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AXIOMATIZATION OF THE LOGIC DETERMINED BY SOME n-ELEMENT RELATIONAL SYSTEMS WITH IDENTITY

A b s t r a c t. Let $\underline{N_n} = \langle N_n, =, 1, \ldots, n\rangle$ be the relational system of n - element set $N_n = \{1, 2, \ldots, n\}$, with identity and with fixed elements: $1, 2, ..., n$. We show that logic $L\left(\underline{N_n}\right)$ determined by the system $\underline{N_n}$ is axiomatizable.

1. Let $\underline{L_n}$ be the first - order language for $\underline{N_n}$ containing the following symbols: $=, \overline{1}, \overline{2}, \ldots, \overline{n}$ interpreted in $\underline{N_n}$ as the identity and natural numbers: $1, 2, ..., n$.

By $\underline{F_n}$ we denote the set of all formulas of $\underline{L_n}$ which will be represented by Latin letters. Moreover, $\underline{F_n}$ is the smallest set such that: $t_1 = t_2 \in \underline{F_n}$ (atomic formulas) and if $A, B \in \underline{F_n}$, then: $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \equiv B$, $\bigwedge_{x_k} A$, $\bigvee_{x_k} A \in \underline{F_n}$.

We shall use the symbols: $\underline{\wedge}$, $\underline{\vee}$, $\underline{\neg}$, $\Rightarrow$, $\Leftrightarrow$, $\forall_{x_k}$, $\exists_{x_k}$, as the well-known propositinal connectives and the quantifiers from the metalanguage.

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The symbol $\underline{N_n} \models A$ will be the abbreviation of the expression: the formula A is true in $\underline{N_n}$.

2. The set of all logical schemes SL is the smallest set such that:

a. $P_k^n(x_{i_1}, ..., x_{i_n}) \in SL$, $k, n = 1, 2, \ldots$

b. if $\alpha, \beta \in SL$ then $\neg\alpha$, $\alpha \to \beta$, $\alpha \wedge \beta$, $\alpha \vee \beta$ $\alpha \equiv \beta$, $\bigwedge_{x_k} \alpha$, $\bigvee_{x_k} \alpha \in SL$.

where P_k^n is n-ary predicate letter.

The set of all atomic schemes $P_k^n(x_{i_1}, \ldots, x_{i_n})$ for $k, n = 1, 2, \ldots$ will be denoted by At.

We define now the set $\underline{V_{N_n}}$ of functions as follows:

$v \in \underline{V_{N_n}} \Leftrightarrow$

a. $v : At \to F_n$

b. $vf\,(v\,(\alpha) = vf\,(\alpha))$

c. $v(\alpha(x_k/x_n)) \approx [v\,(\alpha)]_{k+n}\,(x_k/x_n)$

for every $\alpha \in At \subseteq SL$.

Every function $v \in \underline{V_{N_n}}$ can be extendend to a homomorphism $h^v : SL \to F_n$.

Definition 1. (cf. [3]) A scheme $\alpha \in SL$ is a tautology of the relational system $\underline{N_n} = \langle N_n, =, 1, \ldots, n\rangle$ if and only if for every $v \in \underline{V_{N_n}}$: $\underline{N_n} \models h^v(\alpha)$.

The set of all tautologies of $\underline{N_n}$ is denoted by $L\left(\underline{N_n}\right)$ and is called the logic of the system $\underline{N_n}$.

Then by Definition 1 we have:

Corollary 1. *For every* $\alpha, \beta \in SL$

(i) $L_2 \subseteq L\left(\underline{N_n}\right)$,

(ii) $\alpha \in L\left(\underline{N_n}\right) \underline{\wedge}\, \alpha \to \beta \in L\left(\underline{N_n}\right) \Rightarrow \beta \in L\left(\underline{N_n}\right)$,

(iii) $\alpha \in L\left(\underline{N_n}\right) \Rightarrow \bigwedge_{x_k} \alpha \in L\left(\underline{N_n}\right)$,

(iv) $\alpha \in L\left(\underline{N_n}\right) \Rightarrow \forall_{e \in E_*} h^e(\alpha) \in L\left(\underline{N_n}\right)$, *where L_2 denotes the set of all logical schemes, which are theorems of the classical first-order predicate logic and E_* is the set of functions of substitution for atomic schemes (for details see [2], p.216).*

Every function $e \in E_*$ can be extended to an endomorphism $h^e : SL \to SL$.

We define now the scheme Θ_n as follows:

a. $\Theta_1 = \bigwedge_{x_t} \left[P_2^2(x_1, x_t) \equiv P_2^2(x_2, x_t)\right]$

b.

$$\begin{aligned}\Theta_{n+1} = \Theta_n &\vee \bigwedge_{x_t} \left[P_2^2(x_1, x_t) \equiv P_2^2(x_{n+2}, x_t)\right] \\ &\vee \bigwedge_{x_t} \left[P_2^2(x_2, x_t) \equiv P_2^2(x_{n+2}, x_t)\right] \\ &\vdots \\ &\vee \bigwedge_{x_t} \left[P_2^2(x_{n+1}, x_t) \equiv P_2^2(x_{n+2}, x_t)\right].\end{aligned}$$

Then:

Lemma 1. *For every natural number $n \in N$*

$$\Theta_n \in L\left(\underline{N_n}\right).$$

Thus by Definition 1 we have:

Lemma 2. *If $\underline{M}$ is a relational system of n-element domain with relation of identity then $L\left(\underline{N_n}\right) \subseteq L\left(\underline{M}\right)$, where $L\left(\underline{M}\right)$ is the logic determined by relational system $\underline{M}$.*

Proof. It follows from the fact that $\underline{M}$ is the reduct of the system $\underline{N_n}$ and the logic of the reducted system includes the logic of the system $\underline{N_n}$. ■

Lemma 3. (cf. [3]). *For every* $k, m \in N$

$$k \leq m \Rightarrow L(\underline{N_m}) \subseteq L(\underline{N_k}).$$

Theorem 1. $L\left(\underline{N_n}\right) = L^n$, *where* $L^n = Cn_{\{r_0, r_\forall, r_*\}}(L_2 \cup \{\theta_n\})$, $Cn_{\{r_0, r_\forall, r_*\}}$, *is the consequence operation determined by the rule of detachement* (r_0)*, the rule of generalization* ($r_\forall$) *and the rule of substitution* (r_*)*.*

Proof. The inclusion $L^n \subseteq L(\underline{N_n})$ follows immediately from Lemma 1.

$L\left(\underline{N_n}\right) \subseteq L^n$.

Suppose that there exists a scheme α_0 such that:

(6) $$\alpha_0 \in L\left(N_n\right)$$

(7) $$\alpha_0 \notin Cn_{\{r_0, r_\forall, r_*\}}(L_2 \cup \{\theta_n\}),$$

and furthemore

(8) $$vf\left(\alpha_0\right) = \emptyset$$

Let $T \subseteq SL$ be the set defined as follows

(9) $$T = Cn_{\{r_0, r_\forall\}}(L^n \cup \{\neg\alpha_0\}).$$

Then from (7), (8) and according to Tarski's theorem we infer:

(10) $$T \neq SL.$$

Let $\Pr(\alpha_0)$ be the set of all predicate letters occurring in α_0

(11) $$\Pr(\alpha_0) = \{P_{k_1}^{n_1}, P_{k_2}^{n_2}, ..., P_{k_t}^{n_t}\}.$$

The set $SL(\alpha_0)$ is the smallest set such that

a. $P_{k_1}^{n_1}(x_1, \ldots, x_{n_1}), \ldots, P_{k_t}^{n_t}(x_1, \ldots, x_{n_t}) \in SL(\alpha_0)$

b. if $\alpha, \beta \in SL(\alpha_0)$ then $\neg\alpha$, $\alpha \to \beta$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \equiv \beta$, $\bigwedge_{x_k} \alpha$, $\bigvee_{x_k} \alpha \in SL(\alpha_0)$.

Let now $\leftrightarrow$ be a binary relation over $SL(\alpha_0)$ such that

$$\gamma \leftrightarrow \delta \Leftrightarrow \gamma \to \delta, \delta \to \gamma \in T \tag{12}$$

for every $\delta, \gamma \in SL(\alpha_0)$.

Hence, the relational system $\left\langle SL(\alpha_0)\big/_{\leftrightarrow}, \leq \right\rangle$ - where $SL(\alpha_0)\big/_{\leftrightarrow}$ is the set of all equivalence classes $\|\gamma\|$ $(\gamma \in SL\,(\alpha_0))$ and $\leq$ is the relation given by $\|\gamma\| \leq \|\delta\| \Leftrightarrow \gamma \to \delta \in T$ and the system is a Boolean algebra.

It follows from (8) and (10) that the class $\|\neg\alpha_0\|$ is a non-zero element in $\left\langle SL(\alpha_0)\big/_{\leftrightarrow}, \leq \right\rangle$.

Thus according to Rasiowa–Sikorski Lemma (cf. [4]) there exists an ultrafiter H in $\left\langle SL(\alpha_0)\big/_{\leftrightarrow}, \leq \right\rangle$ which contains $\|\neg\alpha_0\|$;

$$\|\neg\alpha_0\| \in H \tag{13}$$

and preserves all the meets

$$\left\| \bigwedge_{x_k} \alpha \right\| \in H \Leftrightarrow \forall_{n \in N} \, \|\alpha(x_k/x_n)\| \in H \tag{14}$$

Let $\models\!\!\!\dashv$ be a binary relation over N defined as follows

$$k \models\!\!\!\dashv n \Leftrightarrow \|I\,(x_k, x_n)\| \in H, \tag{15}$$

$I\,(x_k, x_n)$ is defined as follows (cf. [1])

$$I\,(x_k, x_n) = [I\,(x, y)]_{k+n}\,(x/x_k)(y/x_n)$$

$k, n = 1, 2, \ldots$, where $I\,(x, y) = \prod_{k \in \{1,2,\ldots,t\}} I^k(x, y)$ and
$I^k\,(x, y) = \prod_{i \in \{1,2,\ldots,n_k\}} I_i^k(x, y)$.
Let $I_{ik}\,(x, y)$ be the abbreviation of the scheme

$$\bigwedge_{x_{k_1}} \cdots \bigwedge_{x_{k_n}} [P_k^{n_k}(x_1, \ldots, x_{n_k})(x_i/x) \equiv P_k^{n_k}(x_1, \ldots, x_{n_k})(x_i/y)]$$

for $k = 1, 2, \ldots, t$ and $i = 1, 2, \ldots, n_k$.

It is easily seen that the relation $\models\!\mid$ is an equivalence relation over N.

We define the relational system as follows: $\underline{N^*} = \langle N^*, =, p_1^{n_1}, \ldots, p_t^{n_t} \rangle$ where $N^* = \{k^* : k \in N\}$ (N^*-is the set of all $\models\!\mid$ equivalence classes $k^* = \{m \in N : m \models\!\mid k\}$ for $k \in N$) and

$$p_i^{n_i}(k_1^*, \ldots, k_{n_i}^*) \Leftrightarrow \left\| P_i^{n_i}(x_{k_1}, \ldots, x_{k_{n_i}}) \right\| \in H \tag{16}$$

$i = 1, 2, ..., t$.

Let now $\underline{L_N^*}$ be the language for $\underline{N^*}$. By $F_{\underline{N}}^*$ we denote the set of all formulas of $\underline{L_N^*}$. Hence $F_{\underline{N}}^*$ is the smallest set such that:

$$x_i = x_j, \overline{p}_1^{n_1}(x_{i_1}, ..., x_{i_{n_1}}), \ldots, \overline{p}_t^{n_t}(x_{i_1}, ..., x_{i_{n_t}}) \in F_{\underline{N}}^*$$

and closed with respect to: $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$ and $\bigwedge_{x_i}$, $\bigvee_{x_i}$.

The symbols $\overline{p}_1^{n_1}, \ldots, \overline{p}_t^{n_t}$ are predicate constants interpreted in $\underline{N^*}$ as the relations: $p_1^{n_1}, \ldots, p_t^{n_t}$.

Let $v_0 : At \rightarrow F_{\underline{N}^*}$ be a function such that

$$v_0(P_i^{n_i}(x_1, ..., x_{n_i})) = \overline{p}_i^{n_i}(x_1, \ldots, x_{n_i}) \tag{17}$$

for $i = 1, 2, ..., t$.

Hence for every $\gamma \in SL(\alpha_0)$ we have

$$\underline{N^*} \models h^{v_0}(\gamma)\,[\Sigma^*] \Leftrightarrow \|\gamma\| \in H \tag{18}$$

where Σ^* is the sequence from N onto N^* such that $\Sigma^*(k) = k^*$ for every $k \in N$.

According to (18) the formula $h^{v_0}(\gamma)$ is satisfied by the sequence Σ^* in the structure $\underline{N^*}$ iff $\|\gamma\| \in H$; this can be proved by the induction on the number of connectives and quantifiers in γ.

First of all

$$\underline{N^*} \models v_0(P_{k_i}^{n_i}(x_1, ..., x_{n_i}))[\Sigma^*] \Leftrightarrow \left\| P_{k_i}^{n_n}(x_1, ..., x_{n_i}) \right\| \in H.$$

Indeed

$$\begin{aligned}\underline{N^*} \models v_0(P^{n_i}_{k_i}(x_1,...,x_{n_i}))\,[\Sigma^*] &\Leftrightarrow p^{n_i}_{k_i}(x_1[\Sigma^*],...,x_{n_i}[\Sigma^*]) \\ &\Leftrightarrow p^{n_i}_{k_i}(\Sigma^*(1),...,\Sigma^*(n_i)) \\ &\Leftrightarrow p^{n_i}_{k_i}(1^*,\dots,n^*) \\ &\Leftrightarrow \left\|p^{n_i}_{k_i}(x_1,\dots,x_{n_i})\right\| \in H.\end{aligned}$$

Let us assume that β, δ are the formulas satisfying the condition. We show that $N^* \models (\beta \wedge \delta)[\Sigma^*] \Leftrightarrow \|\beta \wedge \delta\| \in H$.

Indeed from the fact that H is a filter we have

$$\underline{N^*} \models (\beta \wedge \delta)\,[\Sigma^*] \Leftrightarrow \underline{N^*} \models \beta[\Sigma^*] \wedge \underline{N^*} \models \delta[\Sigma^*]$$

$$\Leftrightarrow \|\beta\| \in H \wedge \|\delta\| \in H \Leftrightarrow \|\beta \wedge \delta\| \in H.$$

Let us assume that δ satisfies the inductive assumption and $\gamma = \bigwedge_{x_k} \delta$. Then

$$\begin{aligned}\underline{N^*} \models (\bigwedge_{x_k} \delta)[\Sigma^*] &\Leftrightarrow \forall_{a\in N/=}\underline{N^*} \models \delta[\Sigma^{*k}_a] \\ &\Leftrightarrow \forall_{m\in N}\underline{N^*} \models \delta[\Sigma^{*k}_{m^*}] \\ &\Leftrightarrow \forall_{m\in N}\underline{N^*} \models \delta(x_k/x_m)[\Sigma^*] \\ &\Leftrightarrow \forall_{m\in N}\,\|\delta(x_k/x_m)\| \in H \\ &\Leftrightarrow \left\|\bigwedge_{x_k} \delta\right\| \in H,\end{aligned}$$

where $\Sigma^{*k}_a(i) = \begin{cases} \Sigma^*(i) & k \neq i \\ a^* & k = i \end{cases}$

Hence, from (18) and (13) we infer:

$$\underline{N^*} \models h^{v_0}(\neg\alpha_0)[\Sigma^*]. \tag{19}$$

Then $\underline{N^*} \not\models h^{v_0}(\alpha_0)[\Sigma^*]$. Therefore

$$\alpha_0 \notin L\,(\underline{N^*})\,. \tag{20}$$

Thus by the fact that $\Theta_n \in L^n$ we have also:

(21) $$card\,(\underline{N^*}) \leq n$$

(22) $$\Theta_n\,(x_{n+1}/x_t) \in L^n \quad \text{for every } t \in N.$$

Thus for the function $s_1 : At \rightarrow SL$ such that

$$s_1(P_k^m(x_{i_1},...,x_{i_m})) = \begin{cases} I(x_{i_1},x_{i_2})\text{if } k=2 \wedge m=2 \\ P_k^m(x_{i_1},...,x_{i_m})\text{if } k \neq 2 \vee m \neq 2 \end{cases}$$

we have

(23) $$h^{s_1}(\Theta_n(x_{n+1}/x_t)) \in L^n \subseteq T$$

(24) $$\bigwedge_y [I(x_1,y) \equiv I(x_2,y)] \vee \ldots \vee \bigwedge_y [I(x_n,y) \equiv I(x_t,y)] \in T$$

(25) $$I\,(x_1,x_2) \vee \ldots \vee I\,(x_n,x_t) \in T,$$

(26) $$\|I\,(x_1,x_2) \vee \ldots \vee I\,(x_n,x_t)\| \in H.$$

From the fact that H is an ultrafilter and from (15) we get

(27) $$1 \boxminus 2 \;\vee\; 2 \boxminus 3 \;\vee\; \ldots \;\vee\; n \boxminus t$$

for every $t \in N$.

Thus $card\,(\underline{N^*}) \leq n$.

From (21) and Lemmas 2, 3 we have

(28) $$L\left(\underline{N_n}\right) \subseteq L\,(\underline{N^*})\,.$$

Hence, from (22) we infer:

(29) $$\alpha_0 \notin L\left(\underline{N_n}\right).$$

This contradicts the hypothesis (6). Thus Theorem 1 is proved. ■

References

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