REPORTS ON MATHEMATICAL LOGIC
32 (1998), 35–52

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A NOTE ON CLASSICAL MODAL RELEVANT ALGEBRAS

A b s t r a c t In this paper we define three classes of classical modal relevant algebras (CR-algebras), and study they representation theory. We also investigate sufficients conditions that make the class be the same. We finish the work characterizing the simple and subdirectly irreducible algebras for a particular variety of CR-algebras.

1. Introduction

R. Meyer and E. Mares [10] studied the logic CNR, a classical relevant logic with a modal connective. Such logic is of type $\mathbf{S}_4$. As it has been noted in [10], the addition of a modal operator $\Box$ in the language of classical relevant logic and the axioms $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ and $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$ do not necessarily produce theorems such that as $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$ or $(\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi)$. These formulas are well known theorems of classical modal logic [9]. We will define here classical relevant algebras with a modal operator. We shall investigate some types of algebras

Received April 24, 1998

that are obtained by addition of distinct axioms. For these algebras we will develop a topological duality. The results of duality are based on the usual duality for modal algebras and the duality recently developed by A Urquhart [15] for relevant algebras by means of Priestley spaces. The Urquhart's duality is an extension of the duality given by Brink [3] for Boolean monoides.

In Section 2 we will give the necessary definitions and results in order to develop this work. In Section 3, we will give the above mentioned duality. Section 4 is devoted to determine some conditions that make the considered algebras identical. At last, in Section 5, we will characterize the simple and subdirectly irreducible algebras for a particular variety of algebras.

2. Preliminaries

Definition 1. A classical relevant algebra, or CR-algebra for short, is an algebra $A = \langle A, \vee, \rightarrow, \circ, \neg, e, 0\rangle$ such that:

1. $\langle A, \vee, \neg, 0\rangle$ is a Boolean algebra,
2. $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$,
3. $(a \vee b) \circ c = (a \circ c) \vee (b \circ c)$,
4. $e \circ a = a$,
5. $a \circ 0 = 0 \circ a = 0$,
6. $a \circ b \leq c \Leftrightarrow a \leq b \rightarrow c$,

for all $a, b, c \in A$.

The variety of CR-algebras is a particular case of the relevant algebras studied in [15]. Some classical relevant algebras are also studied by Brink [3].

Let A be a CR-algebra. The element e is defined to be the neutre element of the algebra A, and it determines the order between the elements of A, i.e. $a \leq b \Leftrightarrow e \leq a \rightarrow b$. The filter generated by a subset $X \subseteq A$ will be denoted by $F(X)$, and the collection of all filters will be denoted by $F_i(A)$.

Let A be a CR-algebra and let $Ul\,(A)$ be the set of all ultrafilters of A. Let X and Y be subsets of A. We define by means of the operation $\circ$, the set

$$X \circ Y = \{a \circ b : a \in X \text{ and } b \in Y\}.$$

We define a ternary relation $T_A \subseteq Ul\,(A)^3$ as follows: for all $P, Q, D \in Ul\,(A)$,

$$(P, Q, D) \in T_A \Leftrightarrow P \circ Q \subseteq D.$$

We will denote by $E\,(A)$ the set of all ultrafilters P such that $e \in P$.

The following results summarize known results on the relation T_A. For a proof of these results the reader is referred to [3], [13] or [15].

Theorem 2. *Let A be a CR-algebra. Then:*

1. *Let $F_1, F_2 \in F_i\,(A)$ and $P \in Ul\,(A)$. Then $F_1 \circ F_2 \in F_i\,(A)$, and if $F_1 \circ F_2 \subseteq P$, then there exist $Q, D \in Ul\,(A)$ such that $F_1 \subseteq Q$, $F_2 \subseteq D$ and $Q \circ D \subseteq P$.*
2. *Let $a, b \in A$ and $P \in Ul\,(A)$. Then $a \circ b \in P$ if and only if there exit $Q, D \in Ul\,(A)$ such that $a \in Q$, $b \in D$ and $Q \circ D \subseteq P$.*
3. *Let $a, b \in A$ and $P \in Ul\,(A)$. Then $a \to b \in P$ if and only if for any $Q, D \in Ul\,(A)$ such that, if $P \circ Q \subseteq D$ and $a \in Q$, then $b \in D$.*
4. *For all P, $Q \in Ul\,(A)$ and for all $E \in E\,(A)$, $E \circ P \subseteq Q$ if and only if $P = Q$.*

Let us recall that a *modal algebra* is an algebra $A = \langle A, \vee, \neg, \Box, 0\rangle$ where $\langle A, \vee, \neg, 0\rangle$ is a Boolean algebra and $\Box$ is an unary operator such that it verifies the identities:

M1 $\Box\,(a \wedge b) = \Box a \wedge \Box b$,

M2 $\Box 1 = 1$.

The dual operator $\Diamond$ is defined as $\Diamond a = \neg\Box\neg a$. Let us denote $a \supset b$ the classical implication. Recall that the following inequalities are verified for all $a, b \in A$,

1. $\Box(a \supset b) \leq \Box a \supset \Box b$,
2. $\Diamond a \supset \Box b \leq \Box(a \supset b)$,
3. $\Diamond(a \supset b) \leq (\Box a \supset \Diamond b)$.

Let A be a modal algebra. For the set $Ul(A)$ we can define a binary relation R_A as follows: for all $P, Q \in Ul(A)$,

$$(P, Q) \in R_A \Leftrightarrow \Box^{-1}(P) \subseteq Q \Leftrightarrow Q \subseteq \Diamond^{-1}(P).$$

The pair $\langle Ul(A), R_A \rangle$ is called the **frame** associated to A.

Lemma 3. *Let A be a modal algebra, let $a \in A$, and let $P \in Ul(A)$. Then*

1. $\Box a \notin P \Leftrightarrow \exists\, Q \in Ul(A)$ *such that* $(P, Q) \in R_A$ *and* $a \notin Q$.
2. $\Diamond a \in P \Leftrightarrow \exists\, Q \in Ul(A)$ *such that* $(P, Q) \in R_A$ *and* $a \in Q$.

In order to obtain a topological duality for the algebras that we shall introduce in the next section, we shall recall some concepts of topological duality for Boolean algebras. Some familiarity with topology, particularly with Boolean spaces and modal spaces, is assumed (see, for example, [14]).

A *Boolean space* X is a topological space that is *compact and totally disconnected*, i.e., given distinct points $x, y \in X$, there is a clopen (closed and open) subset U of X such that $x \in U$ and $y \notin U$. If X is a Boolean space, then the family $Clop(X)$ of clopen subsets is a basis for X and a Boolean algebra under set-theoretical complement and intersection. Also, the application $\varepsilon : X \to Ul(Clop(X))$ given by $\varepsilon(x) = \{U \in Clop(X) : x \in U\}$ is a bijective and continuous function. To each Boolean algebra A, we can associate a Boolean space $Spec(A)$ whose points are the elements of $Ul(A)$ with the topology determined by the basis $\beta(A) = \{S_A(a) : a \in A\}$. By the above we have that, if X is a Boolean space, then $X \cong Spec(Clop(X))$, and if A is a Boolean algebra, then $A \cong Clop(Spec(A))$.

A **modal space** [14] is a pair $\langle X, R\rangle$ where X is a Boolean space and R is a binary relation on X such that:

1. $R(x)$ is a closed set for each $x \in X$.
2. $\Diamond_R(U) = \{x \in X : R(x) \cap U \neq \emptyset\} \in Clop(X)$

The condition 2. above is equivalent to

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\} \in Clop(X)$$

for each $U \in Clop(X)$.

If $\langle X, R\rangle$ is a modal space, then $Clop(X)$ with the unary operator $\Box_R$ is a modal algebra. If A is a modal algebra, the pair $\langle Spec(A), R_A\rangle$ is a modal space.

3. Classical Modal Relevant Algebras

Now, we will consider algebras that are CR-algebras and modal algebras simultaneously. We will give a representation theorem for each class of algebras considered.

Definition 4. A classical modal algebra, or CMR-algebra, is an algebra

$A = \langle A, \vee, \to, \circ, \neg, \Box, e, 0\rangle$ such that $A = \langle A, \vee, \neg, \Box, 0\rangle$ is a modal algebra and $A = \langle A, \vee, \to, \circ, \neg, e, 0\rangle$ is a CR-algebra.

The inequalities $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box 1 = 1$ are not enough to produce formulas similar to formulas 1, 2 and 3 of the above section, with the relevant implication $\to$ instead of the classical implication. This fact motivates the following definition.

Definition 5. Let A be a CMR-algebra. We will say that A is a CMR_1, CMR_2, or CMR_3-algebra, if A satisfies the following conditions respectively,

$\mathbf{M}_1$ $\Box(a \to b) \leq \Box a \to \Box b$,

$\mathbf{M}_2$ $\Diamond a \to \Box b \leq \Box (a \to b)$,

$\mathbf{M}_3$ $\Diamond (a \to b) \leq (\Box a \to \Diamond b)$.

Let us note that $\mathbf{M}_1$ is equivalent to:

$\mathbf{M}_1'$ $\Box a \circ \Box b \leq \Box (a \circ b)$.

Let us assume $\mathbf{M}_1$. Since $a \leq b \to (a \circ b)$, then by $\mathbf{M}_1$ and the monotony of $\Box$, we get $\Box a \leq \Box (b \to (a \circ b)) \leq \Box b \to \Box (a \circ b)$. Hence, $\Box a \circ \Box b \leq \Box (a \circ b)$. Let us now suppose $\mathbf{M}_1'$. Since $(a \to b) \circ a \leq b$, we get $\Box (a \to b) \circ \Box a \leq \Box ((a \to b) \circ a) \leq \Box b$. Thus, $\Box (a \to b) \leq \Box a \to \Box b$.

Let us consider a relational structure $\langle X, R, T, E \rangle$, where R is a binary relation on X, T is a ternary relation on X, and E is a subset of X. Let us also consider the following first-order conditions:

$\mathbf{EM}_1$ $\forall x \forall y \forall z \forall d \, (Txyz \wedge Rzd \Rightarrow \exists x' \exists y' \, (Rxx' \wedge Ryy' \wedge Tx'y'd))$.

$\mathbf{EM}_2$ $\forall x \forall y \forall z \forall d \, (Rxy \wedge Tyzd \Rightarrow \exists z' \exists d' \, (Rz'z \wedge Rd'd \wedge Txz'd'))$.

$\mathbf{EM}_3$ $\forall x \forall y \forall z \forall d \, (Txyz \wedge Rxd \Rightarrow \exists y' \exists z' \, (Ryy' \wedge Rzz' \wedge Tdy'z'))$.

Let us consider the structure $\langle X, R, T, E \rangle$. Let D be a subset of $\mathcal{P}(X)$. For each $U, V \in D$, we define the operations $*$ and $\Rightarrow$ as follows:

$$U * V = \{z \in X : \exists x, y : \ \ Txyz \ \& \ x \in U \ \& \ y \in V\}.$$

$$U \Rightarrow V = \{x \in X : \forall y, z \ : \ Txyz \ \& \ y \in U, \text{ implies } z \in V\}.$$

We shall write $(x, y, z) \in T \Leftrightarrow (x, y) \in T^{-1}(z) \Leftrightarrow z \in T(x, y)$.

If the condition $\mathbf{EM}_i$, $i = 1, 2, 3$, is valid in $\langle X, R, T, E \rangle$, we shall write $\langle X, R, T, E \rangle \vDash \mathbf{EM}_i$. In a similar way , if an inequality $\varphi \leq \psi$ is valid in an algebra A, we will write $A \vDash \varphi \leq \psi$.

The next theorem is the key result of this work.

Theorem 6. *Let A be a CMR-algebra. Then*

1. $A \vDash \mathbf{M}_1 \Leftrightarrow \ \langle Ul(A), R_A, T_A, E(A)\rangle \vDash \mathbf{EM}_1$
2. $A \vDash \mathbf{M}_2 \Leftrightarrow \ \langle Ul(A), R_A, T_A, E(A)\rangle \vDash \mathbf{EM}_2$
3. $A \vDash \mathbf{M}_3 \Leftrightarrow \ \langle Ul(A), R_A, T_A, E(A)\rangle \vDash \mathbf{EM}_3$

Proof. *1.* $\Rightarrow$) Assume that $A \vDash \Box(a \to b) \leq \Box a \to \Box b$ and consider $P, Q, D, Z \in Ul(A)$ such that $P \circ Q \subseteq D$ and $\Box^{-1}(D) \subseteq Z$. We note that $\Box^{-1}(P) \circ \Box^{-1}(Q) \subseteq Z$, because if $\Box a \in P$ and $\Box b \in Q$, then $\Box a \circ \Box b \leq \Box(a \circ b) \in P \circ Q \subseteq D$. Since $\Box^{-1}(P)$ and $\Box^{-1}(Q)$ are filters, by point 1. of Theorem 2, there are ultrafilters P' and Q' such that $\Box^{-1}(P) \subseteq P'$, $\Box^{-1}(Q) \subseteq Q'$ and $P' \circ Q' \subseteq Z$.

$\Leftarrow$) Let us suppose that there exist $a, b \in A$ such that $\Box(a \to b) \nleq \Box a \to \Box b$. Then there is an ultrafilter P such that $\Box(a \to b) \in P$ and $\Box a \to \Box b \notin P$. From point 3. of Theorem 2 and Lemma 3, there are ultrafilters Q, D, and Z such that $P \circ Q \subseteq D$, $\Box a \in Q$, $\Box b \notin D$, $\Box^{-1}(D) \subseteq Z$ and $b \notin Z$. By hypothesis there are ultrafilters P', Q' such that $\Box^{-1}(P) \subseteq P'$, $\Box^{-1}(Q) \subseteq Q'$, and $P' \circ Q' \subseteq Z$. Since $\Box(a \to b) \in P$, we get $a \to b \in P'$, and since $a \in Q'$, then $b \in Z$, which is a contradiction. Therefore $A \vDash \Box(a \to b) \leq \Box a \to \Box b$.

2. $\Rightarrow$) Let $P, Q, D, Z \in Ul(A)$ such that $\Box^{-1}(P) \subseteq Q$ and $Q \circ D \subseteq Z$. Let us consider the filter $F(P \circ \Diamond(D))$. This filter is proper because if we suppose the contrary, then $0 \in F(P \circ \Diamond(D))$. Thus there are elements $p \in P$ and $d \in D$ such that $p \circ \Diamond d = 0$. It follows that $p \leq \Diamond d \to 0$, and since $0 \leq \Box 0$, then $\Diamond d \to 0 \leq \Diamond d \to \Box 0 \leq \Box(d \to 0) \in P$. But since $\Box^{-1}(P) \subseteq Q$ and $Q \circ D \subseteq Z$, we have $0 \in Z$, which is absurd. Thus $F(P \circ \Diamond(D))$ is proper.

Let us consider the set

$$Z_{\Box} = \{\Box z : \neg z \in Z\},$$

and we will prove that

$$F\left(P \circ \Diamond (D)\right) \cap Z_{\Box} = \emptyset. \tag{1}$$

Suppose the contrary. Then there exist elements $p \in P$, $d \in D$, and $\Box z \in Z_{\Box}$ such that

$$p \circ \Diamond d \leq \Box z.$$

It follows that

$$p \leq \Diamond d \to \Box z \leq \Box (d \to z) \in P.$$

Since $\Box^{-1}(P) \subseteq Q$ and $Q \circ D \subseteq Z$, then $z \in Z$, which is impossible. Thus (1) is valid. Now, let us consider the family

$$\mathcal{H} = \left\{F \in F_i(A) : F\left(P \circ \Diamond (D)\right) \subseteq F \text{ and } F \cap Z_{\Box} = \emptyset\right\}.$$

From (1) we have $\mathcal{H} \neq \emptyset$. Moreover, $\mathcal{H}$, ordered by inclusion, is closed under unions of non-empty chains. So, by Zorn's lemma, there is a maximal element Z' in $\mathcal{H}$. We prove that $Z' \in Ul(A)$. Let $a \in A$ and suppose that $a \notin Z'$ and $\neg a \notin Z'$. Then $F_1 = F(Z' \cup \{a\}) \notin \mathcal{H}$ and $F_2 = F(Z' \cup \{\neg a\}) \notin \mathcal{H}$. It follows that $F_1 \cap Z_{\Box} \neq \emptyset$ and $F_2 \cap Z_{\Box} \neq \emptyset$. Then there are elements $d_1, d_2 \in Z'$ and $\neg z_1, \neg z_2 \in Z$ such that $d_1 \wedge a \leq \Box z_1$ and $d_2 \wedge \neg a \leq \Box z_2$. So,

$$(d_1 \wedge d_2) \wedge (a \vee \neg a) \leq (d_1 \wedge a) \vee (d_2 \wedge \neg a)$$

$$\leq \Box z_1 \vee \Box z_2 \leq \Box (z_1 \vee z_2) \in Z'. \tag{2}$$

As $\neg z_1, \neg z_2 \in Z$, $\neg (z_1 \vee z_2) \in Z$, and this implies that $\Box (z_1 \vee z_2) \in Z_{\Box}$. Then, by ((2), $Z_{\Box} \cap Z' \neq \emptyset$, which is a contradiction. Thus, $Z' \in Ul(A)$, and since $Z' \cap Z_{\Box} = \emptyset$, we get

$$\Box^{-1}(Z') \subseteq Z. \tag{3}$$

At last, since $P \circ F(\Diamond (D)) \subseteq Z'$, by point 1. of Theorem 2, there is an ultrafilter D' such that $P \circ D' \subseteq Z'$ and $\Box^{-1}(D') \subseteq D$. By this and (3) we obtain the result.

$\Leftarrow$) Let $a, b \in A$ and let us suppose that $\Diamond a \to \Box b \not\leq \Box(a \to b)$. Then by Theorem 2 and Lemma 3, there exist ultrafilters P, Q, D, and Z such that $\Diamond a \to \Box b \in P$, $\Box(a \to b) \notin P$, $\Box^{-1}(P) \subseteq Q$, $a \to b \notin Q$, $Q \circ D \subseteq Z$, $a \in D$, and $b \notin Z$. Then, by hypothesis, there are ultrafilters D' and Z' such that $P \circ D' \subseteq Z'$, $\Box^{-1}(D') \subseteq D$, and $\Box^{-1}(Z') \subseteq Z$. So, $\Box b \notin Z'$, but since $\Diamond a \to \Box b \in P$ and $\Diamond a \in D'$, we have $\Box b \in Z'$, which is absurd. Therefore, $A \vDash \Diamond a \to \Box b \leq \Box(a \to b)$.

3. $\Rightarrow$) Let us consider $P, Q, D, Z \in Ul(A)$ such that $P \circ Q \subseteq D$ and $\Box^{-1}(P) \subseteq Z$. First, we prove that the filter $F\left(Z \circ \Box^{-1}(Q)\right)$ is proper. If we suppose the contrary, there exits $z \in Z$ and there exits $\Box q \in Q$ such that $z \circ q = 0$, which implies that $z \leq q \to 0$. Then $\Diamond z \leq \Diamond(q \to 0) \leq \Box q \to \Diamond 0 = \Diamond q \to 0 \in P$, because $\Box^{-1}(P) \subseteq Z$. But since $\Box q \in Q$ and $P \circ Q \subseteq D$, we get $0 \in D$, which is impossible. Thus $F\left(Z \circ \Box^{-1}(Q)\right)$ is proper. Now it is easy to check that

$$F\left(Z \circ \Box^{-1}(Q)\right) \cap \Diamond^{-1}(D)^{c} = \emptyset,$$

where $\Diamond^{-1}(D)^{c} = \{x : \neg\Diamond x \in D\}$. Then there is an ultrafilter D' such that

$$Z \circ \Box^{-1}(Q) \subseteq D' \text{ and } \Box^{-1}(D) \subseteq D'.$$

As $\Box^{-1}(Q)$ is a filter, by point 1. of Theorem 2, there is an ultrafilter Q' such that $\Box^{-1}(Q) \subseteq Q'$ and $Z \circ Q' \subseteq D'$.

$\Leftarrow$). It is very similar to the previous proofs and hence left to the reader. ■

Theorem 7 (of Representation). *Let A be a CMR-algebra. Then $\beta(A) = \langle \beta(A), \cap, ^{c}, \Rightarrow, *, \Box_{R_A}, E(A), \emptyset \rangle$ is a CMR-algebra isomorphic to A. Moreover, A is a CMR_i-algebra, $i = 1, 2, 3$, if and only if $\beta(A)$ is a CMR_i-algebra, $i = 1, 2, 3$, respectively.*

Proof. It follows from the results of Urquhart [15], Theorem 6, and the Representation theorem for modal algebras (see [14]). ■

Now, we shall give the topological duality for the varieties

Definition 8. Let X be a Boolean space and let T be a ternary relation on X. We shall say that T is **closed** iff for all $z \in X$, $T^{-1}(z)$ is a closed set of the product space $X \times X$, i.e., for all $z \in X$ if $(x,y) \notin T^{-1}(z)$, then there exist $U, V \in Clop(X)$ such that $T^{-1}(z) \cap (U,V) = \emptyset$.

Definition 9. A **CR-space** is a structure $\langle X, T, E \rangle$ such that X is a Boolean space, T is a ternary relation on X, $E \in Clop(X)$ and:

1. T is closed.
2. $\forall x, y \in X\, (\exists e \in E \wedge Texy \Leftrightarrow x = y)$.
3. For all $U, V \in Clop(X)$, $U * V$, $U \Rightarrow V \ \in Clop(X)$.

If $\langle X, T, E \rangle$ is a $CR-$space, $\langle Clop(X), \cup, \rightarrow, ^{c}, *, E, \emptyset \rangle$ is a CR-algebra. Reciprocally, if A is a CR-algebra, then $\langle Spec(Ul(A)), T_A, E(A) \rangle$ is an RC-space. For more details on these results see [15].

The next result gives an equivalent condition to the condition 1. of the above definition.

Proposition 10. *Let $\langle X, R, T \rangle$ be a CR-space. Then the following conditions are equivalent:*

1. *T is closed.*
2. $(x,y,z) \in R \Leftrightarrow (\varepsilon(x), \varepsilon(y), \varepsilon(z)) \in T_{Clop(X)}$.

Proof. 1. $\Rightarrow$ 2. Let $(\varepsilon(x), \varepsilon(y), \varepsilon(z)) \in T_{Clop(X)}$ and suppose that $(x,y) \notin T^{-1}(z)$. Then, there exist $U, V \in Clop(X)$, such that $(x,y) \in (U,V)$ and $T^{-1}(z) \cap (U,V) = \emptyset$. Then $U * V \in \varepsilon(x) * \varepsilon(y)$ and $U * V \notin \varepsilon(z)$, which is a contradiction.

2. $\Rightarrow$ 1. Let $(x,y) \in \overline{T^{-1}(z)}$ (closure of $T^{-1}(z)$) and suppose that $(x,y) \notin T^{-1}(z)$. Then $\varepsilon(x) * \varepsilon(y) \nsubseteq \varepsilon(z)$. This means that there are $U \in \varepsilon(x), V \in \varepsilon(y)$ such that $U \circ V \in \varepsilon(x) * \varepsilon(y)$ and $U \circ V \notin \varepsilon(z)$. Thus, $T^{-1}(z) \cap (U,V) = \emptyset$. But, since $(x,y) \in (U,V)$ and $(x,y) \in \overline{T^{-1}(z)}$, we obtain $T^{-1}(z) \cap (U,V) \neq \emptyset$, so we have a contradiction. ■

Definition 11. A modal CR-space, or CMR-space, is a structure of the form $\langle X, T, R, E\rangle$ where $\langle X, T, E\rangle$ is a CR-space and $\langle X, R\rangle$ is a modal space.

Lemma 12. *Let $\langle X, R, T, E\rangle$ be CMR-space. Then for all $x, y \in X$, $T(x, y)$ is a closed set of X.*

Proof. Let $x, y, z \in X$ and suppose that $z \notin T(x, y)$. Then $(x, y) \notin T^{-1}(z)$ and since $T^{-1}(z)$ is a closed set of X^2, there exist $U, V \in Clop(X)$ such that $x \in U$, $y \in V$ and $T^{-1}(z) \cap (U, V) = \varnothing$. Thus $z \notin U * V$ and $T(x, y) \subseteq U * V$. Therefore $T(x, y)$ is a closed set of X. ■

Theorem 13. *Let A be a CMR-algebra and let $\langle X, R, T, E\rangle$ be a CMR-space. Then*

1. *$Clop(Spec(A))$ is a CMR-algebra isomorphic to A.*
2. *$Spec\ (Clop(X))$ is a CMR-space homeomorphic to X.*

Proof. These results follow from the results given in [15] for the duality of relevant algebras and the results given in [14] for the duality of modal algebras. ■

Theorem 14. *Let $\langle X, R, T, E\rangle$ be a CMR-space. Then*

1. $\langle X, R, T, E\rangle \vDash \mathbf{EM}_1 \Leftrightarrow Spec(Clop(X)) \vDash \mathbf{M}_1$
2. $\langle X, R, T, E\rangle \vDash \mathbf{EM}_2 \Leftrightarrow Spec(Clop(X)) \vDash \mathbf{M}_2$
3. $\langle X, R, T, E\rangle \vDash \mathbf{EM}_3 \Leftrightarrow Spec(Clop(X)) \vDash \mathbf{M}_3$

Proof. It follows from Theorem 6 and the duality for relevant and modal algebras. ■

By an abuse of notation, we shall denote by **CMR** the category of CMR algebras and homomorphisms of CMR algebras. The category of **CMR**-spaces and p-morphism, as defined by Urquhart in [15l, shall be denoted by **ECMR**. We are going to define functors $\mathcal{F} : \mathbf{CMR} \to \mathbf{ECMR}$ and $\mathcal{P} : \mathbf{ECMR} \to \mathbf{CMR}$ as follows. For each algebra A we consider the space $\mathcal{F}(A) = \langle X(A), R_A, T_A, E(A)\rangle$. If $h : A_1 \to A_2$ is a homomorphism of **CMR** algebras, then $\mathcal{F}(h) : \mathcal{F}(A_2) \to \mathcal{F}(A_1)$ given by $\mathcal{F}(h)(P) = f^{-1}(P)$ is a p-morphism. If $\langle X, R, T, E\rangle$ is a **CMR** space, then $\mathcal{P}(\langle X, R, T, E\rangle) = Spec(Clop(X))$ is a CMR algebra. If $f : \langle X_1, R_1, T_1, E_1\rangle \to \langle X_2, R_2, T_2, E_2\rangle$ is a p-morphism, then the function $\mathcal{P}(f) : Spec(Clop(X_2)) \to Spec(Clop(X_1))$ given by $\mathcal{P}(f)(U) = f^{-1}(U)$ is a homomorphism of CMR algebras. From results on the duality between the category of Boolean algebras and the category of modal spaces, and the results on the duality between relevant algebras and relevant spaces [15], we have that the categories **CMR** and the category **ECMR** are equivalents. This duality is easy to extend to the CMR_i-algebras.

4. CMR-algebras with negation

Now, we will study the following question. Under which conditions the inequalities $\mathbf{M}_1$, $\mathbf{M}_2$ or $\mathbf{M}_3$ are equivalent? To answer this question we will consider algebras with a De Morgan negation.

We will say that $\langle A, \vee, \to, \circ, e, \neg, \sim, 0\rangle$ is a $CMR_{\sim}$-algebra if $\langle A, \vee, \to, \circ, \Box, e, \neg, 0\rangle$ is a CMR-algebra, $\langle A, \vee, , \sim, 0\rangle$ is a De Morgan algebra, and the following condition is verified:

Mo $A \vDash a \to\sim b \leq b \to\sim a$.

It is easy to check that **Mo** is equivalent to:

Mo' $a \circ c \leq b \Rightarrow a\circ \sim b \leq\sim c$, for all $a, b, c \in A$.

Let us recall that a symmetrical Boolean algebra $\langle A, \alpha\rangle$ is a Boolean algebra A such that α is an involutive automorphism (see [1]) , i.e. $\alpha^2 x = x$, for all $x \in A$. If A is a $CMR_{\sim}$-algebra, then the equation $\alpha x = \neg \sim x$ defines an involutive automorphism. We note that α is a homomorphism of Boolean algebra but not necessary is a homomorphism for the operations $\circ$, $\to$ or $\Box$.

If A is a $CMR_{\sim}$-algebra, then we can define an involution g on $Ul(A)$ given by: $g(P) = \{x :\sim x \notin P\}$, for $P \in Ul(A)$. Since for all $x \in A$, $\sim \neg x = \neg \sim x$, then $g(P) = \alpha(P)$ for all $P \in Ul(A)$.

Theorem 15. *Let A be a $CMR_{\sim}$-algebra. Then the following conditions are equivalent:*

1. $\neg\Box\neg a = \sim\Box\sim a$,
2. $\alpha\Box a = \Box\alpha a$,
3. $\Box(\sim a \vee b) \leq \sim\Box a \vee \Box b$.

Proof. 1. $\Rightarrow$ 2. Let $a \in A$. Then

$$\alpha\Box a = \neg\sim\Box a = \neg\sim\Box\sim\sim a = \neg\neg\Box\neg\sim a = \Box\alpha a$$

2. $\Rightarrow$ 3. Let us recall that for any Boolean algebra the following inequality is verified: $\Box(\neg a \vee b) \leq \neg\Box \vee \Box b$, for all $a, b \in A$. So,

$$\begin{aligned}\Box(\sim a \vee b) =& \Box\neg\neg(\sim a \vee b) = \Box(\neg\alpha a \vee b)\\ \leq& \neg\Box\alpha a \vee \Box b = \neg\alpha\Box a \vee \Box b\\ =& \sim\Box a \vee \Box b.\end{aligned}$$

3. $\Rightarrow$ 1. Since $1 = a \vee \neg a = \sim\sim a \vee \neg a$, then

$$1 = \Box 1 = \Box(\sim\sim a \vee \neg a) \leq \sim\Box\sim a \vee \Box\neg a.$$

It follows that $\neg\Box\neg a \leq \sim\Box\sim a$.

Since $1 = \neg\sim a \vee \sim a = \sim\neg a \vee \sim a$, then

$$1 = \Box 1 = \Box(\neg\sim a \vee \sim a) \leq \sim\Box\neg a \vee \Box\sim a.$$

So, $\Box\neg a \wedge \sim\Box\sim a = 0$. Thus, $\sim\Box\neg a \leq \neg\Box\neg a$. ■

Now, we shall give an equivalent condition to the conditions of Theorem 15 but in terms of $\langle Ul(A), R_A, g\rangle$.

Theorem 16. *Let A be a $CMR_{\sim}$-algebra. Then the following conditions are equivalent*

1. *For all $a \in A$, $\alpha \Box a = \Box \alpha a$*
2. *For all $P, Q \in Ul(A)$, if $(P, Q) \in R_A$, then $(g(P), g(Q)) \in R_A$.*

Proof. Assume that for all $a \in A$, we have $\alpha \Box a = \Box \alpha a$, and let $P, Q \in Ul(A)$ such that $\Box^{-1}(P) \subseteq Q$. If $a \in \Box^{-1}(g(P))$, then $\alpha \Box a = \Box \alpha a \in Q$. It follows that $a \in g(Q)$.

Assume that $(P, Q) \in R_A$, implies that $(g(P), g(Q)) \in R_A$. Suppose that $\alpha \Box a \neq \Box \alpha a$. Then there exists $P \in Ul(A)$ such that $\alpha \Box a \in P$ and $\Box \alpha a \notin P$. Then, by Lemma 3, there is an ultrafilter Q such that $(P, Q) \in R_A$ and $\alpha a \notin Q$. Since $\Box^{-1}(g(P)) \subseteq g(Q)$ and $\alpha \Box a \in P$, we get $a \in g(Q)$, which is a contradiction. Therefore $\alpha \Box a \leq \Box \alpha a$. The proof of $\Box \alpha a \leq \alpha \Box a$ is similar. ∎

Definition 17. Let A be a $CMR_{\sim}$-algebra. We shall say that A is symmetrical if α is a modal homomorphism.

Let A be a $CMR_{\sim}$-algebra. Let us recall the following equivalences between conditions defined on A and conditions defined on $\langle Ul(A), T_A \rangle$.

N $A \vDash a \to\sim b \leq b \to\sim a$ iff $\forall P, Q, D \in Ul(A)$, $(P, Q, D) \in T_A$ implies that $(P, g(D), g(D)) \in T_A$.

C $A \vDash a \circ b = b \circ a$ iff $\forall P, Q, D \in Ul(A)$, $(P, Q, D) \in T_A$ implies that $(Q, P, D) \in T_A$.

We shall say that A is commutative if it verifies **C**.

Theorem 18. *Let A be a symmetrical CMR$_{\sim}$-algebra that is commutative and verifies* **N**. *Then* $\mathbf{M}_1$ *and* $\mathbf{M}_3$ *are equivalent.*

Proof. Let A be a $CMR_{\sim}$-algebra and let us consider

$$\langle Ul(A), R_A, T_A, E(A) \rangle .$$

By Theorem 6 it is enough to prove that the conditions $\mathbf{EM}_1$ and $\mathbf{EM}_3$ are equivalent. Suppose that the condition $\mathbf{EM}_1$ is verified and we prove $\mathbf{EM}_3$. Let $x, y, z, d \in X = Ul(A)$ such that $(x, d) \in R_A$ and $(x, y, z) \in T_A$. Then, by Theorem 16, we have

$$(y, x, z) \in T_A \text{ and } (g(x), g(d)) \in R_A,$$

and by condition $\mathbf{N}$, we have

$$(y, g(z), g(x)) \in T_A \text{ and } (g(x), g(d)) \in R_A.$$

Then there exist $y', z' \in X$ such that $(y, y') \in R_A$, $(g(z), z') \in R_A$, and $(y', z', g(d)) \in T_A$. By condition $\mathbf{N}$ and condition $\mathbf{C}$, $(d, y', g(z')) \in T_A$, and by Theorem 16, $(z, g(z')) \in R_A$. Then $\mathbf{EM}_3$ is valid.

The proof that $\mathbf{EM}_3$ implies $\mathbf{EM}_1$ is similar. ■

Theorem 19. *Let A be a $CMR_{\sim}$-algebra such that $\Diamond\Box a \leq a$, for all $a \in A$. Then $\mathbf{M}_2$ and $\mathbf{M}_3$ are equivalent.*

Proof. Let us recall $A \vDash \Diamond\Box a \leq a$ iff R_A is symmetrical relation. By Theorem 6 it is sufficient to prove that the conditions $\mathbf{EM}_2$ and $\mathbf{EM}_3$ are equivalent.

Assume $\mathbf{EM}_2$ and let $x, y, z, d \in Ul(A)$ such that $(x, d) \in R_A$ and $(x, y, z) \in T_A$. Since R_A is symmetrical, $(d, x) \in R_A$. By the assumption, there exist y', $z' \in Ul(A)$ such that

$$(z, z') \in R_A, \ (y, y') \in R_A \text{ and } (d, y', z') \in T_A.$$

So, $(z', z) \in R_A$ and $(y', y) \in R_A$, because R_A is symmetrical. Thus $\mathbf{EM}_3$ is valid.

The proof of that $\mathbf{EM}_3$ implies $\mathbf{EM}_2$ is similar. ■

Corollary 20. *Let A be a commutative and symmetrical $CMR_{\sim}-$algebra which verifies $\mathbf{N}$ and $\Diamond\Box a \leq a$. Then the conditions $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are equivalent.*

5. Some simple and subdirectly irreducible algebras

Let us recall that an $R^{\neg}$-algebra ([3]) is a classical relevant algebra with negation and it verifies **N**, it is commutative, the operation $\circ$ is associative and $a \leq a \circ a$, for all $a \in A$. A *modal* $R^{\neg}$-algebra, or $MR^{\neg}$-algebra, is a pair $\langle A, \Box \rangle$ such that A is an $R^{\neg}$-algebra, is a modal algebra, and $\Box e = e$. We will characterize the simple and subdirectly irreducibles $MR^{\neg}$-algebras.

Let A be an $R^{\neg}$-algebra. A *relevant filter* of A is a filter F such that $e \in F$. It is known (see [5], [6] and [7]) that the lattice of relevant filters is isomorphic to the lattice of congruences of A

Let A be a modal algebra. An *open filter* of A is a filter F such that $\Box a \in F$ for each $a \in F$. We know that the lattice $\mathcal{F}_n(A)$ of open filters of A is isomorphic to the lattice of congruences of A (see [14]). Thus, by these observations we have that:

Proposition 21. *Let A be a $MR^{\neg}$-algebra. Then the lattice $\mathcal{D}_s(A)$ of the filters that are open and relevant is isomorphic to the lattice $Con(A)$ of congruences of A.*

Let A be a modal algebra and let $a \in A$. The open filter generated by a is $F_n(a) = \{x \in A : t_n(a) \leq x\}$, where $t_n(a) = a \wedge \Box a \wedge \ldots \Box^n a$. The following results are part of the folklore of the modal algebras.

Proposition 22. *Let A be a modal algebra. Then*

1. *A is simple iff for all $x \in A$, if $x \neq 1$ then $t_n(x) = 0$ for some $n \geq 0$.*
2. *A is subdirectly irreducible iff there is an element $a \in A$, $a \neq 1$ such that for all $x \neq 1$, $t_n(x) \leq a$, for some $n \geq 0$.*

Let A be an $MR^{\neg}$-algebra. It is well known that the ideal $I(e)$ is a Boolean algebra. Let us consider the modal algebra $B_e = \langle I(e), \Box \rangle$.

Theorem 23. *Let A be an $MR^{\neg}$-algebra. Then $\mathcal{D}_s(A) \cong \mathcal{F}_n(B_e)$.*

Proof. Let $F \in \mathcal{D}_s(A)$. It is clear that $F \cap I(e) \in \mathcal{F}_n(B_e)$. Then we can define the function

$$\varphi : \mathcal{D}_s(A) \to \mathcal{F}_n(B_e)$$

given by $\varphi(F) = F \cap I(e)$. We prove that φ is bijective. Let $F_1, F_2 \in \mathcal{D}_s(A)$ such that $\varphi(F_1) = \varphi(F_2)$ and let $a \in F_1$. Since $e \in F_1$, $a \wedge e \in F_1 \cap I(e) = F_2 \cap I(e)$. Hence $a \in F_2$ because F_2 is a filter. Thus $F_1 \subseteq F_2$. Similarly, $F_2 \subseteq F_1$. Therefore φ is injective.

Let $H \in \mathcal{F}_n(B_e)$ and let $F_n(H)$ be the open filter generated by H in A. We prove that $\varphi(F_n(H)) = F_n(H) \cap I(e) = H$. Let $a \in F_n(H) \cap I(e)$. Since $H \subseteq F_n(H)$ and $e \in H$, we have $e \in F_n(H)$. So, $F_n(H)$ is a relevant filter. Moreover, if $a \in F_n(H)$, there exists $x \in H$ and some $n \geq 0$ such that $t_n(x) \leq a$. Since $x \in H$, we get $t_n(x) \in H$. Then $a \in H$. This implies that $F_n(H) \cap I(e) \subseteq H$. The other inclusion is immediate. So φ is surjective. Thus $\mathcal{D}_s(A) \cong \mathcal{F}(B_e)$. ■

Corollary 24. *Let A be a $MR^{\neg}$-algebra. Then*

1. *A is simple iff B_e is simple.*
2. *A is subdirectly irreducible iff B_e is subdirectly irreducible.*

Proof. It follows from Propositions 21, 22 and 23. ■

Studies in progress generalize some results presented in this work to relevant algebras with operators.

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