$

We have then $\forall a \in A_{2t}, a \neq 1 : d^{2t+1} a = 0 = d^{2t} a$, and, obviously, $d^{2t+1} 1 = 1 = d^{2t} 1$. So, $\mathbf{A}_{2t} \models \delta_{2t}$ which contradicts the assumption that $\mathbf{A}_n \not\models \delta_n$.

This way we have established that:

$$\exists a \in A_{2t}, a \neq 1 : d^t a \not\leq b_{2t}.$$

The very same argument, with nothing but stylistic modifications, shows that:

$$\exists a \in A_{2t+1}, a \neq 1 : d^t a \not\leq b_{2t+1}.$$

We can put these two together as:

$$\forall n < \omega \exists a \in A_n, a \neq 1 : d^{[\frac{n}{2}]} a_n \not\leq b_n \qquad (\dagger)$$

where $[\frac{n}{2}]$ denotes the greatest integer $\leq \frac{n}{2}$.

Now, let's pick out from each coordinate an element $a_n \neq 1_{\mathbf{A}_n}$ with the property (†). Consider $\overline{a} = \langle a_n : n < \omega \rangle / U$, clearly $\overline{a} \neq 1_{\mathbf{A}}$, and hence, by (∗), we should have a $k < \omega$ with $d^k \overline{a} \leq \overline{b}$. But, for every $m > 2k$ we have $d^k a_m \nleq b_m$, so, by the properties of ultraproducts, $d^k \overline{a} \nleq \overline{b}$, for any k. This contradiction finishes the whole proof. ■

3. A few remarks

Some of the equivalences that have just been shown to hold follow from general theorems on EDPC varieties proved by W. Blok and D. Pigozzi in [4]. Among these, notably, the equivalence of (i), (ii), and (iii), under the assumption of (v). These follow from (1) the fact (proved in [1], see also [2] and [4]) that congruence-permutable EDPC varieties that are semisimple coincide with discriminator varieties, and (2) Theorem 5.3. of [4].

Yet, the implication from (v) to (i), could not be derived from Blok and Pigozzi's work; indeed, it does not hold in general. The variety of *Tarski algebras,* i.e., implicational subreducts of boolean algebras, is one out of quite a number of counterexamples. However, the "standard" counterexamples, including the above, fail to be discriminator because they are not congruence-permutable. To see that (v) does not imply (i) even restricted to boolean algebras with conjugate operators consider the following example.

Example 2.5. For every natural number n, let us define an algebra $\mathbf{A}_n = \langle A; \wedge, \vee, \neg, 0, 1, l_k (k \in \omega) \rangle$, such that $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ is the four-element boolean algebra and, for each $k \in \omega$, l_k is a unary operation defined by the conditions:

$$l_k x = x, \quad \text{if } k \neq n;$$

$$l_n x = \begin{cases} 1 \text{ for } x = 1 \\ 0 \text{ for } x \neq 1 \end{cases}.$$

Clearly, each l_k is a normal modal operator, and each $\mathbf{A}_n$ is simple. Moreover, each l_k is its own conjugate, i.e., $x \leq l_k \neg l_k \neg x$ holds. Consider

$\mathcal{V} = \mathcal{V}(\{\mathbf{A}_n : n \in \omega\})$. By Jónsson Theorem the class of si algebras in $\mathcal{V}$ is contained in $HSP_U(\{\mathbf{A}_n : n \in \omega\})$. Since the underlying boolean algebras of $\mathbf{A}_n$'s have four elements each, the same is also true of any ultraproduct of these. Moreover, for each $\mathbf{A}_n$ and l_k we have: $l_k x = x \Leftrightarrow l_k \neg x = \neg x$, and $l_k x \leq x$. These carry over to ultraproducts as well, and thus, each member of $P_U(\{\mathbf{A}_n : n \in \omega\})$ is either simple or directly decomposable as a product of two two-element algebras. It follows that $\mathcal{V}$ is semisimple. We will show that not every member of $P_U(\{\mathbf{A}_n : n \in \omega\})$ is simple. Let F be any non-principal ultrafilter on ω; take $\mathbf{A} = \prod_{n\in\omega} \mathbf{A}_n/F$. Let $\overline{a}/F = \langle a_n : n \in \omega\rangle$ with $0 < a_n < 1$ for each n. For any k, the set $\{n \in \omega : l_k a_n \neq a_n\} = \{n \in \omega : l_k a_n = 0_n\} = \{k\}$ and thus, does not belong to F. Therefore, $l_k(\overline{a}/F) = \overline{a}/F$, for every k, and, thus, we get that $\mathbf{A}$ is not simple. Hence, $\mathcal{V}$ cannot be a discriminator variety, as in these ultraproducts of simple algebras are simple.

Thus, Theorem 2.4. can be seen as a strengthening of Blok and Pigozzi's result for the particular case of varieties of tense algebras.

Corollary 2.6. *Let $\mathcal{V} \subseteq \mathcal{T}$ be generated by its finite members, and let $\mathcal{E}_n = \mathcal{V} \cap \mathcal{D}_n$, then:*

$$\mathcal{V} = \bigvee_{n\in\omega} \mathcal{E}_n,$$

in particular:

$$\mathcal{T} = \bigvee_{n\in\omega} \mathcal{D}_n.$$

Proof. We claim that $\mathcal{V} = HSP(\bigcup_{n\in\omega} \mathcal{E}_n)$. Since $\mathcal{V}$ is generated by its finite algebras, and each of these is a member of $\mathcal{E}_n$, for some n, the claim is proved. ■

Corollary 2.7. *Each of the varieties $\mathcal{D}_n$ is generated by its finite simple members.*

Proof. The proof proceeds through a standard filtration argument, and could be conveniently sketched as follows: Suppose we have a term ϕ with $\mathcal{D}_n \models \phi \neq 1$. Then, as $\mathcal{D}_n$ is semisimple, there exists a simple algebra

$\mathbf{A} \in \mathcal{D}_n$ with $\mathbf{A} \models \phi \neq 1$. Let $\Sigma(\phi)$ be the set of all subterms of ϕ, and let $\overline{a} = \langle a_0, \ldots, a_{k-1} \rangle \in A^k$ be such that $\phi^{\mathbf{A}}(\overline{a}) \neq 1$. Further, let S stand for $\{\sigma^{\mathbf{A}}(\overline{a}) : \sigma \in \Sigma(\phi)\}$. Observe, that for each $g(\sigma) \in \Sigma(\phi)$, $\bigwedge g^{-1}(\neg\sigma(\overline{a}))$ is well-defined, by Fact 1.12. Put $S^+ = S \cup \{\bigwedge g^{-1}(\neg\sigma(\overline{a})) : g(\sigma) \in \Sigma(\phi)\}$. Now, let $\mathbf{B}_0$ be the subalgebra of the boolean reduct of $\mathbf{A}$, generated by S^+. Then, endow $\mathbf{B}_0$ with two unary operations H and G, defined as follows:

- $Hx = \begin{cases} hx, & \text{iff } hx \in B_0, \\ \bigvee\{hy : y \leq x, hy \in B_0\}, & \text{otherwise;} \end{cases}$
- $Gx = \neg\bigwedge H^{-1}[\neg x)$.

We verify, that the finite algebra $\mathbf{B}$ obtained this way is a tense algebra. First we need to show that Hx defined above is a normal operator. This presents no difficulty. Then, observe that $\bigwedge H^{-1}[x)$ is well-defined, as our algebra is finite. This, by Fact 1.12., yields the desired conclusion. Moreover, from the definition of S^+ it is clear that "the original" $\phi(\overline{a})$ has been preserved on $\mathbf{B}$, in particular, that $\phi^{\mathbf{B}}(\overline{a}) \neq 1$. Further still, the definition of H makes it clear that $Hx \leq hx$—in the sense $e(Hx) \leq hx$, where e is the natural (boolean) embedding of $\mathbf{B}_0$ into the boolean reduct of $\mathbf{A}$. By the same token, we have: for each $y \in B_0$, $H^{-1}[y) \subseteq h^{-1}[y)$, and thus $\bigwedge H^{-1}[y) \geq \bigwedge h^{-1}[y)$. By taking $y = \neg x$, we get that $Gx \leq gx$ as well, and we can conclude that $Dx = Hx \wedge x \wedge Gx \leq dx$.

Now, $\mathbf{A}$ is a simple algebra in $\mathcal{D}_n$. Thus, $d^n a = 0$, if only $a \neq 1$. Hence, whenever $a \neq 1$, $a \in B_0$, we have $D^n a = 0$, which shows that $\mathbf{B} \in \mathcal{D}_n$ as well. This ends the proof of the first statement. ■

4. The varieties $\mathcal{D}_n$ in terms of frame span

Throughout this section we will be working with frames rather than with algebras, attempting to provide another characterisation of the varieties $\mathcal{D}_n$, understood here as varieties of tense algebras *sensu stricto.*

Definition 2.11. Let $\boldsymbol{F} = \langle W, R, R^{-1} \rangle$ be a finite, connected frame. By a *path* (from a to b) of length n, we will mean a finite sequence

$\langle w_0, \dots, w_n\rangle$, such that $w_0 = a$, $w_n = b$, and for all $i \in \{0, \dots, n-1\}$, $w_i S w_{i+1}$, where $S = R \cup R^{-1}$. The *span* of $\boldsymbol{F}$ is the smallest $n \in \omega$ such that for any two points $a, b \in W$ there is a path from a to b of length not exceeding n.

As it is seen from the definition, the notion is intended to apply only to finite connected frames, although it could be easily generalised to cover also infinite connected frames. Yet, since finite connected frames correspond to finite simple algebras, and these latter have been shown to generate $\mathcal{D}_n$, for any n, this is sufficient for a characterisation. Intuitively, the span of a frame is the smallest number of steps that suffices to pass from any point in the frame to any other point. Thus, for instance all linear frames (i.e., satisfying $\forall x, y : xRy \vee yRx \vee x = y$) have span equal to 1.

Theorem 2.12. *Let $\boldsymbol{F}$ be a connected frame. The algebra $\mathbf{Alg}(\boldsymbol{F})$ belongs to $\mathcal{D}_0$ iff the span of $\boldsymbol{F}$ equals 0. The algebra $\mathbf{Alg}(\boldsymbol{F})$ belongs to $\mathcal{D}_{n+1} \setminus \mathcal{D}_n$ iff the span of $\boldsymbol{F}$ equals $n+1$.*

Proof. For the first part, it suffices to notice that the span of $\boldsymbol{F}$ is 0 iff $\boldsymbol{F}$ is a one-element frame. There are precisely two algebras corresponding to such frames, **Ver** and **Tr**. Both of these belong to $\mathcal{D}_0$.

For the second part, we have to show that (i) $\boldsymbol{F} \models d^{n+2}x \leftrightarrow d^{n+1}x$ and (ii) $\boldsymbol{F} \not\models d^{n+1}x \leftrightarrow d^n x$ iff the span of $\boldsymbol{F}$ equals $n+1$. Recall that we have adopted a convention of identifying tense-logical formulas with terms of (the language of) tense algebras.

($\Rightarrow$) Since $\boldsymbol{F} \models d^{n+1}x \to d^n x$ is always the case, if the equivalence fails to hold, we must have $\boldsymbol{F} \not\models d^n x \to d^{n+1}x$. This means, that for some assignment v and some element $w \in W$, we have $w \not\models_v d^n x \to d^{n+1}x$. Thus, $w \models_v d^n x$ yet $w \not\models_v d^{n+1}x$.

Now, put $aS^0 b$ iff $a = b$, and $aS^{n+1}b$ iff $\langle a, b\rangle \in S \circ S^n$. An easy inductive argument proves that $w \models_v d^k x$ iff $\forall i \leq k \forall w' \in W : w' S^i w \Rightarrow w' \models_v x$.

In our case, we have that $\forall i \leq n \forall w' \in W : w' S^i w \Rightarrow w' \models_v x$, and, that there are a natural number $j \leq n+1$ and an element w' of W such that $w'S^jw$ and $w' \not\models_v x$. It follows that j must be equal to $n+1$, and thus the path from w' to w has length at least $n+1$.

Suppose, that the span of $\boldsymbol{F}$ exceeds $n+1$, i.e., that there are $a, b \in W$ such that any path connecting them has length at least $n+2$. Let v be an assignment such that $v(x_0) = W \setminus \{a\}$ for some fixed variable x_0. Thus, in particular: $aS^{n+2}b$ and $a \not\models_v x_0$. Hence, $b \not\models_v d^{n+2}x_0$. On the other hand, $b \models_v d^{n+1}x_0$, since the only point falsifying x_0, namely a, is $n+2$ steps away. This yields $b \not\models_v d^{n+2}x_0 \leftrightarrow d^{n+1}$, i.e., $\boldsymbol{F} \not\models d^{n+2}x \leftrightarrow d^{n+1}x$, which contradicts the assumption (i) and finishes the proof in this direction.

($\Leftarrow$) To argue for *reductio*, assume that the span of $\boldsymbol{F}$ is $n+1$, and, if $\boldsymbol{F} \models d^{n+2}x \leftrightarrow d^{n+1}x$, then $\boldsymbol{F} \models d^{n+1}x \leftrightarrow d^nx$.

Now, notice that the antecedent of the above implication is always true. For, if it were false, we would have an assignment v and an element a verifying $d^{n+1}x$ but falsifying $d^{n+2}x$, i.e., $a \models_v d^{n+1}x$, yet $a \not\models_v d^{n+2}x$. This would, in turn, amount to stating that there exists some $b \in W$ such that $bS^{n+2}a$ and $b \not\models_v x$. Yet, since the span of $\boldsymbol{F}$ is $n+1$, there must exist a shorter path from b to a, say, $bS^{n+1}a$. Then, however, it follows that $b \models_v x$, contradicting the assumption.

In view of the above, we obtain that $\boldsymbol{F} \models d^{n+1}x \leftrightarrow d^nx$. Pick out a, b from W such that they are precisely $n+1$ steps apart. Such elements must exist, by the definition of span. Take an assignment v, such that $v(x_0) = W \setminus \{a\}$, for a fixed variable x_0. Clearly, $a \not\models_v x_0$. Thus, $b \not\models d^{n+1}x_0$. Yet, $b \models d^nx_0$, since for any c that is n or less steps apart from b, $c \neq a$, and hence $c \models_v x_0$. Therefore, $b \not\models_v d^nx \to d^{n+1}x$, and thus, $\boldsymbol{F} \not\models d^nx \to d^{n+1}x$ either. This, however, contradicts the assumption of $\boldsymbol{F} \models d^{n+1}x \leftrightarrow d^nx$, and finishes the whole proof. ■

Corollary 2.13. *The variety $\mathcal{D}_n$, for any $n \in \omega$, is generated by the algebras* $\mathbf{Alg}(\boldsymbol{F})$, $\boldsymbol{F} \in \mathcal{F}_n$, *where $\mathcal{F}_n$ is the set of all finite frames whose spans do not exceed n. In particular, the logics $\mathbf{D}_n$ are complete with respect to Kripke frames.*

Proof. Follows immediately from Corollary 2.7. and Theorem 2.12. ■

Part Two. Minimal varieties

1. Outline

This part is devoted to investigating the bottom of the lattice $L^{\mathcal{V}}(\mathcal{T})$ of subvarieties of $\mathcal{T}$. It is shown that $L^{\mathcal{V}}(\mathcal{D}_0)$ has two atoms, $L^{\mathcal{V}}(\mathcal{D}_1)$ has countably many atoms, and for $n > 1$, $L^{\mathcal{V}}(\mathcal{D}_n)$ has continuum atoms. The proof of the second of these three statements involves a rather detailed description of zero-generated simple algebras in $\mathcal{D}_1$. As a by-product we obtain that the lattice of subvarieties of $\mathcal{V}(\mathbf{Lin}_t)$ has countably many atoms.

Rendered from the point of view of logics, the content of this chapter amounts to investigating maximal consistent members of the lattice of all normal extensions of the logic $\mathbf{K}_t$. These should be contrasted with what was commonly called a Post-complete extension of a modal (or tense) logic in a series of papers from 70s (cf. e.g., [18], or [12]), namely a maximal consistent extension with respect to the classical consequence operation with detachment as its sole inference rule, without the requirement of closure under necessitation. A fuller discussion of Post-completeness and maximal consistence with respect to a given consequence operation is presented in [16].

It is well-known that, contrary to the situation in the lattice of normal modal logics, where it follows from a theorem of D. Makinson (cf. [11]) that there are precisely two co-atoms (both of which are Post-complete logics), in the lattice of tense logics we have continuum of co-atoms (of which only two are Post-complete).

We show that the lattice of all normal extensions of $\mathbf{Lin}_t$ has countably many co-atoms. And when we descend "just a bit" further down, namely, below the logic $\mathcal{D}_1$ we will find as many of them as can be, i.e., continuum.

As a matter of interest, it can be added that the logics naturally associated with the algebras from the family construed in section 5, and also all these associated with infinite simple algebras from section 4, all enjoy a property which could be dubbed anti-finite model property, namely: they do not have any finite models.

2. General strategy and the inside of $\mathcal{D}_0$

A variety $\mathcal{V}$ is said to be *minimal* iff the only proper subvariety of $\mathcal{V}$ is the trivial variety. Thus, minimal subvarieties of a given variety are the atoms in the lattice of its subvarieties. For finding minimal subvarieties of $\mathcal{T}$, the following fact is crucial.

Fact 3.1. *Let* $\mathbf{A}$ *be a zero-generated simple algebra in* $\mathcal{T}$*. Then,* $\mathcal{V}(\mathbf{A})$ *is a minimal variety. Moreover, if* $\mathbf{A}$, $\mathbf{B}$ *are non-isomorphic zero-generated simple algebras in* $\mathcal{T}$*, then* $\mathcal{V}(\mathbf{A})$, $\mathcal{V}(\mathbf{B})$ *are different.*

Proof. By Jónsson Theorem, all si algebras in $\mathcal{V}(\mathbf{A})$ are in $HSP_U(\mathbf{A})$. Thus, if $\mathbf{B}$ is a si algebra in $\mathcal{V}(\mathbf{A})$, then there is an algebra $\mathbf{C} \in SP_U(\mathbf{A})$ and a congruence $\Theta \in \mathrm{Con}\mathbf{C}$ such that $\mathbf{B} \cong \mathbf{C}/_\Theta$. Since $\mathbf{A}$ is zero-generated, we have $\mathbf{A} \subseteq \mathbf{C}$. As $\mathbf{A}$ is simple, the restriction of Θ to A must be the identity. Thus, $\mathbf{A} \subseteq \mathbf{C}/_\Theta \cong \mathbf{B}$ as well, and, therefore, $\mathbf{A} \subseteq \mathbf{B}$ which yields $\mathbf{A} \in \mathcal{V}(\mathbf{B})$. It follows, that for any si algebra $\mathbf{B}$ in $\mathcal{V}(\mathbf{A})$, $\mathbf{A}$ and $\mathbf{B}$ generate the same variety, hence $\mathcal{V}(\mathbf{A})$ is minimal. This finishes the first part.

For the second, let $\mathbf{A}$, $\mathbf{B}$ be as in the assumptions. Suppose $\mathcal{V}(\mathbf{A})$, $\mathcal{V}(\mathbf{B})$ are identical. This implies, in particular, that $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ and $\mathbf{A} \in \mathcal{V}(\mathbf{B})$. Since both $\mathbf{A}$ and $\mathbf{B}$ are simple (and thus, si), by the previous part of the argument we get that $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$. Thus, $\mathbf{A} \cong \mathbf{B}$, which is a contradiction. ■

The variety $\mathcal{D}_0$ is defined by the equation $dx = x$, so its only simple members can be two-element algebras (for, if there is an element $1 \neq a \neq 0$, we have $da = a$, and thus, by Fact 1.16., we obtain that the algebra in question cannot be simple). These are, up to isomorphism, two:

- $\mathbf{Tr} = \langle\{0,1\}; \wedge, \neg, h, g, 0, 1\rangle$, with $h0 = 0 = g0$; and
- $\mathbf{Ver} = \langle\{0,1\}; \wedge, \neg, h, g, 0, 1\rangle$, with $h0 = 1 = g0$.

thus, we can state:

Fact 3.2. *The varieties $\mathcal{V}(\mathbf{Tr})$ and $\mathcal{V}(\mathbf{Ver})$ are the only atoms of $L^{\mathcal{V}}(\mathcal{D}_0)$. Indeed, $\mathcal{D}_0 = \mathcal{V}(\mathbf{Tr} \times \mathbf{Ver})$.*

3. Zero-generated simple algebras in $\mathcal{D}_1$

Let $\mathbf{A}$ be a zero-generated simple algebra in $\mathcal{D}_1$. We define: $h^0 0 =_{\mathrm{df}} 0$, $h^{n+1}0 =_{\mathrm{df}} hh^n 0$, and similarly for g.

Lemma 3.3. *Assume $h^{n+1}0 > h^n 0$, for a certain n. Then the following hold:*

(i) $g(h^n 0 \vee \neg h^{n+1} 0) = \neg h^n 0$,

(ii) $h(h^n 0 \vee \neg h^{n+1} 0) = h^{n+1} 0$,

(iii) $h^n 0 \vee \neg h^{n+1} 0$ *is a co-atom,*

(iv) $h^{n+1} 0$ *covers* $h^n 0$,

(v) $h \neg h^n 0 = h0$, *and* $g \neg h^{n+1} 0 = \neg h^n 0$,

(vi) $h^n 0 \leq \neg g0$, *and* $gh^n 0 = g0$.

Proof. The fact that $\mathbf{A}$ is simple and belongs to $\mathcal{D}_1$ implies that, for any $a \neq 1$ in A, we have $da = 0$. This will be frequently made use of in the sequel. Since the defining identities of $\mathcal{D}_1$ are symmetric with respect to h and g, we will assume that all the facts proved on the way also hold with h and g interchanged. We will frequently, and without further notice, employ the facts that $h0 = 1$ iff $g0 = 1$, and that $h\neg g0 = 1$, as well as $\neg x \leq g\neg hx$.

By monotonicity of h we obtain that $h^n 0 \leq h^{n+1} 0$ always, so the assumption of the lemma amounts to stating that the chain $\langle h^k 0 : k \in \omega\rangle$ is strictly increasing for $k < n+1$. In particular, $h^n0 \vee \neg h^{n+1}0 \neq 1$, and thus, $d(h^n0 \vee \neg h^{n+1}0) = 0$. Calculate: $g(h^n0 \vee \neg h^{n+1}0) \geq gh^0 0 \vee g\neg h^{n+1}0 \geq g\neg h^{n+1}0 \geq \neg h^n0$. On the other hand, $0 = d(h^n0 \vee \neg h^{n+1}0) = g(h^n0 \vee \neg h^{n+1}0) \wedge (h^n0 \vee \neg h^{n+1}0) \wedge h(h^n0 \vee \neg h^{n+1}0) \geq g(h^n0 \vee \neg h^{n+1}0) \wedge (h^n0 \vee \neg h^{n+1}0) \wedge (hh^n0 \vee h\neg h^{n+1}0) \geq g(h^n0 \vee \neg h^{n+1}0) \wedge ((h^n0 \vee \neg h^{n+1}0) \wedge h^{n+1}0) = g(h^n0 \vee \neg h^{n+1}0) \wedge h^n0$. Thus, $g(h^n0 \vee \neg h^{n+1}0) \leq \neg h^n0$, and both the inequalities yield: $g(h^n0 \vee \neg h^{n+1}0) = \neg h^n0$, which proves (i).

Having established that, we calculate again: $0 = d(h^n0 \vee \neg h^{n+1}0) = g(h^n0 \vee \neg h^{n+1}0) \wedge (h^n0 \vee \neg h^{n+1}0) \wedge h(h^n0 \vee \neg h^{n+1}0)$, but this time we leave out the last conjunct and employ the previously established equality to obtain: $0 = \neg h^n0 \wedge (h^n0 \vee \neg h^{n+1}0) \wedge h(h^n0 \vee \neg h^{n+1}0) = \neg h^{n+1}0 \wedge h(h^n0 \vee \neg h^{n+1}0)$. This yields: $h(h^n0 \vee \neg h^{n+1}0) \leq h^{n+1}0$. On the other hand, $h(h^n0 \vee \neg h^{n+1}0) \geq h^{n+1}0$, so they are equal. This proves (ii).

Now, suppose there is an element $a \in A$, with $1 > a > h^n0 \vee \neg h^{n+1}0$. We, thus, have $ha \geq h(h^n0 \vee \neg h^{n+1}0) = h^{n+1}0$, $ga \geq g(h^n0 \vee \neg h^{n+1}0) = \neg h^n0$. Calculate: $da \geq h^{n+1}0 \wedge \neg h^n0 \wedge a$. But, $da = 0$. Thus, $a \leq \neg(\neg h^{n+1}0 \vee h^n0)$, and this is a contradiction which finishes the proof of (iii).

As for (iv), it follows from (iii) by boolean algebra. To see it, notice that the intervals $I[h^n0 \vee \neg h^{n+1}0, 1]$, and $I[h^n0, h^{n+1}0]$ are projective, hence, isomorphic.

For the first part of (v), we have: $\neg h0 \geq \neg h^n0$, thus $h\neg h0 \geq h\neg h^n0$. From (ii), taking $n = 0$, we obtain $h\neg h0 \leq h0$, and since $h\neg h0 \geq h0$ by monotonicity, we get $h\neg h0 = h0$. We conclude that $h\neg h^n0 = h0$. For the second part of (v), we employ (i) to get: $\neg h^n0 = g(h^n0 \vee \neg h^{n+1}0) \geq g\neg h^{n+1}0 \geq \neg h^n0$. Hence, $g\neg h^{n+1}0 = \neg h^n0$.

To prove (vi), let's consider three cases:

(1) If $g0 = 0$, there is nothing to show for the first part. For the second, it holds trivially for $n = 0$, so calculate assuming $n > 0$: $gh^n0 \wedge g\neg h^n0 = g0 = 0$. Thus, $gh^n0 \leq \neg g\neg h^n0$, which, by (v), yields $gh^n0 \leq h^{n-1}0$. On the other hand, since $h^n0 \neq 1$, we have: $0 = dh^n0 = h^{n+1}0 \wedge h^n0 \wedge gh^n0 =$

$h^n0 \wedge gh^n0$. Thus, $gh^n0 \leq \neg h^n0$. Finally, we get: $gh^n0 \leq \neg h^n0 \wedge h^{n-1}0 = 0 = g0$.

(2) If $g0 = 1$, then $h0 = 1$ as well, and thus the assumption is satisfied only for $n = 0$, in which case we have: $h^00 = 0 \leq 0 = \neg g0$, and $gh^00 = 1 = g0$, so both parts hold.

(3) If neither of the above is the case, we have: $\neg g0 < 1$, so $0 = d\neg g0 = h\neg g0 \wedge \neg g0 \wedge g\neg g0 = \neg g0 \wedge g\neg g0$. Thus, $g\neg g0 \leq g0$, and so $g\neg g0 = g0$. Now, according to our assumption, $h^n0 \neq 1$, thus $0 = dh^n0 = h^{n+1}0 \wedge h^n0 \wedge gh^n0 = h^n0 \wedge gh^n0$. We obtain, then: $h^n0 \leq \neg gh^n0 \leq \neg g0$. Hence, $gh^n0 \leq g\neg g0 = g0$. This finishes the whole proof. ■

The above describes the "part" of **A** generated by strictly ascending chains of h^n0 and g^m0. Situations not covered by Lemma 3.3. will be dealt with in the next three lemmas.

Lemma 3.4. *Assume $h^{n+1}0 = h^n0 \neq 1$, for some $n \in \omega$. Let n be the smallest such. Then the following hold:*

- *if $n > 0$, then:*
  - (i) $h\neg h^n0 = h0$, *and* $g\neg h^n0 = \neg h^{n-1}0$,
  - (ii) $h^n0 \leq \neg g0$, *and* $gh^n0 = g0$,
- *if $n = 0$, then:*
  - (iii) $h\neg h^n0 = 1$, $g\neg h^n0 = 1$, *and* $gh^n0 = g0$.

Proof. Assume $n > 0$. Then, all that has been proved in Lemma 3.3. holds for $k = n - 1$. In particular, $g\neg h^n0 = g\neg h^{k+1}0$, which, by Lemma 3.3., equals $\neg h^k0 = \neg h^{n-1}0$. This proves the second part of (i). For the first part, we have: $\neg h0 \geq \neg h^n0$, thus $h\neg h0 \geq h\neg h^n0$. By the assumption that $n > 0$, we get $h0 > 0$, i.e, $\neg h0 < 1$. This is all we need to calculate: $0 = d\neg h0 = h\neg h0 \wedge \neg h0 \wedge g\neg h0 = h\neg h0 \wedge \neg h0$. It follows that $h\neg h0 \leq h0$, and thus, since $\neg h^n0 \leq \neg h0$ and $h\neg h^n0 \leq h\neg h0$, we obtain $h\neg h^n0 = h0$, as required.

The proof of (ii) mimics that of Lemma 3.3.(vi), namely:

(1) If $g0 = 0$, nothing needs to be shown for the first part. For the second, as we have $n > 0$, we calculate: $gh^n0 \wedge g\neg h^n0 = g0 = 0$. Thus, $gh^n0 \leq \neg g\neg h^n0$, which, by (i) above, yields $gh^n0 \leq h^{n-1}0$. On the other hand, since $h^n0 \neq 1$, we have: $0 = dh^n0 = h^{n+1}0 \wedge h^n0 \wedge gh^n0 = h^n0 \wedge gh^n0$. Thus, $gh^n0 \leq \neg h^n0$. Finally, we get: $gh^n0 \leq \neg h^n0 \wedge h^{n-1}0 = 0 = g0$.

(2) If $g0 = 1$, then $h0 = 1$. The proof of Lemma 3.3.(vi) (2) applies.

(3) Neither of the above is the case. Lemma 3.3.(vi) (3) applies.

The remaining (iii) hardly needs a proof. ■

Lemma 3.5. *For any $m, n \in \omega$, if $g^m0 \vee h^n0 \neq 1$, then $g^m0 < \neg h^n0$. Otherwise, $g^m0 \geq \neg h^n0$.*

Proof. The assumption enables us to calculate: $d(g^m0 \vee h^n0) \geq (hg^m0 \vee h^{n+1}0) \wedge (g^m0 \vee h^n0) \wedge (g^{m+1}0 \vee gh^n0)$. This equals further, by Lemma 3.3.(vi) and/or Lemma 3.4.(ii), $(h^{n+1}0 \wedge (g^m0 \vee h^n0)) \wedge g^{m+1}0 = ((h^{n+1}0 \wedge g^m0) \vee h^n0) \wedge g^{m+1}0 = (h^{n+1}0 \wedge g^m0) \vee (h^n0 \wedge g^{m+1}0) \geq h^n0 \wedge g^{m+1}0$. It follows that $\neg h^n0 \geq g^{m+1}0$. Now, if both the equalities $\neg h^n0 = g^{m+1}0$ and $g^m0 = g^{m+1}0$ were satisfied we would have $g^m0 = \neg h^n0$, i.e., $g^m0 \vee h^n0 = 1$ contradicting the assumption. Thus, we must have $\neg h^n0 > g^{m+1}0$, or $g^m0 < g^{m+1}0$. In either case, we obtain $\neg h^n0 > g^m0$, as needed. The "otherwise" part simply restates that $g^m0 \vee h^n0 = 1$. ■

Lemma 3.6. *Assume that there exist $n, k \in \omega$ with $h^{n+1}0 = h^n0 \neq 1$, $g^{k+1}0 = g^k0 \neq 1$. Let n, k be the smallest such. Then:*

(i) $g^k0 \vee h^n0 \neq 1$,

(ii) $h(g^k0 \vee h^n0) = h^n0$, $g(g^k0 \vee h^n0) = g^k0$,

(iii) $g^k0 \vee h^n0$ *is a co-atom.*

Proof. For (i), notice that if $g^k0 \vee h^n0 = 1$ then $h^n0 \geq \neg g^k0$, and thus $h^{n+k}0 = h^n0 \geq h^k0\neg g^k0 = 1$. Then, $h^n0 = 1$, a contradiction.

For the first part of (ii), observe that $(g^k0 \vee h^n0) \wedge \neg g^k0 = (g^k0 \wedge \neg g^k0) \vee (h^n0 \wedge \neg g^k0) = h^n0 \wedge \neg g^k0$ and this equals h^n0 by Lemma 3.5., and (i) of the present lemma. Thus, $h(g^k0 \vee h^n0) \wedge h\neg g^k0 = h^{n+1}0 = h^n0$, notice that we can assume that $n, k > 0$; if any of these is 0 the claim is either trivial or follows from Lemma 3.4.(ii). Then, by Lemma 3.4.(i) we can rewrite the last equality as $h(g^k0 \vee h^n0) \wedge \neg g^{k-1}0 = h^n0$. Further, $(h(g^k0 \vee h^n0) \wedge \neg g^{k-1}0) \wedge g^{k-1}0 = h^n0 \wedge g^{k-1}0$, which yields $h(g^k0 \vee h^n0) \wedge \neg g^{k-1}0 = h^n0 \wedge g^{k-1}0$, and finally,

$$h(g^k0 \vee h^n0) \leq h^n0 \wedge g^{k-1}0. \qquad (*)$$

On the other hand, $d(g^k0 \vee h^n0) = 0$, i.e., $g(g^k0 \vee h^n0) \wedge (g^k0 \vee h^n0) \wedge h(g^k0 \vee h^n0) = 0$, thus, $g(g^k0 \vee h^n0) \wedge (g^k0 \vee h^n0) \leq \neg h(g^k0 \vee h^n0)$, and since $g(g^k0 \vee h^n0) \geq g^k0$, we obtain that $g^k0 \wedge (g^k0 \vee h^n0) \leq \neg h(g^k0 \vee h^n0)$, which is

$$g^k0 \leq \neg h(g^k0 \vee h^n0). \qquad (**)$$

¿From $(*)$ and $(**)$ we obtain $h(g^k0 \vee h^n0) \leq (h^n0 \vee g^{k-1}0) \wedge \neg g^k0$, and then, $h(g^k0 \vee h^n0) \leq (h^n0 \wedge \neg g^k0) \vee (g^{k-1}0 \wedge \neg g^k0)$. Now, since $g^{k-1}0 < g^n0$ and $h^n0 < \neg g^k0$, we obtain $h(g^k0 \vee h^n0) \leq h^n0$. Yet, obviously $h(g^k0 \vee h^n0) \geq h^n0$ as well, so we get the desired $h(g^k0 \vee h^n0) = h^n0$. The remaining part of (ii) follows by symmetry.

To prove (iii) we will show that for any $a \in A$ we have either $a \leq g^k0 \vee h^n0$ or $a \geq \neg g^k0 \wedge \neg h^n0$. The proof proceeds by induction the length of the generation of a. More precisely, let $\langle X_i : i \in \omega \rangle$ be defined as follows: $X_0 = \{0, 1\}$, $X_{i+1} = X_i \cup \{f(X_i) : f \in \{\wedge, \neg, h, g\}\}$. Clearly, $A = \bigcup_{i \in \omega} X_i$. If $a \in X_0$, the claim obviously holds. Let then $a \in X_{n+1} \setminus X_n$.

- if $a = b \wedge c$, or $a = \neg b$, the claim follows by the inductive hypothesis and the fact that (the appropriate reduct of) **A** is a boolean algebra;
- if $a = hb$, then, for $b \leq g^k0 \vee h^n0$ we have, by (i) of the present lemma, $hb \leq h(g^k0 \vee h^n0) = h^n0 \leq g^k0 \vee h^n0$. For $b \geq \neg g^k0 \wedge \neg h^n0$ we employ Lemma 3.3.(iii) to get that: either $b \leq h^l0 \vee \neg h^{l+1}0$, for some $l < n$, in which case $hb \leq h(h^l0 \vee \neg h^{l+1}0) = h^{l+1}0 \leq h^n0 \leq g^k0 \vee h^n0$, as desired; or else $b \geq \neg h^l0 \wedge h^{l+1}0$ for all $l < n$. The latter amounts to $b \geq (\neg h^00 \wedge h^10) \vee (\neg h^10 \wedge h^20) \vee \ldots \vee (\neg h^{n-1}0 \wedge h^n0) = h^n0$. Thus, we get that $b \geq (\neg g^k0 \wedge \neg h^n0) \vee h^n0 = \neg g^k0 \vee h^n0 = \neg g^k0$, since $g^k0 \vee h^n0 \neq 1$ by (i) of the present lemma and Lemma 3.5. Hence, $hb \geq h\neg g^k0 = \neg g^{k-1}0 > \neg g^k0 \geq \neg g^k0 \wedge \neg h^n0$, as needed. ■

Definition 3.7. For a zero-generated simple algebra $\mathbf{A} \in \mathcal{D}_1$ define: $N = \{n \in \omega : h^n 0 < h^{n+1} 0\}$, $M = \{m \in \omega : g^m 0 < g^{m+1} 0\}$, and then let $C_h = \{h^n 0 \vee \neg h^{n+1} 0 : n \in N\}$, $C_g = \{g^m 0 \vee \neg g^{m+1} 0 : m \in M\}$. Then, if both N and M have largest elements (i.e., are finite) let $l = max(N) + 1$ and $m = max(M) + 1$; moreover, if neither of $g^m 0$, $h^l 0$ equals 1, put $C_* = \{g^m 0 \vee h^l 0\}$. Otherwise set $C_* = \emptyset$. Finally, define $C = C_h \cup C_g \cup C_*$, and $\overline{C} = \{\neg x : x \in C\}$.

Lemma 3.8. *Let $a \in A \setminus \{0, 1\}$; and C, $\overline{C}$ be as above. Then:*

(i) *$a = \bigwedge D$, for some finite $D \subset C$, or $a = \bigvee E$, for some finite $E \subset \overline{C}$;*

(ii) *C, and $\overline{C}$ are, respectively, the set of all co-atoms and the set of all atoms of $\mathbf{A}$.*

Proof. We have already shown that all elements of C are co-atoms, thus, to prove (ii) it suffices to show that C exhausts their set, which will be done shortly. First, we show that (i) holds, by induction of the length of generation of the element a. Let X_i $(i \in \omega)$ be as in the proof of Lemma 3.6.(iii).

The claim clearly holds for X_0. Thus, assume $a \in X_{n+1} \setminus X_n$. We need to consider some cases:

Case 1. If $a = b \wedge c$ or $a = \neg b$, the claim follows by inductive hypothesis, and the fact that (the appropriate reduct of) $\mathbf{A}$ is a boolean algebra.

Case 2. Suppose $a = hb$ and $b = \bigwedge D$, $D \subset_{\text{fin}} C$. Observe, that $D = D_g \cup D_h \cup D_*$, where $D_g \subseteq C_g$, $D_h \subseteq C_h$, $D_* \subseteq C_*$, with at least one of these nonempty. Thus $b = \bigwedge D_g \wedge \bigwedge D_h \wedge \bigwedge D_*$, with $\bigwedge D_q = 1$ whenever $D_q = \emptyset$, $(q \in \{h, g, *\})$. Then $hb = h(\bigwedge D_g) \wedge h(\bigwedge D_h) \wedge h(\bigwedge D_*)$. Now, from Lemma 3.3. it follows that: if $D_g \neq \emptyset$, then $h(\bigwedge D_g) = \neg g^k 0$, for the greatest k such that $g^k 0 \vee \neg g^{k+1} 0 \in D_g$; similarly, if $D_h \neq \emptyset$, then $h(\bigwedge D_h) = h^{l+1} 0$, for the greatest l such that $h^l 0 \vee \neg h^{l+1} 0 \in D_h$. Finally, from Lemma 3.6., we have that if $D_* \neq \emptyset$, then $D_* = \{g^m 0 \vee h^n 0\}$ such that n, m are the smallest with $g^m 0 = g^{m+1} 0 \neq 1$ and $h^n 0 = g^{n+1} 0 \neq 1$, thus $h(\bigwedge D_*) = h^n 0$, with $n > l$, where l is as above.

By the above remarks, we obtain: $hb = p \wedge q$ where:

$$p = \begin{cases} h^n 0 & \text{if } D_* \neq \emptyset \\ h^{l+1} 0 & \text{if } D_* = \emptyset,\ D_h \neq \emptyset \\ 1 & \text{if } D_* \cup D_h = \emptyset \end{cases}$$

$$q = \begin{cases} \neg g^k 0 & \text{if } D_g \neq \emptyset \\ 1 & \text{if } D_g = \emptyset \end{cases}$$

Moreover, we have:

$$h^j 0 = (\neg h^0 0 \wedge h^1 0) \vee \ldots \vee (\neg h^{j-1} 0 \wedge h^j 0) \tag{\dagger}$$

$$\neg g^k 0 = (g^0 0 \vee \neg g^1 0) \wedge \ldots \wedge (g^{k-1} 0 \vee \neg g^k 0) \tag{\ddagger}$$

where j is equal to either of $l+1, n$ and k from above. Thus, it only remains to check that our claim holds for $hb = h^j 0 \wedge \neg g^k 0$. This, however, by Lemma 3.5., can be equal either to $h^j 0$ or to $\neg g^k 0$, which finishes the proof of this case.

Case 3. Suppose $a = hb$ and $b = \bigvee E$, $E \subset_{\text{fin}} \overline{C}$. Let k be the smallest natural number such that $\forall i < k : \neg h^i 0 \wedge h^{i+1} 0 \in E$—such a number exists, as E is finite—and let $E_0 = \{\neg h^i 0 \wedge h^{i+1} 0 : i < k\}$. We will show that:

$$hb = h^{k+1} 0.$$

We have $b = \bigvee (E \setminus E_0) \vee \bigvee E_0$. From now on we fix the set E_0 and the number k with the properties as above. Take $\neg h^k 0 \wedge h^{k+1} 0$. This cannot belong to E, for if it did, it would contradict the choice of E_0 and k; hence, $b \ngeq \neg h^k 0 \wedge h^{k+1} 0$. Since $\neg h^k 0 \wedge h^{k+1} 0$ is an atom, we obtain $b \leq \neg(\neg h^k 0 \wedge h^{k+1} 0) = h^k 0 \vee \neg h^{k+1} 0$. Thus, $hb \leq h(h^k 0 \vee \neg h^{k+1} 0) = h^{k+1} 0$. On the other hand, $hb = h(\bigvee (E \setminus E_0) \vee \bigvee E_0) \geq h(\bigvee E_0)$. Now, by the definition of E_0, we get: $\bigvee E_0 = (\neg h^0 0 \wedge h^1 0) \vee (\neg h^1 0 \wedge h^2 0) \vee \ldots \vee (\neg h^{k-1} 0 \wedge h^k 0) = h^k 0$. Then, $hb \geq h(\bigvee E_0) = h(h^k 0) = h^{k+1} 0$. Together, these yield the desired $hb = h^{k+1} 0$.

Case 4. For $a = gb$ the claim follows by the symmetry of the situation.

Now, turning to (ii), let's suppose there is a co-atom c with $c \notin C$. By (i) we obtain that $c = \bigvee E$, for some finite $E \subseteq \overline{C}$. Now, the only atom that is not under c is $\neg c$, which, by definition, does not belong to $\overline{C}$. Thus, we obtain that $E = \overline{C}$, i.e, $c = \bigvee \overline{C}$. In particular, $\overline{C}$ and C are finite, and, hence, there are n, k, the smallest such that $h^{n+1}0 = h^n 0$, and $g^{k+1}0 = g^k 0$.

Now, the set $\overline{C}$ contains all elements $\neg h^i 0 \wedge h^{i+1}0$, with $0 \leq i < n$; and all elements $\neg g^j 0 \wedge g^{j+1}0$, with $0 \leq j < k$. Thus, by (†) and (‡) above, we get $\bigvee \overline{C} \geq h^n 0 \vee g^k 0$. If $h^n 0 \vee g^k 0$ were equal to 1, we would have $c = 1$, which contradicts the supposition of c being a co-atom. Thus, $h^n 0 \vee g^k 0 \neq 1$, which yields, in particular that neither of the disjuncts equals to 1. Thus, the assumptions of Lemma 3.6. are satisfied; hence, $h^n 0 \vee g^k 0$ is a co-atom. This yields, $c = h^n 0 \vee g^k 0$, but this belongs to C, contrary to the supposition. This finishes the proof of (ii). ■

We can summarise the content of this section in the following:

Corollary 3.9. *Let* $\mathbf{A} \in \mathcal{D}_1$ *be zero-generated and simple. Then:*

(i) $\mathbf{A}$ *is atomic, and generated as a boolean algebra by its atoms;*

(ii) $\mathbf{A}$ *is either finite, or isomorphic as a boolean algebra to the algebra of finite and cofinite subsets of* ω.

Proof. The (i) part is a restatement of Lemma 3.8. For (ii), it suffices to notice that, as $\mathbf{A}$ zero-generated, if it is infinite, it must be countable. The rest follows by (i). ■

4. Minimal subvarieties of $\mathcal{D}_1$

Having completed a bit unwieldy and quite tedious preparatory work, we may now introduce some order into the area. Let's start with yet another definition.

Definition 3.10. For a zero-generated simple algebra $\mathbf{A} \in \mathcal{D}_1$, let $H_{\mathbf{A}} = \{h^n 0 : n \in \omega\}$, $G_{\mathbf{A}} = \{g^n 0 : n \in \omega\}$.

Lemma 3.11. $\mathbf{A}$ *is generated as a boolean algebra by the set* $G_{\mathbf{A}} \cup H_{\mathbf{A}}$*; thus,* $\mathbf{A}$ *is finite iff both* $G_{\mathbf{A}}$ *and* $H_{\mathbf{A}}$ *are. Moreover, the functions* h *and* g *are completely determined by* $G_{\mathbf{A}} \cup H_{\mathbf{A}}$.

Proof. The first statement follows from Corollary 3.9. and the fact that all the atoms (remind that these can be of three types: $\neg h^n 0 \wedge h^{n+1}0$, $\neg g^k 0 \wedge g^{k+1}0$, or $\neg h^m 0 \wedge \neg g^l 0$) are generated by $G_{\mathbf{A}} \cup H_{\mathbf{A}}$.

To show that the second statement holds, observe first that, by Lemma 3.8., an element $b \in A \setminus \{0, 1\}$ can only be a join of finitely many atoms or a meet of finitely many of co-atoms. Now, h and g on finite meets of co-atoms are determined by what happens on co-atoms, since h and g distribute over (finite) meets. This suffices to determine them completely if both $G_{\mathbf{A}}$ and $H_{\mathbf{A}}$ are finite. If at least one of them is infinite, we have to consider the case in which $b = \bigvee E$, with E a finite set of atoms. Now, hb in such a case has been determined in Lemma 3.8.Case 3, namely, $hb = h^{k+1}0$, where k is the smallest such that $\forall i < k : \neg h^i 0 \wedge h^{i+1} 0 \in E$. Similarly, gb has been determined in Case 4. ■

Finite zero-generated simple algebras in $\mathcal{D}_1$ can be of two types:

- *Type 1.* For some $n \in \omega$, $h^n 0 = 1$. Equivalently, for the same n, $g^n 0 = 1$.

The conditions above are indeed equivalent, for suppose $h^n 0 = 1$ and n is the smallest such. By Lemma 3.3.(i), we obtain $g0 = g\neg h^n 0 = \neg h^{n-1} 0$. Repeating this n times we get $g^n 0 = g^n \neg h^n 0 = 1$, as desired.

- *Type 2.* For some $n, m \in \omega$, $h^n 0 = h^{n+1} 0 \neq 1$ and $g^m 0 = g^{m+1} 0 \neq 1$.

Obviously, there are only countably many algebras of type 1 and type 2, since these are finite.

- *Type 3a.* For some $n \in \omega$, $h^n 0 = h^{n+1} 0 \neq 1$, but for all $m \in \omega$, $g^m 0 < g^{m+1} 0$.

Algebras **A**, **B** of this type are isomorphic iff $|H_{\mathbf{A}}| = |H_{\mathbf{B}}|$ iff the number n for both is the same. This follows from the fact that the natural bijection between the sets $G_{\mathbf{A}} \cup H_{\mathbf{A}}$ and $G_{\mathbf{B}} \cup H_{\mathbf{B}}$ extends, by Lemma 3.11., to isomorphism.

There are only countably many algebras of this type, one for each n.

- *Type 3b.* For some $m \in \omega$, $g^m 0 = g^{m+1} 0 \neq 1$, but for all $n \in \omega$, $h^n 0 < h^{n+1} 0$.

This is completely analogous to type 3a: if the algebras are viewed as non-indexed (i.e., without distinguishing "numerically" between operations), type 3a and 3b are identical.

- *Type 4.* For all $n \in \omega$, $h^n 0 < h^{n+1} 0$ and $g^n 0 < g^{n+1} 0$.

There is only one—up to the isomorphism defined by the natural bijection $f : G_{\mathbf{A}} \cup H_{\mathbf{A}} \mapsto G_{\mathbf{B}} \cup H_{\mathbf{B}}$—algebra of type 4.

Let us now list frames corresponding to algebras of each of the above types. Calculations needed to establish the facts presented below are entirely standard, although sometimes can be fairly long. Since the main result does not depend on these, we skip them.

Type 1 algebras are precisely the canonical algebras of these finite frames $\langle W, R, R^{-1} \rangle$ where R is irreflexive, transitive and linear, and n is the number of elements of W.

Type 2 algebras are precisely the canonical algebras of these finite frames which are like the former, except that precisely one point in the frame is reflexive. In this case, $|W| = n + 1 + m$ and the reflexive point is the n'th successor of the initial point and m'th predecessor of the final point.

Algebras of type 3a, 3b, and 4 cannot be canonical algebras of any Kripke frame, since no such algebra can be countably infinite. Yet, the following is the case.

Type 3a algebras are precisely the zero-generated subalgebras of the canonical algebras of the Kripke frames $\langle W, R, R^{-1}\rangle$ where W is the ordinal $\omega + n$ with R being its natural strict ordering.

Type 3b algebras are precisely the zero-generated subalgebras of the canonical algebras of the duals of the former.

Finally, the unique type 4 algebra is the zero-generated subalgebra of the canonical algebra of the Kripke frame $\langle W, R, R^{-1}\rangle$ where W is the order type $\omega + \omega^*$ (ω^* being the dual of ω), and R is its natural strict ordering.

Corollary 3.12. *There are countably many minimal subvarieties of* $\mathcal{D}_1$*. These are precisely all minimal subvarieties of* $\mathcal{V}(\mathbf{Lin}_t)$*.*

Proof. The classification presented in the section 4.2. comprises all zero-generated simple algebras in $\mathcal{D}_1$, and we have seen that there are countably many (non-isomorphic) of these. Since all minimal subvarieties of $\mathcal{D}_1$ are generated by such algebras, the proof of the first part is finished.

For the second part, recall that $\mathcal{V}(\mathbf{Lin}_t)$ is a proper subvariety of $\mathcal{D}_1$ (cf. remarks preceding Corollary 2.8.). Now, it is readily verified that all zero-generated simple algebras in $\mathcal{D}_1$ belong to $\mathcal{V}(\mathbf{Lin}_t)$; it suffices to notice that all the relevant frames are transitive and linear. ■

5. Outside $\mathcal{D}_1$

Once we leave $\mathcal{D}_1$, the number of minimal subvarieties increases to continuum. To show this, we will present a construction producing $2^{\aleph_0}$ non-isomorphic zero-generated simple algebras in $\mathcal{D}_2$.

Consider the boolean algebra $\mathcal{F}(\omega)$ of all finite and cofinite sets of natural numbers with standard set-theoretical operations. The complement of X ($\omega \setminus X$) will be denoted by $\overline{X}$. Let I be any subset of ω. We define an operation h on the universe of $\mathcal{F}(\omega)$:

Definition 3.14.

(i) For the unit and zero elements of $\mathcal{F}(\omega)$ we take: $h(\omega) = \omega$, $h(\emptyset) = \{0\}$;

(ii) For a finite, nonempty $X \subset \omega$, we take: $h(X) = \{0, 2, \ldots, 2n\}$, where n is the largest such that $\forall k \in \omega, k < n : 2k \in X$;

(iii) For a single co-atom of the form $\overline{\{2n+1\}}$ we take: $h(\overline{\{1\}}) = \omega$, and $h(\overline{\{2n+1\}}) = \overline{\{1, 3, \ldots, 2n-1\}}$, for $n \geq 1$;

(iv) For a single co-atom of the form $\overline{\{2n\}}$ we set: $h(\overline{\{2n\}}) = \{0, 2, \ldots, 2n\}$, if $n \in I$, and $h(\overline{\{2n\}}) = \{0, 2, \ldots, 2n, 2n+1\}$ otherwise;

(v) For any other cofinite $X = \overline{\{n_1, \ldots, n_k\}}$, we extend (iii) and (iv) by putting $h(X) = h(\overline{\{n_1\}}) \cap \ldots \cap h(\overline{\{n_k\}})$.

The definition above produces $2^{\aleph_0}$ structures $\langle \mathcal{F}(\omega), h \rangle_i$, $i \in I$, such that for each pair $i \neq j$, $\langle \mathcal{F}(\omega), h \rangle_i$ differs from $\langle \mathcal{F}(\omega), h \rangle_j$, i.e., the function h_i is different than h_j. Moreover, the function h has been chosen so that the inverse h-image of a principal filter is a principal filter as well:

Lemma 3.15. *For any $x \in \mathcal{F}(\omega)$, the set $h^{-1}[x)$ is a principal filter.*

Proof. Notice that the elements y, with $y = hx$ for some x can only be of the form $\{0, 2, \ldots, 2n\}$, $\{0, 2, \ldots, 2n, 2n+1\}$, or $\overline{\{1, 3, \ldots, 2n-1\}}$, or, finally, ω. In particular, if for some co-atom c, $hc = \{0, 2, \ldots, 2n, 2n+1\}$ and for some co-atom $d \neq c$, $hd = \{0, 2, \ldots, 2m, 2m+1\}$, then we have $n \neq m$, and, thus $h(c \cap d) = \{0, 2, \ldots, 2k\}$ with $k = \min\{n, m\}$. Thus, the elements $\{0, 2, \ldots, 2n, 2n+1\}$, if they are h-images at all, they are images of precisely one co-atom.

Further, it is not difficult to see from the construction that the elements $\overline{\{1, 3, \ldots, 2n-1\}}$ and ω are h-images of finite intersections of co-atoms. As for the elements $\{0, 2, \ldots, 2n\}$, these are h-images of the elements from $\{0, 2, \ldots, 2n-2\}$ (including the case $h(\emptyset) = \{0\}$), and no smaller elements x have $hx \supseteq \{0, 2, \ldots, 2n\}$.

Thus, for the elements just mentioned, their inverse h-images have smallest elements, which proves the claim in their case.

For all other elements of $\mathcal{F}(\omega)$ it suffices to notice that any cofinite $y \subset \mathcal{F}(\omega)$ contains the smallest subset x of the form $\overline{\{1,3,\dots,2n-1\}}$, and any finite $z \subset \mathcal{F}(\omega)$ contains the smallest subset u of the form $\{0,2,\dots,2n,2n+1\}$ (and, for that matter, $\{0,2,\dots,2n\}$; choose this which is actually the value of h in our case), to conclude that the claim holds for these as well. ■

Definition 3.16. For any of the structures $\langle \mathcal{F}(\omega), h\rangle_i$ defined above, put $gx = \neg \bigwedge h^{-1}[\neg x)$. Then define $\mathbf{A}_i = \langle \mathcal{F}(\omega), h, g\rangle_i$.

This way, we have obtained $2^{\aleph_0}$ structures. The next lemma ensures that they are exacly the algebras we need.

Lemma 3.17. *For any $i \in I$:*

(i) $\mathbf{A}_i$ *is a tense algebra;*

(ii) $\mathbf{A}_i$ *is zero-generated;*

(iii) *if $i \neq k$, then $\mathbf{A}_i$ is not isomorphic to $\mathbf{A}_k$;*

(iv) *if $a \in A_i$ and $a \neq 1^{\mathbf{A}_i}$, then $d^2a = 0^{\mathbf{A}_i}$;*

(v) $\mathbf{A}_i$ *satisfies $d^3x = d^2x$.*

Proof. It is readily checked that h is a normal modal operator. By Definition 3.14. and Fact 1.12., we obtain that $\mathbf{A}_i$ is a tense algebra, as claimed in (i).

For (ii), notice that we have: $h(\emptyset) = \{0\}$, $h(\{0\}) = \{0,2\}$, $h(\{0,2\}) = \{0,2,4\}, \dots, h(\{0,2,\dots,2n\}) = \{0,2,\dots,2n,2n+2\}, \dots$. Thus, the atom $\{0\}$ is generated by $h(\emptyset)$, and the atom $\{2n+2\}$ is generated as a complement of $\{0,2,\dots,2n\}$ in the interval $I[\emptyset,\{0,2,\dots,2n,2n+2\}]$. Hence, all the even atoms are zero-generated. To see that all the odd atoms are zero-generated as well, observe that from the definition of g it follows that: $g(\emptyset) = \{1\}$, $g(\{1\}) = \{1,3\}$, $g(\{1,3\}) = \{1,3,5\}, \dots, g(\{1,3,\dots,2n+1\}) = \{1,3,\dots,2n+1,2n+3\}, \dots$.

It is straightforward from the construction that (iii) holds. For (iv), consider the worst possible case, i.e., that of a co-atom $\overline{\{2n\}}$ with $h(\overline{\{2n\}}) =$

$\{0, 2, \ldots, 2n, 2n+1\}$. We have: $g\overline{\{2n\}} = \neg \bigwedge h^{-1}[\neg\overline{\{2n\}}) = \neg \bigwedge h^{-1}[\{2n\})$, which equals $\neg\{0, 2, \ldots, 2n-2\}$ (or $\emptyset$, if $n = 0$). This, in turn, equals $\overline{\{0, 2, \ldots, 2n-2\}}$ (or, ω, if $n = 0$). So, $d(\overline{\{2n\}}) = \{0, 2, \ldots, 2n, 2n+1\} \cap \overline{\{2n\}} \cap \overline{\{0, 2, \ldots, 2n-2\}}$, which equals $\{2n+1\}$ (or, $d(\overline{\{0\}}) = \{1\} \cap \overline{\{0\}} \cap \omega = \{1\}$). Now, in both cases, we obtain that $d(\overline{\{2n\}})$ is an atom. Then, it is immediate from the definitions that the next iteration will land in 0. The case of a co-atom $\overline{\{2n+1\}}$ goes exactly the same, by the definition of g.

The remaining (v) is obvious, in view of (iv) and $d(\omega) = \omega$. ■

Thus, we have $2^{\aleph_0}$ simple non-isomorphic zero-generated algebras in $\mathcal{D}_2$. This, by Fact 3.1., amounts to:

Corollary 3.18. *There are $2^{\aleph_0}$ minimal subvarieties of $\mathcal{D}_2$. Thus, there are $2^{\aleph_0}$ minimal subvarieties of the variety of tense algebras.*

Remark 3.19. The second part of Corollary 3.18. was proved by W. Blok as early as in 1970s. Yet, the class of algebras construed in his proof is too broad to fall within $\mathcal{D}_2$ (cf. [5]).

Corollary 3.19. *Varieties $\mathcal{V}(\mathbf{A}_i)$ have no finite members.*

Proof. It is generally the case with respect to minimal varieties that either they are generated by a finite algebra, or have no non-trivial finite members. ■

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