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C.J. VAN ALTEN and J.G. RAFTERY

ON QUASIVARIETY SEMANTICS OF FRAGMENTS OF INTUITIONISTIC PROPOSITIONAL LOGIC WITHOUT EXCHANGE AND CONTRACTION RULES

A b s t r a c t. Let H be the Hilbert-style intuitionistic propositional calculus without exchange and contraction rules (as given by Ono and Komori). An axiomatization of H with the separation property is provided. Of the superimplicational fragments of H, it is proved that just two fail to be finitely axiomatized, and that all are algebraizable. The paper is a study of these fragments, their equivalent algebraic (quasivariety) semantics and their axiomatic extensions.

Introduction

In various formal approaches to intuitionistic propositional logic, a hierarchy of logical systems results from the presence of all, some or none of the structural rules of weakening, exchange and contraction: see, e.g. [26]. Among such 'substructural' logics are two Gentzen-style sequent calculi studied by Ono and Komori in [22], which they called L_{BCC} and L_{BCK}.

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These are essentially the logics obtained from Gentzen's calculus LJ (defined in [15]) by removing, respectively, the contraction and exchange rules, and the contraction rule alone. (See [10], [11], [12] also.)

The present paper concerns the logic L_{BCC}, its superimplicational fragments and their algebraic model classes. We abbreviate L_{BCC} as L. An equivalent Hilbert-style deductive system H_{BCC} (which we call H) was defined in [22] by a finite set I of (axioms and) inference rules. This axiomatization does not provide a 'separation theorem': in particular, for a subset C of the connectives $\supset, \&, \vee, \wedge, \bot$ containing $\supset$ and $\vee$ but not $\wedge$, not all theorems of the C–fragment of H may be derived from the rules of I that use only connectives from C.

Answering a question raised in [22], we provide (in Section 2) an axiomatization J of H for which the separation theorem does hold. Thus, for any C containing $\supset$, the C–fragment of H is axiomatized by the rules of J that use only connectives from C. The axiomatization J is unavoidably infinite, however: we prove in Section 3 that the $\{\supset, \vee\}$– and $\{\supset, \vee, \bot\}$–fragments of H are not finitely axiomatizable.

Whereas the 'C–rules' of I axiomatize a nonalgebraizable logic when C contains $\supset$ and $\vee$ but not $\wedge$, we observe in Section 3 that for any C containing $\supset$, the C–fragment of H is algebraizable in the sense of [5]. For its equivalent quasivariety semantics, we prefer a dual (additive) notation in which a $C^* \subseteq \{\dot{-}, \oplus, \sqcap, \sqcup, 1\}$ replaces $C \subseteq \{\supset, \&, \vee, \wedge, \bot\}$. This quasivariety, now explicitly axiomatized, is denoted $\mathcal{H}_{C^*}$ and consists of suitable subreducts of left residuated integral ordered monoids with well behaved bounded lattice operations: the left residuation operation $\dot{-}$ corresponding to (reversed) implication is always in C^* and $x \dot{-} x$ defines the identity 0 of the monoid operation $\oplus$, which is also the least element of the order $\leq$.

Sections 4–6 of the paper concern algebraic properties of such a quasivariety $\mathcal{H}_{C^*}$. It is always relatively congruence distributive. A new simple characterization of the 'filters' (with respect to H) of algebras $\mathbf{A} \in \mathcal{H}_{C^*}$ is given; there is a lattice isomorphism between these and the relative congruences of $\mathbf{A}$. The class $\mathcal{H}_{C^*}$ is a variety if and only if C^* contains (in addition to $\dot{-}$) $\oplus$ and at least one of $\sqcap$, $\sqcup$ (a result of P.M. Idziak). When C^* contains $\dot{-}$, $\oplus$ and $\sqcap$, this variety is arithmetical.

In Section 5, we characterize relative subvarieties of $\mathcal{H}_{C^*}$ with (i) the relative congruence (i.e. filter) extension property (FEP) and (ii) equationally definable principal relative congruences (i.e. filters), which amounts to saying which axiomatic extensions of the C–fragment of H have a (i) local, and (ii) full deduction theorem. For each $n \in \omega$, these include the subquasivarieties $\mathcal{H}^n_{C^*}$ of $\mathcal{H}_{C^*}$ satisfying $x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0$, i.e., the axiomatic extensions in which $\vdash q \supset^n ((q \supset p) \supset p)$.

We focus on the classes $\mathcal{H}^n_{C^*}$ in Section 6. They encompass all BCK–algebras (take $C^* = \{\dot{-}\}$ and $n = 1$) and, up to anti-isomorphism, ordinals not exceeding ω^ω, considered as right residuated well ordered additive monoids (take $n = 2$). For C^* excluding both $\oplus$ and 1, the relative subvarieties of the $\mathcal{H}^n_{C^*}$s are characterized by the FEP together with a certain weak finiteness condition. In particular, we prove that all locally finite relative subvarieties of $\mathcal{H}_{C^*}$ with the FEP lie in some $\mathcal{H}^n_{C^*}$. We characterize the finitely subdirectly irreducible members of $\mathcal{H}^n_{C^*}$. If $\sqcap \in C^*$ then $\mathcal{H}^n_{C^*}$ has 'equationally definable principal meets' (EDPM); we also prove (constructively) that the quasivariety generated by the linearly ordered members of $\mathcal{H}^n_{C^*}$ is a relative subvariety of $\mathcal{H}^n_{C^*}$ with EDPM, regardless of whether C^* contains $\sqcap$. We indicate how these facts illustrate a general result of Czelakowski and Dziobiak [9].

2. Hilbert-Style Formulation of L with Separation

A fixed denumerable set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots\}$ of *variables* is given, over which lower case Roman letters, possibly with integer subscripts, shall range as metavariables. We prefer the metavariables $p, q, r, \ldots$ in logical contexts (i.e., these denote 'propositional' variables) and $x, y, z, \ldots$ in algebraic ones. In either context, lower case Greek letters shall be used to denote *formulas* (i.e. terms) arising in the usual way from variables and the *connectives* (i.e.

fundamental operation symbols) of a sentential (i.e. algebraic) language; and upper case Greek letters to denote (possibly empty) sets or sequences of formulas. *Sequents* are written as $\Gamma \rightarrow \alpha$; recall that in such expressions Γ is required to be a finite sequence of formulas. We assume a basic knowledge of Gentzen-style sequent calculi (henceforth, briefly, Gentzen systems): see, e.g., [27].

In their paper [22], Ono and Komori study the Gentzen system L_{BCC}, which we denote throughout this paper by L. The language of L consists of the binary connectives $\supset$, &, $\vee$, $\wedge$ and the constant $\bot$. Initial sequents of L are either of the form $\bot \rightarrow \alpha$ for any formula α, or of the form $p \rightarrow p$ for any propositional variable p.

Rules of inference of L are as follows: [1]

$$\frac{\Gamma, \Delta \rightarrow \gamma}{\Gamma, \alpha, \Delta \rightarrow \gamma}(\text{weakening}) \qquad \frac{\Gamma \rightarrow \alpha \qquad \Delta, \alpha, \Sigma \rightarrow \gamma}{\Delta, \Gamma, \Sigma \rightarrow \gamma}(\text{cut})$$

$$\frac{\Gamma, \alpha \rightarrow \beta}{\Gamma \rightarrow \alpha \supset \beta}(\rightarrow \supset) \qquad \frac{\Gamma \rightarrow \alpha \qquad \Delta, \beta, \Sigma \rightarrow \gamma}{\Delta, \alpha \supset \beta, \Gamma, \Sigma \rightarrow \gamma}(\supset \rightarrow)$$

$$\frac{\Gamma \rightarrow \alpha}{\Gamma \rightarrow \alpha \vee \beta}(\rightarrow \vee 1) \qquad \frac{\Gamma \rightarrow \beta}{\Gamma \rightarrow \alpha \vee \beta}(\rightarrow \vee 2)$$

$$\frac{\Gamma, \alpha, \Delta \rightarrow \gamma \qquad \Gamma, \beta, \Delta \rightarrow \gamma}{\Gamma, \alpha \vee \beta, \Delta \rightarrow \gamma}(\vee \rightarrow)$$

$$\frac{\Gamma, \alpha, \Delta \rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \rightarrow \gamma}(\wedge \rightarrow 1) \qquad \frac{\Gamma, \beta, \Delta \rightarrow \gamma}{\Gamma, \alpha \wedge \beta, \Delta \rightarrow \gamma}(\wedge \rightarrow 2)$$

$$\frac{\Gamma \rightarrow \alpha \qquad \Gamma \rightarrow \beta}{\Gamma \rightarrow \alpha \wedge \beta}(\rightarrow \wedge)$$

$$\frac{\Gamma \rightarrow \alpha \qquad \Delta \rightarrow \beta}{\Gamma, \Delta \rightarrow \alpha \,\&\, \beta}(\rightarrow \&) \qquad \frac{\Gamma, \alpha, \beta, \Delta \rightarrow \gamma}{\Gamma, \alpha \,\&\, \beta, \Delta \rightarrow \gamma}(\& \rightarrow).$$

[1] The weak conjunction & is called 'fusion' (of premisses). If we restore exchange and contraction to L then & duplicates $\wedge$.

Note that L is *not* equipped with the structural rules

$$\frac{\Gamma, \alpha, \alpha, \Delta \to \gamma}{\Gamma, \alpha, \Delta \to \gamma}(\text{contraction}) \qquad \frac{\Gamma, \alpha, \beta, \Delta \to \gamma}{\Gamma, \beta, \alpha, \Delta \to \gamma}(\text{exchange}).$$

The exchange rule is derivable in the extension of L by the contraction rule.

Theorem 2.1. [22] *The cut elimination theorem holds for* L.

Let $\{\supset\} \subseteq C \subseteq \{\supset, \&, \vee, \wedge, \bot\}$. A formula α is a C*–formula* if any logical connective appearing in α belongs to C. A sequent consisting of only C–formulas is called a C*–sequent*. Let C–L denote the Gentzen system dealing with C–sequents whose initial sequents are $p \to p$ for any propositional variable p and, when $\bot \in C$, also $\bot \to \alpha$ for any C–formula α, and whose rules of inference are those of L that use only connectives from C. Theorem 2.1 has the following corollary: see, e.g., [27, Thm. 6.3].

Corollary 2.2. *If a sequent* $\Gamma \to \alpha$ *is derivable in* L *then it is derivable in* C–L*, whenever* C *contains* $\supset$ *and all the connectives occurring in* $\Gamma \cup \{\alpha\}$.

We therefore call C–L the C*–fragment* of L.

By a Hilbert-style deductive system (briefly, a Hilbert system), we mean a deductive system S as defined, e.g., in [5]. Although defined as a language together with a (consequence) relation $\vdash_S$ having certain properties, such a system is often specified by a set of (axioms and) inference rules. (We may regard axioms as inference rules with empty sets of premisses.) Recall that S is, by definition, *structural*, i.e., whenever $\Gamma \cup \{\varphi\}$ is a set of formulas, f an assignment of formulas to variables (extended compatibly to all formulas in the standard way) and $f[\Gamma] = \{f(\gamma) : \gamma \in \Gamma\}$, we have

$$\Gamma \vdash_S \varphi \quad \text{implies} \quad f[\Gamma] \vdash_S f(\varphi).$$

Also, by definition, S is *finitary*, i.e., whenever $\Gamma \vdash_S \varphi$, we have $\Gamma_1 \vdash_S \varphi$ for some finite $\Gamma_1 \subseteq \Gamma$.

Ono and Komori also introduced a Hilbert system H_{BCC}, with the same connectives as L. We shall use H throughout as an abbreviation of

H_{BCC} and we reserve the symbol $\vdash$ for its consequence relation. We adopt the convention for $\supset$ that omitted parentheses are associated to the right, e.g., $\alpha \supset \beta \supset \gamma$ abbreviates $\alpha \supset (\beta \supset \gamma)$.

The system $H = H_{\mathrm{BCC}}$ is defined by the axioms

(H1) $p \supset q \supset p$

(H2) $\bot \supset p$

(H3) $(p \supset q) \supset (r \supset p) \supset r \supset q$

(H4) $((p \supset r) \wedge (q \supset r)) \supset (p \vee q) \supset r$

(H5) $p \supset (p \vee q)$

(H6) $q \supset (p \vee q)$

(H7) $(p \wedge q) \supset p$

(H8) $(p \wedge q) \supset q$

(H9) $((r \supset p) \wedge (r \supset q)) \supset r \supset (p \wedge q)$

(H10) $p \supset q \supset (p \wedge q)$

(H11) $(p \supset q \supset r) \supset (p \,\&\, q) \supset r$

(H12) $p \supset q \supset (p \,\&\, q)$.

and the following two additional inference rules:

$$p,\ p \supset q\ \vdash\ q \quad \text{(m.p.1)} \quad \text{and} \quad q,\ p \supset q \supset r\ \vdash\ p \supset r \quad \text{(m.p.2)}$$

Let I be the above *axiomatization* of H, i.e., the set consisting of (H1)–(H12), (m.p.1) and (m.p.2). Let $\{\supset\} \subseteq C \subseteq \{\supset, \&, \vee, \wedge, \bot\}$. The *$C$–fragment* of H, denoted C–H, is the Hilbert system whose language consists of the connectives of C and whose consequence relation, denoted $\vdash_C$, is defined by:

$$\Gamma \vdash_C \varphi \text{ if and only if } \Gamma \vdash \varphi \text{ and } \Gamma \cup \{\varphi\} \text{ is a set of } C\text{–formulas.}$$

Here our nomenclature differs from that of [22]. What Ono and Komori refer to as the 'C–fragment' of H will be called here the '$\langle C, I\rangle$–subsystem'. This notion, unlike the one just defined, depends not only on H and C but also on the axiomatization I. In particular, for C as above, the $\langle C, I\rangle$–*subsystem* of H, denoted $\langle C, I\rangle$–H, is a Hilbert system whose language consists of the connectives of C; it is axiomatized by the formulas among (H1)–(H12) that use only connectives from C, and (m.p.1) and (m.p.2). We denote its consequence relation by $\vdash_{C,I}$. (Our usage of 'fragment' is consistent, for example, with [5]. No corresponding notational distinction need be drawn for L, in view of Corollary 2.2) A *superimplicational* fragment shall mean a C–fragment, for some C containing $\supset$.

Let G and S be a Gentzen and a Hilbert system, respectively, with a common language that includes a specified binary connective $\supset$. Suppose that for all formulas $\alpha_1, \ldots, \alpha_n, \gamma$, the sequent $\alpha_1, \ldots, \alpha_n \to \gamma$ is derivable in G exactly when the formula $\alpha_1 \supset \alpha_2 \supset \ldots \supset \alpha_n \supset \gamma$ is provable in S. Then we say that G and S are *logically equivalent*. We say that the *separation theorem* holds for an axiomatization I' of S provided that for any provable formula α of S, there exists a derivation of α (with respect to I') in which all occurring connectives other than $\supset$ are among the connectives occurring in α.

Theorem 2.3. [22, Cor. 2.8.2] *L and H are logically equivalent. Also, for any subset C of $\{\supset, \&, \vee, \wedge, \bot\}$ which contains $\supset$ and either does not contain $\vee$ or does contain $\wedge$, the systems C–L and $\langle C, I\rangle$–H are logically equivalent.*

In [22, §9, Remark 1], Ono and Komori ask whether there is a Hilbert-style formulation of L for which the separation theorem holds. We shall present such a formulation here.

Let J be the set obtained by removing from I the axiom (H4) and adding

$$(X_0)\quad (q \supset s) \supset (r \supset s) \supset (q \vee r) \supset s,$$

$$(X_n)\quad (p_1 \supset \ldots \supset p_n \supset q \supset s) \supset$$
$$(p_1 \supset \ldots \supset p_n \supset r \supset s) \supset p_1 \supset \ldots \supset p_n \supset (q \vee r) \supset s,$$

for each natural number n. Let $H' = H'_{\mathrm{BCC}}$ be the Hilbert system with the same language as H that is axiomatized by J and let $\vdash'$ denote its consequence relation. For $\{\supset\} \subseteq C \subseteq \{\supset, \&, \vee, \wedge, \bot\}$, we define the C–fragment and $\langle C, J\rangle$–subsystem of H' just as for H and I. We denote them by C–H' and $\langle C, J\rangle$–H', and their consequence relations by $\vdash'_C$ and $\vdash'_{C,J}$, respectively.

Lemma 2.4. *$\langle\{\supset, \vee, \wedge\}, I\rangle$–$H$ and $\langle\{\supset, \vee, \wedge\}, J\rangle$–$H'$ coincide. Thus, H and H' are the same Hilbert system.*

Proof. First we show that each (X_n) is provable in $\langle\{\supset, \vee, \wedge\}, I\rangle$–$H$. We begin with the following observation:

$$v \supset w,\ t \supset u \supset v \vdash_{\{\supset\},I} t \supset u \supset w. \tag{1}$$

For, by (H3),

$$\vdash_{\{\supset\},I} (v \supset w) \supset (u \supset v) \supset u \supset w,$$

and

$$\vdash_{\{\supset\},I} ((u \supset v) \supset (u \supset w)) \supset (t \supset u \supset v) \supset t \supset u \supset w,$$

so (1) follows by (m.p.1). By (H4),

$$\vdash_{\{\supset,\vee,\wedge\},I} ((q \supset s) \wedge (r \supset s)) \supset (q \vee r) \supset s,$$

and by (H10),

$$\vdash_{\{\supset,\vee,\wedge\},I} (q \supset s) \supset (r \supset s) \supset ((q \supset s) \wedge (r \supset s)).$$

Thus, by (1), (X_0) is a theorem of $\langle\{\supset, \vee, \wedge\}, I\rangle$–$H$. Using (H9), (1) and induction, we obtain

$$\vdash_{\{\supset,\vee,\wedge\},I} ((p_1 \supset \ldots \supset p_n \supset q \supset s) \wedge (p_1 \supset \ldots \supset p_n \supset r \supset s)) \supset$$

$$p_1 \supset \ldots \supset p_n \supset ((q \supset s) \wedge (r \supset s)).$$

Using (H4), (H3) and induction, we have

$$\vdash_{\{\supset,\vee,\wedge\},I} (p_1 \supset \ldots \supset p_n \supset ((q \supset s) \wedge (r \supset s))) \supset p_1 \supset \ldots \supset p_n \supset (q \vee r) \supset s$$

whence, by (H3) and (m.p.1),

$$\vdash_{\{\supset,\vee,\wedge\},I} ((p_1 \supset \ldots \supset p_n \supset q \supset s) \wedge (p_1 \supset \ldots \supset p_n \supset r \supset s)) \supset$$
$$p_1 \supset \ldots \supset p_n \supset (q \vee r) \supset s.$$

Now, by (H10),

$$\vdash_{\{\supset,\vee,\wedge\},I} (p_1 \supset \ldots \supset p_n \supset q \supset s) \supset (p_1 \supset \ldots \supset p_n \supset r \supset s) \supset$$
$$((p_1 \supset \ldots \supset p_n \supset q \supset s) \wedge (p_1 \supset \ldots \supset p_n \supset r \supset s)),$$

so, using (1), we obtain (X_n) as a theorem of $\langle\{\supset,\vee,\wedge\},I\rangle$–$H$.

Next, we derive (H4) in $\langle\{\supset,\vee,\wedge\},J\rangle$–$H'$. Substituting $(q \supset s) \wedge (r \supset s)$ for p in (X_1), we get

$$\vdash'_{\{\supset,\vee,\wedge\},J} (((q \supset s) \wedge (r \supset s)) \supset (q \supset s)) \supset (((q \supset s) \wedge (r \supset s)) \supset (r \supset s))$$
$$\supset ((q \supset s) \wedge (r \supset s)) \supset (q \vee r) \supset s,$$

so, using (H7), (H8) and (m.p.1), we may prove (H4).

This establishes the first assertion of the lemma. The second follows because the axioms by which H and H' differ use only the connectives $\supset$, $\vee$ and $\wedge$. ■

Henceforth, therefore, we shall drop the $'$ from H', $\vdash'$, etc. Note that in deriving (H4) in $\langle\{\supset,\vee,\wedge\},J\rangle$–$H'$, we used only (X_1), (H7), (H8) and (m.p.1). Thus, H is also axiomatized by the finite set J_1 obtained from J by replacing the $(X_n), n \in \omega$, by the single formula (X_1), while $\langle\{\supset,\vee,\wedge\},I\rangle$–$H$ is axiomatized by the corresponding subset of J_1. As we shall see, however, H does not have the separation theorem for J_1, nor for any axiomatizating subset of J in which only finitely many of the (X_n)'s are present.

Evidently, if C contains $\supset$ and either contains $\wedge$ or does not contain $\vee$, we also have $\langle C,I\rangle$–$H = \langle C,J\rangle$–H. In this case, by Theorem 3.2, C–L and $\langle C,J\rangle$–H are logically equivalent. We show that the same is true in the other cases.

Lemma 2.5. *$\langle\{\supset,\vee\},J\rangle$–$H$ and $\{\supset,\vee\}$–L are logically equivalent. Moreover, if C is a subset of $\{\supset,\&,\vee,\wedge,\bot\}$ that contains $\supset$ and $\vee$ but not $\wedge$, then $\langle C,J\rangle$–H and C–L are logically equivalent.*

Proof. We first show that the sequent

$$p_1 \supset \ldots \supset p_n \supset q \supset s,\;\; p_1 \supset \ldots \supset p_n \supset r \supset s,\;\; p_1,\ldots,p_n, q \vee r \to s,$$

which corresponds to the axiom (X_n), is derivable in $\{\supset,\vee\}$–L. Consider the following derivation in $\{\supset,\vee\}$–L. Each step is an application of the rule $(\supset\,\to)$.

$$\dfrac{\dfrac{p_{n-1} \to p_{n-1} \qquad \dfrac{p_n \to p_n \qquad \dfrac{q \to q \qquad s \to s}{q \supset s, q \to s}}{p_n \supset q \supset s, p_n, q \to s}}{\vdots}}{p_1 \supset \ldots \supset p_n \supset q \supset s,\;\; p_1,\ldots,p_n, q \to s}$$

Similarly we can derive

$$p_1 \supset \ldots \supset p_n \supset r \supset s,\;\; p_1,\ldots,p_n, r \to s.$$

Let Γ denote the sequence

$$p_1 \supset \ldots \supset p_n \supset q \supset s,\quad p_1 \supset \ldots \supset p_n \supset r \supset s,\quad p_1,\ldots,p_n.$$

By (weakening), we derive $\Gamma, q \to s$ and $\Gamma, r \to s$, and then, by $(\vee\,\to)$, also $\Gamma, q \vee r \to s$.

The sequents corresponding to the other axioms of $\langle\{\supset,\vee\},J\rangle$–$H$ are easily derivable in $\{\supset,\vee\}$–L. Now suppose that $\vdash_{\{\supset,\vee\},J} \alpha_1 \supset \ldots \supset \alpha_n \supset \gamma$. It is straightforward to prove, by induction on the length of a derivation of $\alpha_1 \supset \ldots \supset \alpha_n \supset \gamma$, that the sequent $\alpha_1,\ldots,\alpha_n \to \gamma$ is derivable in $\{\supset,\vee\}$–L.

Conversely, suppose $\alpha_1,\ldots,\alpha_n \to \gamma$ is derivable in $\{\supset,\vee\}$–L. To show that $\alpha_1 \supset \ldots \supset \alpha_n \supset \gamma$ is a theorem of $\langle\{\supset,\vee\},J\rangle$–$H$, we proceed by induction on the length of a derivation of $\alpha_1, \ldots, \alpha_n \to \gamma$. Suppose that in some such derivation, $\alpha_1, \ldots, \alpha_n \to \gamma$ occurs as the lower sequent of the rule $(\vee\ \to)$. Then

$$\alpha_1, \ldots, \alpha_n = \beta_1, \ldots, \beta_m, \zeta \vee \eta, \delta_1, \ldots, \delta_r,$$

where $m + r + 1 = n$, and

$$\frac{\beta_1, \ldots, \beta_m, \zeta, \delta_1, \ldots, \delta_r \to \gamma \qquad \beta_1, \ldots, \beta_m, \eta, \delta_1, \ldots, \delta_r \to \gamma}{\beta_1, \ldots, \beta_m, \zeta \vee \eta, \delta_1, \ldots, \delta_r \to \gamma}(\vee\ \to).$$

By the induction hypothesis,

$$\vdash_{\{\supset,\vee\},J} \beta_1 \supset \ldots \supset \beta_m \supset \zeta \supset \delta_1 \supset \ldots \supset \delta_r \supset \gamma$$

and

$$\vdash_{\{\supset,\vee\},J} \beta_1 \supset \ldots \supset \beta_m \supset \eta \supset \delta_1 \supset \ldots \supset \delta_r \supset \gamma.$$

Thus, using the axiom (X_m) of $\langle\{\supset, \vee\}, J\rangle$–$H$ and (m.p.1), we obtain

$$\vdash_{\{\supset,\vee\},J} \beta_1 \supset \ldots \supset \beta_m \supset (\zeta \vee \eta) \supset \delta_1 \supset \ldots \supset \delta_r \supset \gamma,$$

as required. If there exists a derivation in which $\alpha_1, \ldots, \alpha_n \to \gamma$ occurs as the lower sequent of any of the other inference rules (other than the cut rule), the same result follows easily. The rest of the lemma is proved similarly. ∎

In conjunction with Lemma 2.4, the above lemma gives us the following:

Corollary 2.6. *For each subset C of $\{\supset, \&, \vee, \wedge, \perp\}$ containing $\supset$, the systems $\langle C, J\rangle$–H and C–L are logically equivalent.*

Corollary 2.7. *The separation theorem holds for the axiomatization J of H, i.e., $\langle C, J\rangle$–$H = C$–H whenever C contains $\supset$.*

Proof. Let C be a subset of $\{\supset, \&, \vee, \wedge, \perp\}$ containing $\supset$ and suppose that a C–formula α is provable in H. By Corollary 2.6, the sequent $\to \alpha$

is derivable in L. In view of Corollary 2.2, this derivation can be assumed to belong to C–L. By Corollary 2.6 again, α is provable in $\langle C, J\rangle$–H. ∎

3. Algebraizability and Axiomatization of Fragments of H

In this section we investigate the algebraizability, in the sense of Blok and Pigozzi [5], of the Hilbert systems of Section 2.

In an algebra $\mathbf{A}$ of type $\mathcal{L}$, the interpretation of an operation symbol α of $\mathcal{L}$ shall be denoted $\alpha^{\mathbf{A}}$ (or just α, if $\mathbf{A}$ is understood). For consistency, we continue to refer to the terms of $\mathcal{L}$ as 'formulas'. For classes $\mathcal{M}$ of similar algebras, we make standard use (see, e.g., [8]) of the class operator symbols I, H, S, P, and $\mathrm{P_U}$ (for ultraproducts) and we denote the least quasivariety containing $\mathcal{M}$ by $\mathrm{Q}(\mathcal{M})$.

For a quasivariety $\mathcal{K}$ and an algebra $\mathbf{A}$ of the same type, the $\mathcal{K}$–*congruences* (or *relative congruences*, if $\mathcal{K}$ is understood) of $\mathbf{A}$ are the congruences θ of $\mathbf{A}$ for which $\mathbf{A}/\theta \in \mathcal{K}$. We use $\mathrm{Con}\,\mathbf{A}$ [resp. $\mathrm{Con}_{\mathcal{K}}\mathbf{A}$] to denote the set of all [resp. all relative] congruences of $\mathbf{A}$. When ordered by set inclusion, these become algebraic lattices $\mathbf{Con}\,\mathbf{A}$ and $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$. The least $\mathcal{K}$–congruence of $\mathbf{A}$ containing $X \subseteq A^2$ is denoted by $\Theta^{\mathbf{A}}_{\mathcal{K}}(X)$, or by $\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b)$ if $X = \{\langle a,b\rangle\}$.

Recall from [5] that a Hilbert system S over a language $\mathcal{L}$ is said to be *algebraizable* if there exist $m, r \in \omega$, a family $\mathbf{\Delta} = \{\Delta_j : j < m\}$ of binary formulas and families $\{\delta_t : t < r\}$ and $\{\varepsilon_t : t < r\}$ of unary formulas of $\mathcal{L}$ such that for any connective α (of rank n, say) and any formulas $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n, \varphi, \psi, \zeta$ of $\mathcal{L}$, the following conditions hold for $j < m$: [2]

(i) $\vdash_S \Delta_j(\varphi,\varphi)$

(ii) $\{\Delta_i(\varphi,\psi) : i < m\} \vdash_S \Delta_j(\psi,\varphi)$

(iii) $\{\Delta_i(\varphi,\psi) : i < m\} \cup \{\Delta_i(\psi,\zeta) : i < m\} \vdash_S \Delta_j(\varphi,\zeta)$

[2] Here $\Gamma \vdash_S \Sigma$ means $\Gamma \vdash_S \varphi$ for all $\varphi \in \Sigma$ and $\Gamma \dashv\vdash_S \Sigma$ abbreviates 'both $\Gamma \vdash_S \Sigma$ and $\Sigma \vdash_S \Gamma$'; a similar convention applies to equational consequence relations of classes of algebras.

(iv) $\{\Delta_i(\varphi_k,\psi_k) : i < m ; k = 1,\dots,n\} \vdash_S \Delta_j(\alpha(\varphi_1,\dots,\varphi_n),\alpha(\psi_1,\dots,\psi_n))$

(v) $\zeta \dashv\vdash_S \{\Delta_i(\delta_t(\zeta),\varepsilon_t(\zeta)) : i < m ; t < r\}$

In this case, there is a unique quasivariety $\mathcal{K}$ of algebras of type $\mathcal{L}$ in which, for any set $\Gamma \cup \{\varphi,\psi\}$ of formulas and for $l < r$, we have

$(*)$ $\Gamma \vdash_S \varphi$ iff $\{\delta_t(\zeta) \approx \varepsilon_t(\zeta) : \zeta \in \Gamma ; t < r\} \models_{\mathcal{K}} \delta_l(\varphi) \approx \varepsilon_l(\varphi)$,

$(**)$ $\varphi \approx \psi \mathrel{=\!\mid\models}_{\mathcal{K}} \{\delta_t(\Delta_i(\varphi,\psi)) \approx \varepsilon_t(\Delta_i(\varphi,\psi)) : i < m ; t < r\}$.

Given any axiomatization $\mathrm{Ax}\cup\mathrm{Ir}$ of S, where Ax is a set of axioms and Ir a set of inference rules with nonempty sets of premisses, the aforementioned quasivariety $\mathcal{K}$ is axiomatized by the identities

$$\delta_t(\varphi) \approx \varepsilon_t(\varphi),\ t < r ;\ \varphi \in \mathrm{Ax},$$
$$\delta_t(\Delta_j(x,x)) \approx \varepsilon_t(\Delta_j(x,x)),\ j < m ;\ t < r$$

together with the quasi-identities

$$\Big(\bigwedge_{\xi\in\Xi}\bigwedge_{t<r}\delta_t(\xi) \approx \varepsilon_t(\xi)\Big) \;\Rightarrow\; \delta_l(\varphi) \approx \varepsilon_l(\varphi),$$
$$l < r,\ \langle \Xi,\varphi\rangle \in \mathrm{Ir};$$
$$\Big(\bigwedge_{j<m}\bigwedge_{t<r}\delta_t(\Delta_j(x,y)) \approx \varepsilon_t(\Delta_j(x,y))\Big) \;\Rightarrow\; x \approx y$$

[5, Theorems 2.17, 4.7]. Following [5], we call $\mathcal{K}$ the *equivalent quasivariety semantics* of S. The formulas in $\mathbf{\Delta}$ and the equations $\delta_t \approx \varepsilon_t$, $t < r$, are called *equivalence formulas* and *defining equations* for S and $\mathcal{K}$. They are unique up to interderivability over S and $\mathcal{K}$, respectively. If, in addition, $p,q \vdash_S \Delta_j(p,q)$ for all $j < m$, we say that S has the *Gödel Rule*. In this case $\mathcal{K}$ satisfies $\Delta_i(x,x) \approx \Delta_j(y,y)$ for all i,j and is *relatively point regular* at the constant formula $\mathrm{T} = \Delta_i(x,x)$, i.e., for any $\mathbf{A} \in \mathcal{K}$ and any $\mathcal{K}$–congruences θ_1,θ_2 of $\mathbf{A}$, if $\mathrm{T}/\theta_1 = \mathrm{T}/\theta_2$ then $\theta_1 = \theta_2$. If S is an algebraizable Hilbert system whose equivalent quasivariety semantics $\mathcal{K}$ is a variety, we say that S is *strongly algebraizable* with *equivalent variety semantics* $\mathcal{K}$.

A quasivariety $\mathcal{K}$ satisfying $(*)$ (but not necessarily $(**)$) is called an *algebraic semantics* for S. Algebraic semantics for superimplicational fragments of H are given implicitly in [22, Thm. 8.1]. The next three lemmas sharpen this result.

Lemma 3.1. [25, Prop. 2] *$\{\supset\}$–H and $\{\supset, \&\}$–H are algebraizable Hilbert systems with the Gödel Rule, with defining equation $p \approx p \supset p$ and equivalence formulas $\mathbf{\Delta} = \{p \supset q, q \supset p\}$.*

Lemma 3.2. *$\{\supset, \wedge\}$–H is algebraizable with defining equation $p \approx p \supset p$ and equivalence formulas $\mathbf{\Delta} = \{p \supset q, q \supset p\}$.*

Proof. By Lemma 3.1, it is sufficient to show that

$$p \supset q,\ q \supset p,\ r \supset s,\ s \supset r \vdash_{\{\supset,\wedge\}} (p \wedge r) \supset (q \wedge s),\ (q \wedge s) \supset (p \wedge r).$$

Using (H3), (H7), (H8), (H9), (H10) and (m.p.1) we obtain

$$p \supset q,\ r \supset s \vdash_{\{\supset,\wedge\}} (p \wedge r) \supset (q \wedge s).$$

By symmetry of variables, the result follows. ∎

Lemma 3.3. *$\{\supset, \vee\}$–H is algebraizable with defining equation $p \approx p \supset p$ and equivalence formulas $\mathbf{\Delta} = \{p \supset q, q \supset p\}$.*

Proof. Again, by Lemma 3.1, we need only show that

$$p \supset q,\ q \supset p,\ r \supset s,\ s \supset r \vdash_{\{\supset,\vee\}} (p \vee r) \supset (q \vee s),\ (q \vee s) \supset (p \vee r).$$

Using (H3), (H5), (H6), (X_0) and (m.p.1) we obtain

$$p \supset q,\ r \supset s \vdash_{\{\supset,\vee\}} (p \vee r) \supset (q \vee s),$$

and the result follows by symmetry of the variables. ∎

Corollary 3.4. *For each set C of connectives of H that contains $\supset$, C–H (i.e., $\langle C, J\rangle$–H) is algebraizable with the Gödel Rule, with defining equation $p \approx p \supset p$ and equivalence formulas $\mathbf{\Delta} = \{p \supset q, q \supset p\}$.*

When discussing the equivalent quasivariety semantics of the above Hilbert systems, we replace $\supset, \&, \vee, \wedge, \perp$ by the more natural algebraic (respective) symbols $\dot{-}$, $\oplus$, $\sqcap$, $\sqcup$, 1. Let $C \subseteq \{\supset, \&, \vee, \wedge, \bot\}$. By C^* we mean the subset of $\{\dot{-}, \oplus, \sqcap, \sqcup, 1\}$ got by replacing each connective in C by its corresponding algebraic symbol. For each C–formula α we define a C^*–formula α^* as follows: first replace $\bot$ by 1. If α is a variable, let α^* be α. Suppose β^* and γ^* have been defined. If α is $\beta \supset \gamma$, let α^* be $\gamma^* \dot{-} \beta^*$; if α is $\beta \,\&\, \gamma$, let α^* be $\beta^* \oplus \gamma^*$; if α is $\beta \vee \gamma$, let α^* be $\gamma^* \sqcap \beta^*$; if α is $\beta \wedge \gamma$, let α^* be $\gamma^* \sqcup \beta^*$. Our convention for $\supset$ of omitting parentheses by association to the right reverses for $\dot{-}$.

In this formalism, the equivalent quasivariety semantics of H, denoted $\mathcal{H}$, is a class of algebras of type $\langle 2,2,2,2,0\rangle$, with fundamental operation symbols $\dot{-}$, $\oplus$, $\sqcap$, $\sqcup$, 1. For a subset C of the connectives containing $\supset$, the equivalent quasivariety semantics of C–H, denoted $\mathcal{H}_{\mathcal{C}^*}$, is the class of subalgebras of the C^*–reducts of members of $\mathcal{H}$ [5, Cor. 2.12]. By the Gödel Rule, a constant is defined over $\mathcal{H}_{\mathcal{C}^*}$ by $0 = x \dot{-} x$.

For convenience, *we shall assume throughout that* 0 *is in the language of each* $\mathcal{H}_{\mathcal{C}^*}$. Also, C^* *shall* **always** *denote a subset of* $\{\dot{-}, \oplus, \sqcap, \sqcup, 1\}$ *containing* $\dot{-}$. (Similarly, C *shall* **always** *contain* $\supset$.)

By $(*)$, for all sets Γ of C–formulas and all C–formulas α,

$$\Gamma \vdash_C \alpha \quad \text{if and only if} \quad \{\beta^* \approx 0 : \beta \in \Gamma\} \models_{\mathcal{H}_{\mathcal{C}^*}} \alpha^* \approx 0.$$

An explicit axiomatization of each $\mathcal{H}_{\mathcal{C}^*}$ follows from the separation theorem and algebraizability. By [25, Propositions 1,2], $\mathcal{H}_{\{\dot{-},\oplus\}}$ is axiomatized more economically by:

(L1) $x \dot{-} 0 \approx x$

(L2) $0 \dot{-} x \approx 0$

(L3) $x \dot{-} y \dot{-} (z \dot{-} y) \dot{-} (x \dot{-} z) \approx 0$

(L4) $x \dot{-} y \approx 0$ and $y \dot{-} x \approx 0 \;\Rightarrow\; x \approx y$

(L5) $x \dot{-} (y \oplus z) \approx x \dot{-} z \dot{-} y$.

We noted after Lemma 2.4 that H is axiomatized by (H1)–(H12), excluding (H4), together with (X_1), (m.p.1) and (m.p.2). By algebraizability, an axiomatization of $\mathcal{H}$ is therefore given by (L1)–(L5) and the following:

(L6) $x \dot{-} 1 \approx 0$

(L7) $(x \sqcap y) \dot{-} x \approx 0$

(L8) $(x \sqcap y) \dot{-} y \approx 0$

(L9) $x \dot{-} (y \sqcap z) \dot{-} w \dot{-} (x \dot{-} y \dot{-} w) \dot{-} (x \dot{-} z \dot{-} w) \approx 0$

(L10) $x \dot{-} (x \sqcup y) \approx 0$

(L11) $y \dot{-} (x \sqcup y) \approx 0$

(L12) $(x \sqcup y) \dot{-} z \approx (x \dot{-} z) \sqcup (y \dot{-} z)$

(L13) $(x \sqcup y) \dot{-} y \dot{-} x \approx 0$

The class $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ is axiomatized by (L1)–(L4), (L7), (L8) and the identities

$$(Y_n)\ \ x \dot{-} (y \sqcap z) \dot{-} w_1 \dot{-} \ \ldots\ \dot{-} w_n \dot{-} (x \dot{-} y \dot{-} w_1 \dot{-} \ \ldots\ \dot{-} w_n) \\ \dot{-} (x \dot{-} z \dot{-} w_1 \dot{-} \ \ldots\ \dot{-} w_n) \approx 0$$

(corresponding to (X_n)) for each natural number $n \geq 1$. If we set $w_1 = 0$ in (Y_1) (i.e., (L9)), we get the algebraic equivalent, (Y_0), of (X_0). In fact, we can derive (Y_n) from (Y_{n+1}) and (L1) in general. The class $\mathcal{H}_{\{\dot{-}, \sqcap, \infty\}}$ is axiomatized by the axioms of $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ together with (L6). In all other cases, $\mathcal{H}_{\mathcal{C}^*}$ is axiomatized by (L1)–(L4) and those identities among (L5)–(L13) that use only the connectives in C^*. In fact, the proof of Lemma 2.4 shows that when C^* contains both $\sqcap$ and $\sqcup$, the identities $(Y_n), n \geq 2$, are redundant in our axiomatization of $\mathcal{H}_{C^*}$, whereas if C^* contains both $\sqcap$ and $\oplus$, the (Y_n)'s ($n \geq 2$) may be derived from (L5) and (Y_1) in the following way:

$$\begin{aligned} & x \dot{-} (y \sqcap z) \dot{-} w_1 \dot{-} \ \ldots\ \dot{-} w_n \dot{-} (x \dot{-} y \dot{-} w_1 \dot{-} \ \ldots\ \dot{-} w_n) \\ & \quad \dot{-} (x \dot{-} z \dot{-} w_1 \dot{-} \ \ldots\ \dot{-} w_n) \\ \approx\ & x \dot{-} (y \sqcap z) \dot{-} (w_n \oplus \ldots \oplus w_1) \dot{-} (x \dot{-} y \dot{-} (w_n \oplus \ldots \oplus w_1)) \\ & \quad \dot{-} (x \dot{-} z \dot{-} (w_n \oplus \ldots \oplus w_1)) \ \approx\ 0. \end{aligned}$$

It follows that $\{\supset, \&, \vee\}$–H is axiomatized by (H1), (H3), (H5), (H6), (H11), (H12), (X_1), (m.p.1) and (m.p.2).

Observe that for each C^*, the members of $\mathcal{H}_{C^*}$ admit a definable partial order $\leq$ with least element 0, where $a \leq b$ if and only if $a \dot{-} b = 0$. If $\oplus \in C^*$ then the members of $\mathcal{H}_{C^*}$ are ordered monoids with respect to $\leq$ and $\oplus$,[3] with identity 0, and they are left residuated in the sense that $a \dot{-} b$ is always the least element c for which $a \leq c \oplus b$ (see [22], [33]). It follows that $\dot{-}$ preserves and reverses order in its first and second arguments, respectively, and that we always have $a \dot{-} b \leq a$. On the other hand, if $\sqcap$ [resp. $\sqcup$] $\in C^*$, then $\leq$ is a semilattice order whose meet [resp. join] operation is $\sqcap$ [resp. $\sqcup$]. Of these facts, the former makes essential use of the axiom (Y_0).

By Corollary 2.6, $\mathcal{H}$ [resp. $\mathcal{H}_{C^*}$] coincides with the class of *full–BCC–algebras* [resp. 'C^**–BCC–algebras*'] of [22]. Several embedding theorems for these algebras appear in [22], exemplifying a general correspondence between fragments of an algebraizable Hilbert system S and subreduct classes of the equivalent quasivariety semantics of S [5, Cor. 2.12]. Up to isomorphism, there is a unique two-element algebra in $\mathcal{H}_{C^*}$, which we denote by $\mathbf{2}_{C^*}$.

We present some examples that distinguish the strengths of the identities (Y_n), $n \in \omega$, establishing that no finite subsequence of the (Y_n)'s may replace the whole sequence in our axiomatization of $\mathcal{H}_{\{\dot{-}, \sqcap\}}$. The same applies to $\mathcal{H}_{\{\dot{-}, \sqcap, 1\}}$, since all algebras constructed will have top elements. Then the Compactness Theorem implies the next result, whose corollary follows immediately by algebraizability.

Theorem 3.5. *The quasivarieties* $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ *and* $\mathcal{H}_{\{\dot{-}, \sqcap, \infty\}}$ *are not finitely axiomatizable.*

Corollary 3.6. *The* $\{\supset, \vee\}$*– and* $\{\supset, \vee, \perp\}$*–fragments of* H *are not finitely axiomatizable.*

First we define an algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$ of type $\langle 2, 2, 0\rangle$ that satisfies (L1)–(L4), (L7), (L8) and (Y_0) but does not satisfy (Y_1). Let $\langle A; \leq\rangle$ be the partially ordered set depicted in Figure 1. Let $\sqcap$ be the

[3] We imply by this that $\oplus$ preserves order in both of its arguments.

meet operation on A determined by $\leq$ and let $\dot{-}$ be defined on A by the table in Figure 1. One checks that $\mathbf{A}$ satisfies (L1)–(L4), (L7), (L8) and (Y_0), but $\mathbf{A}$ does not satisfy (Y_1), since

$$\begin{aligned} &1 \mathbin{\dot{-}} (b \sqcap c) \mathbin{\dot{-}} a \mathbin{\dot{-}} (1 \mathbin{\dot{-}} b \mathbin{\dot{-}} a) \mathbin{\dot{-}} (1 \mathbin{\dot{-}} c \mathbin{\dot{-}} a) \\ &\qquad = 1 \mathbin{\dot{-}} a \mathbin{\dot{-}} (c \mathbin{\dot{-}} a) \mathbin{\dot{-}} (b \mathbin{\dot{-}} a) \\ &\qquad = d \mathbin{\dot{-}} 0 \mathbin{\dot{-}} 0 \\ &\qquad = d \neq 0. \end{aligned}$$

Figure 1.

Next, for any fixed $n \geq 1$, we define an algebra $\mathbf{A}$ over $\{\dot{-}, \sqcap, 0\}$ that satisfies (L1)–(L4), (L7), (L8) and (Y_n) (hence also (Y_m) for each $m < n$) but does not satisfy (Y_{n+1}). Consider the structure $\langle B; \dot{-}', d; \leq' \rangle$ where $\langle B; \leq' \rangle$ is the four-element poset depicted in Figure 2 and $\mathbf{B} = \langle B; \dot{-}', d \rangle$ the algebra of type $\langle 2, 0 \rangle$ defined by $s \mathbin{\dot{-}'} t = d$ if $s \leq' t$; $s \mathbin{\dot{-}'} d = s$ $(s, t \in B)$; $a \mathbin{\dot{-}'} b = c \mathbin{\dot{-}'} b = c$ and $a \mathbin{\dot{-}'} c = b \mathbin{\dot{-}'} c = b$. Then $\mathbf{B}$ is isomorphic to $\mathbf{2}_{\{\dot{-}\}} \times \mathbf{2}_{\{\dot{-}\}} \in \mathcal{H}_{\{\dot{-}\}}$.

Figure 2.

Now we define $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0 \rangle$ as follows. Let N be the set of positive natural numbers and A the following union of five mutually disjoint sets:

$$A = \{a_i : i \in N\} \cup \{b_i : i \in N\} \cup \{c_i : i \in N\} \cup \{d_i : i \in N\} \cup \{e, f, 0\}.$$

Let $\leq$ be the partial order on A depicted in Figure 3 and $\sqcap$ the meet semilattice operation on A induced by $\leq$. For all $u, v \in A$, let $u \dot{-} 0 = u$ and if $u \leq v$, define $u \dot{-} v = 0$. For all $i, j \in N$ and all $s, t \in \{a, b, c, d\}$, define

$$s_i \dot{-} t_j = \begin{cases} (s \dot{-}' t)_{i(n+1)} & \text{if } i < j \\ (s \dot{-}' t)_{(i+1)(n+1)} & \text{if } i \geq j \text{ and } s \not\leq' t \\ 0 & \text{if } i \geq j \text{ and } s \leq' t, \end{cases}$$

$$a_{2n} \dot{-} f = e \text{ and } s_i \dot{-} f = s_{i+1} \text{ for } s_i \neq a_{2n},$$

$$e \dot{-} a_{2n+1} = d_{2n(n+1)},$$

$$e \dot{-} u = a_{2n+1} \dot{-} u \text{ for } u \neq a_{2n+1}, \text{ and}$$

$$u \dot{-} e = u \dot{-} a_{2n+1} \text{ for } u \neq e.$$

One checks that **A** satisfies (L1)–(L4), (L7), (L8) and (Y_n), but **A** does not satisfy (Y_{n+1}) because

$$\begin{aligned} (a_1 \dot{-} (b_2 \sqcap c_2) \dot{-} nf \dot{-} a_{2n+1}) \dot{-} (a_1 \dot{-} b_2 \dot{-} nf \dot{-} a_{2n+1}) \\ \dot{-} (a_1 \dot{-} c_2 \dot{-} nf \dot{-} a_{2n+1}) \\ = (e \dot{-} a_{2n+1}) \dot{-} (c_{2n+1} \dot{-} a_{2n+1}) \dot{-} (b_{2n+1} \dot{-} a_{2n+1}) \\ = d_{2n(n+1)} \dot{-} 0 \dot{-} 0 = d_{2n(n+1)} \neq 0. \end{aligned}$$

Figure 3.

Corollary 3.6 contrasts with results of [22] showing that the finite set I (of Section 2) together with an exchange-like rule axiomatizes (with separation) a Hilbert system H_{BCK}, logically equivalent to L_{BCK}. In particular, all superimplicational fragments of H_{BCK} are finitely axiomatized.

Let S and $\mathbf{A}$ be a Hilbert system and an algebra over the same language $\mathcal{L}$ and let $F \subseteq A$. We call F an *S–filter* of $\mathbf{A}$ provided that whenever $\Gamma \vdash_S \varphi$, the following is true: for every assignment $\vec{a}$ of elements of A to the variables of $\Gamma \cup \{\varphi\}$, if $\psi^{\mathbf{A}}(\vec{a}) \in F$ for all $\psi \in \Gamma$ then $\varphi^{\mathbf{A}}(\vec{a}) \in F$. It clearly suffices to check this closure property for the (axioms and) inference rules $\Gamma \vdash_S \varphi$ in some given axiomatization of S. The set $\mathrm{Fi}^S\mathbf{A}$ of all S–filters of $\mathbf{A}$ becomes an algebraic lattice $\mathbf{Fi}^S\mathbf{A}$ when ordered by set inclusion. Let $X \subseteq A$. The least S–filter of $\mathbf{A}$ containing X shall be denoted by $\langle X \rangle_{\mathbf{A}}$; if $X = \{a\}$ for some $a \in A$, this is called a *principal S–filter* and is denoted by $\langle a \rangle_{\mathbf{A}}$.

We say that $\Phi \in \mathrm{Con}\mathbf{A}$ is *compatible* with $F \subseteq A$ provided that $b \in F$ whenever both $a \in F$ and $\langle a, b\rangle \in \Phi$. Let $\Omega_{\mathbf{A}}F$ denote the largest congruence on $\mathbf{A}$ that is compatible with F. The map $\Omega_{\mathbf{A}} : \mathrm{Fi}^S\mathbf{A} \to \mathrm{Con}\mathbf{A}$ thus defined is called the *Leibniz operator* of $\mathbf{A}$. If S is algebraizable with equivalent quasivariety semantics $\mathcal{K}$ then $\Omega_{\mathbf{A}}$ is a lattice isomorphism from $\mathbf{Fi}^S\mathbf{A}$ onto $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ [5, Thm. 5.1] and the actions of $\Omega_{\mathbf{A}}$ and its inverse are determined by the defining equations and equivalence formulas for S and $\mathcal{K}$. By this result and Corollary 3.4, for any $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$, the maps

$$F \mapsto \Omega_{\mathbf{A}}F = \{\langle a, b\rangle \in A^2 : a \dot{-} b,\ b \dot{-} a \in F\} \quad (F \in \mathrm{Fi}^S\mathbf{A});$$

$$\theta \mapsto 0^{\mathbf{A}}/\theta = \{a \in A : \langle a, 0^{\mathbf{A}}\rangle \in \theta\} \quad (\theta \in \mathrm{Con}_{\mathcal{K}}\mathbf{A})$$

are mutually inverse isomorphisms between $\mathbf{Fi}^S\mathbf{A}$ and $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$.

3.1. Nonalgebraizable Subsystems.

It is natural to ask whether the $\langle C, I\rangle$–subsystems of H are algebraizable. If C contains $\supset$ and either contains $\wedge$ or does not contain $\vee$ then $\langle C, I\rangle$–H = $\langle C, J\rangle$–H is the algebraizable fragment C–H of H. We show that $\langle C, I\rangle$–H is *not* algebraizable in any of the remaining cases.

Proposition 3.7. *Let C be a subset of $\{\supset, \&, \vee, \wedge, \bot\}$ containing $\supset$ and $\vee$ but not $\wedge$. Then $\langle C, I\rangle$–H is not algebraizable.*

Proof. Consider $\langle\{\supset, \vee\}, I\rangle$–$H$. Let $\langle A; \leq\rangle$ be the four-element poset depicted in Figure 4. For $s, t \in A$, set $s \dot{-} t = 0$ if $s \leq t$ and $s \dot{-} 0 = s$; $1 \dot{-} a = b \dot{-} a = b$ and $1 \dot{-} b = a$. Also set $1 \sqcap 1 = 1$ and $s \sqcap t = 0$ otherwise. Then $\{0\}$, $\{0, a\}$ and A are $\langle\{\supset, \vee\}, I\rangle$–$H$–filters of $\mathbf{A} := \langle A;\ \dot{-}\ ,\ \sqcap\ , 0\rangle$, but $\mathbf{A}$ is a simple algebra, i.e., $|\,\mathrm{Con}\,\mathbf{A}\,| = 2$. The Leibniz operator $\Omega_{\mathbf{A}}$ is therefore not injective, so $\langle\{\supset, \vee\}, I\rangle$–$H$ is not algebraizable. The same example may be used to show nonalgebraizability of $\langle\{\supset, \vee, \bot\}, I\rangle$–$H$.

Figure 4. **Figure 5.**

Consider $\langle\{\supset, \&, \vee\}, I\rangle$–$H$. Let $\langle A; \leq\rangle$ be the five-element set poset depicted in Figure 5. For $s, t \in A$, set $0 \oplus s = s = s \oplus 0$, $c \oplus c = c$, $a \oplus c = a$, $b \oplus c = b$ and $s \oplus t = 1$ otherwise. Also, set $s \dot{-} t = 0$ if $s \leq t$; $s \dot{-} 0 = s$, $1 \dot{-} a = 1 \dot{-} b = a \dot{-} b = b \dot{-} a = c$, $1 \dot{-} c = 1$, $a \dot{-} c = a$ and $b \dot{-} c = b$. Let $\sqcap$ be the meet operation on A determined by $\leq$. Then $\{0\}$, $\{0, c\}$ and A are $\langle\{\supset, \&, \vee\}, I\rangle$–$H$–filters of $\mathbf{A} := \langle A; \dot{-}, \oplus, \sqcap, 0\rangle$ but again, $\mathbf{A}$ is simple. It follows, just as above, that neither $\langle\{\supset, \&, \vee\}, I\rangle$–$H$ nor $\langle\{\supset, \&, \vee, \bot\}, I\rangle$–$H$ is algebraizable. ∎

This confirms, e.g., that $\{\supset, \vee\}$–H is a proper extension of $\langle\{\supset, \vee\}, I\rangle$–$H$ and that $\langle\{\supset, \vee\}, I\rangle$–$H$ is not logically equivalent to $\{\supset, \vee\}$–L. For an algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that $\langle A; \dot{-}, 0\rangle \in \mathcal{H}_{\{\dot{-}\}}$ and $\mathbf{A}$ satisfies (L7), (L8) (which correspond to (H5), (H6)), the operation $\sqcap^{\mathbf{A}}$ may bear little relation to the partial order '$a \leq b$ iff $a \dot{-} b = 0$' on A. The latter is not generally a meet semilattice order. Even when it is, the meet operation that it induces on A need not be $\sqcap^{\mathbf{A}}$: see the first construction of the previous proof.

4. Algebraic Properties of the Quasivarieties $\mathcal{H}_{\mathcal{C}^*}$

In [22, §9, Remark 7], Ono and Komori asked which of the classes $\mathcal{H}_{\mathcal{C}^*}$ are varieties. (We continue to assume that $\dot{-} \in C^*$.) In other words, which superimplicational fragments of H are *strongly* algebraizable? The next proposition answers this question. The authors thank Professor H. Ono for pointing out that this proposition is an unpublished result of P.M. Idziak. We include a proof the sake of completeness and for future reference.

Proposition 4.1. (Idziak) *The quasivariety $\mathcal{H}_{\mathcal{C}^*}$ is a variety if and only if C^* contains $\oplus$ and at least one of $\sqcap$, $\sqcup$. That is, C–H is strongly algebraizable if and only if C contains & and at least one of $\vee, \wedge$.*

Proof. Define an algebra $\mathbf{A} = \langle A; \dot{-}, \sqcap, \sqcup, 1, 0\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$ as follows. Let $A = \{0, a, b, 1\}$ be a four-element set and $\leq$ the linear order of A given by $0 < a < b < 1$. For $s, t \in A$, set $s \dot{-} t = 0$ if $s \leq t$; $s \dot{-} 0 = s$, $1 \dot{-} a = 1$, $1 \dot{-} b = a$ and $b \dot{-} a = b$. Let $\sqcap$ and $\sqcup$ be, respectively, the meet and join operations on A determined by the order $\leq$. Then $\mathbf{A} \in \mathcal{H}_{\{\dot{-}, \sqcap, \sqcup, \infty\}}$.[4] The union Θ of $\{\langle 0, a\rangle, \langle a, 0\rangle\}$ and $\{\langle s, s\rangle : s \in A\}$ is a congruence on $\mathbf{A}$, but $\mathbf{A}/\Theta$ violates (L4). Thus, $\mathcal{H}_{\{\dot{-}, \sqcap, \sqcup, \infty\}}$ is not a variety. Considering reducts of the same algebra, we infer that $\mathcal{H}_{\mathcal{C}^*}$ is not a variety whenever $\{\dot{-}\} \subseteq C^* \subseteq \{\dot{-}, \sqcap, \sqcup, 1\}$. (The result for $\{\supset\}$ was observed in [20].)

The quasivariety of 'pocrims' (also known as 'BCK–algebras with operation (S)') is the class of all members of $\mathcal{H}_{\{\dot{-}, \oplus\}}$ whose monoid operations $\oplus$ are commutative. Higgs exhibited in [17] a pocrim that has a homomorphic image in which (L4) fails, whence $\mathcal{H}_{\{\dot{-}, \oplus\}}$ is not a variety.

Now suppose that C^* contains $\dot{-}$, $\oplus$ and $\sqcap$. Since $\mathcal{H}_{\mathcal{C}^*}$ is axiomatized by identities and the single quasi-identity (L4), it suffices to check that any homomorphic image $\mathbf{B}$ of an algebra in $\mathcal{H}_{\mathcal{C}^*}$ satisfies (L4). We have noted that any member of $\mathcal{H}_{\mathcal{C}^*}$ has a reduct that is (elementarily) equivalent to a left residuated meet-semilattice ordered monoid whose definable semilattice and residuation operations are the fundamental operations $\sqcap$ and $\dot{-}$. It follows that $\mathcal{H}_{\mathcal{C}^*}$ (and hence also $\mathbf{B}$) satisfies the identity

$$x \sqcap ((x \dot{-} y) \oplus y) \approx x. \qquad (2)$$

[4] The $\langle \dot{-}, \sqcap, \sqcup, 0\rangle$–reduct of $\mathbf{A}$ is isomorphic to the corresponding subreduct on $\{0, 1, \omega, \omega + 1\}$ of the ordinal ω^2 considered as a *right* residuated well ordered monoid with ordinal addition as $\oplus$.

Since $\langle B; \oplus, 0\rangle$ is a monoid, we deduce from (2) that

$$\mathbf{B} \models x \dot{-} y \approx 0 \Rightarrow x \sqcap y \approx x.$$

Since $\sqcap^{\mathbf{B}}$ is commutative, it follows by symmetry of the variables that $\mathbf{B}$ satisfies (L4). (Thus, (2) may replace (L4) in the axiomatization of $\mathcal{H}_{\mathcal{C}^*}$.) A similar consideration of the identity

$$x \sqcup ((x \dot{-} y) \oplus y) \approx (x \dot{-} y) \oplus y.$$

establishes the result when C^* contains $\dot{-}$, $\oplus$ and $\sqcup$. ∎

The above result contrasts with findings of Idziak [18] (also see [31], [17]) essentially concerning the Hilbert system H_{BCK}: all superimplicational C–fragments of H_{BCK} are algebraizable; the strongly algebraizable ones are just those where C contains at least one of $\vee, \wedge$.

Observe that for $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$, a nonempty subset F of A is a C–H–filter of $\mathbf{A}$ if and only if the following are true for all $a, b, c \in A$: $a \in F$ whenever b, $a \dot{-} b \in F$; and $a \dot{-} c \in F$ whenever b, $a \dot{-} b \dot{-} c \in F$. Since this characterization is independent of C, i.e., the C–H–filters of $\mathbf{A}$ coincide with the $\{\supset\}$–H–filters of the $\langle \dot{-} , 0\rangle$–reduct of $\mathbf{A}$, we shall speak simply of the *filters* of $\mathbf{A}$.

A quasivariety $\mathcal{K}$ is *relatively congruence distributive* if the lattice $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ is distributive for all algebras $\mathbf{A} \in \mathcal{K}$.

Proposition 4.2. *Each of the quasivarieties $\mathcal{H}_{\mathcal{C}^*}$ is relatively congruence distributive. Thus, when C^* contains $\oplus$ and at least one of $\sqcap$, $\sqcup$, the variety $\mathcal{H}_{\mathcal{C}^*}$ is congruence distributive.*

Proof. For $C^* = \{ \dot{-} \}$, the relative distributivity of $\mathcal{H}_{\mathcal{C}^*}$ was proved in [25, Prop. 13]; the remaining cases follow from the isomorphism between relative congruences and filters, since the latter don't depend on C^*. ∎

Proposition 4.3. *If C^* contains $\{\, \dot{-}\,, \oplus, \sqcap\,\}$ then the variety $\mathcal{H}_{\mathcal{C}^*}$ is arithmetical. If C^* contains $\{\, \dot{-}\,, \oplus, \sqcup\,\}$ then the variety $\mathcal{H}_{\mathcal{C}^*}$ is congruence 4–permutable.*

Proof. Let C^* contain $\{\, \dot{-}\,, \oplus, \sqcap\,\}$. For the formula

$$\alpha(x, y, z) = ((z \dot{-} y) \oplus x) \sqcap ((x \dot{-} y) \oplus z),$$

$\mathcal{H}_{\mathcal{C}^*}$ satisfies $\alpha(x, y, y) \approx x \approx \alpha(y, y, x)$, and is therefore congruence permutable. Since $\mathcal{H}_{\mathcal{C}^*}$ is also congruence distributive, it is arithmetical. Now let C^* contain $\{\, \dot{-}\,, \oplus, \sqcup\,\}$. If we define

$$\beta_1(x, y, z) = (z \dot{-} y) \oplus x,$$

$$\beta_2(x, y, z) = ((y \dot{-} z) \oplus z) \sqcup ((y \dot{-} x) \oplus x),$$

$$\beta_3(x, y, z) = (x \dot{-} y) \oplus z,$$

then $\mathcal{H}_{\mathcal{C}^*}$ satisfies $x \approx \beta_1(x, y, y)$, $\beta_i(x, x, y) \approx \beta_{i+1}(x, y, y)$ $(i = 1, 2)$, and $\beta_3(x, x, y) \approx y$, the standard criterion for 4–permutability (see [16]). ∎

Every subvariety of $\mathcal{H}_{\{\,\dot{-}\,\}}$ is congruence 3–permutable [25, Cor. 16]. When $\oplus \in C^*$, every locally finite[5] variety generated by members of $\mathcal{H}_{\mathcal{C}^*}$ is a subvariety of $\mathcal{H}_{\mathcal{C}^*}$ [25, Prop. 14] and therefore congruence distributive; such a variety is also congruence 3–permutable [28, Prop. 4.12].

The following internal characterization of filters in $\mathcal{H}_{\mathcal{C}^*}$ will be useful:

Lemma 4.4. *Let $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$ and let F be a nonempty subset of A. The following conditions are equivalent:*

(i) *F is a filter of A.*

(ii) *For all $a, b, c, d \in A$, if $a \dot{-} b, b \in F$ then $a \in F$; and if $d \in F$, then $c \dot{-} (c \dot{-} d) \in F$.*

(iii) *For all $a, b, c \in A$, if $a, b \in F$ then $c \dot{-} (c \dot{-} a \dot{-} b) \in F$.*

[5] In fact it is sufficient here that every 2-generated algebra in the variety be finite.

Moreover, for any filter F of A and $n \geq 1$, if $a_1, \ldots, a_n \in F$ and $c \in A$ then $c \dot{-} (c \dot{-} a_1 \dot{-} \ldots \dot{-} a_n) \in F$.

Proof. The equivalence of (i) and (ii) is proved in [25]. Assume (ii) and let $a, b \in F$. Then $c \dot{-} (c \dot{-} a) \in F$ and $c \dot{-} a \dot{-} (c \dot{-} a \dot{-} b) \in F$. Now, by (L3),

$$c \dot{-} (c \dot{-} a \dot{-} b) \dot{-} (c \dot{-} a \dot{-} (c \dot{-} a \dot{-} b)) \dot{-} (c \dot{-} (c \dot{-} a)) = 0.$$

Trivially, $0 \in F$, so $c \dot{-} (c \dot{-} a \dot{-} b) \in F$. Conversely, assume (iii) and let $a \dot{-} b$, $b \in F$. Then $a = a \dot{-} 0 = a \dot{-} (a \dot{-} b \dot{-} (a \dot{-} b)) \in F$. Also, if $d \in F$, then $c \dot{-} (c \dot{-} d) = c \dot{-} (c \dot{-} d \dot{-} 0) \in F$ since $0 \in F$, so (ii) holds. The remaining statement of the lemma follows by a simple inductive proof. ■

Let $x_0, x_1, x_2, \ldots$ be a fixed denumerable sequence of (distinct) variables. We define sets of formulas T_i as follows: $T_0 = \{x_0, 0\}$ and for each $n \in \omega$,

$$T_{n+1} = \{x_i \dot{-} (x_i \dot{-} \psi \dot{-} \varphi) : i \leq n+1 \text{ and } \psi, \varphi \in T_n\}.$$

Finally, $T = \bigcup\{T_n : n \in \omega\}$. Using Lemma 4.4,(iii), we obtain:

Corollary 4.5. *For $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$ and $a \in A$,*

$$\langle a \rangle_{\mathbf{A}} = \{\tau^{\mathbf{A}}(a, a_1, \ldots, a_n) : \tau(x_0, x_1, \ldots, x_n) \in T \text{ and } a_1, \ldots a_n \in A\}.$$

5. Relative Congruence Extensibility and Deduction Theorems

Recall that a quasivariety $\mathcal{K}$ has the *relative congruence extension property* (RCEP) if for every relative congruence θ of a subalgebra $\mathbf{B}$ of any $\mathbf{A} \in \mathcal{K}$, there is a relative congruence θ' of $\mathbf{A}$ with $\theta' \cap (B \times B) = \theta$. (We drop the qualification 'relative' and speak of the CEP if $\mathcal{K}$ is a variety.) When $\mathcal{K}$ is the equivalent quasivariety semantics of an algebraizable Hilbert system

S, the RCEP clearly amounts to the *filter extension property* (FEP), i.e., the requirement that every S–filter of a subalgebra $\mathbf{B}$ of any $\mathbf{A} \in \mathcal{K}$ be the intersection with B of some S–filter of $\mathbf{A}$. It is also equivalent [4] to the condition that S possess a *local deduction detachment theorem* (LDDT). This last condition is that there is a family $\mathcal{E} = \{\Sigma_i(p,q) : i \in Y\}$ of finite sets $\Sigma_i = \Sigma_i(p,q) = \{\zeta^j(p,q) : j \leq n_i\}$ of binary formulas ζ^j of S such that for any set $\Gamma \cup \{\alpha, \beta\}$ of formulas of S,

$$\Gamma, \alpha \vdash_S \beta \quad \text{if and only if} \quad \Gamma \vdash_S \Sigma_i(\alpha, \beta) \quad \text{for some } i \in Y.$$

In this case, $\mathcal{E}$ is called a *local deduction detachment system* for S. If in addition, we may choose $|\,Y\,| = 1$, say $\mathcal{E} = \{\Sigma\}$, we call Σ a *deduction detachment set* for S and say that S has a *deduction detachment theorem.*

When C^* is $\{\,\dot{-}\,\}$ or $\{\,\dot{-}\,, \oplus\}$, the quasivariety $\mathcal{H}_{C^*}$ lacks the RCEP. More strongly, for such C^*, given any proper filter F of *any* $\mathbf{B} \in \mathcal{H}_{C^*}$, where $|\,F\,| > 1$, there exists $\mathbf{A} \in \mathcal{H}_{C^*}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ but F is not the intersection with B of any filter of $\mathbf{A}$.[6] The proof of this given in [25, Prop. 6] shows that the ordered set underlying $\mathbf{A}$ may be chosen to consist of that of $\mathbf{B}$ together with a three-element chain exceeding all elements of B. This last fact makes it effortless to extend the argument to cases where $\{\,\dot{-}\,\} \subseteq C^* \subseteq \{\,\dot{-}\,, \oplus, \sqcap\,, \sqcup\,\}$ but this approach fails when 1 is included.

On the other hand, for any C^* containing 1, consider the $C^* \setminus \{1\}$–reduct $\mathbf{B}$ of an algebra $\mathbf{E} \in \mathcal{H}_{C^*}$. Regardless of the value of C^*, there is, up to isomorphism, a unique extension of $\mathbf{B}$ by a new top element f such that $f \,\dot{-}\, s = f$ whenever $s \in B$. Enriching this extension by the designated constant f, we have an algebra $\mathbf{B}_1 \in \mathcal{H}_{C^*}$ whose proper filters coincide with the filters of $\mathbf{B}$ (or of $\mathbf{E}$). Choosing $\mathbf{E}$ to have a proper filter F with at least two elements, and using the previously mentioned construction, we obtain a superalgebra $\mathbf{A}$ of $\mathbf{B}$ such that $\mathbf{B}$, F and $\mathbf{A}$ witness failure of the RCEP for $\mathcal{H}_{C^* \setminus \{1\}}$. Then $\mathbf{B}_1$, F and $\mathbf{A}_1$ witness failure of the RCEP in $\mathcal{H}_{C^*}$. We conclude:

[6] One can always choose $\mathbf{B}$ and F as described: if $\mathbf{D} \in \mathcal{H}_{C^*}$ is nontrivial, choose $\mathbf{B} = \mathbf{D} \times \mathbf{D}$ and $F = D \times \{0\}$.

Proposition 5.1. *No superimplicational fragment of H possesses a local deduction detachment theorem.*

In contrast, every superimplicational fragment of H_{BCK} has an LDDT. (This follows easily from [4, Example 2.1].) We now investigate axiomatic extensions of C–H that have an LDDT (for various C containing $\supset$).

A subquasivariety $\mathcal{K}'$ of a quasivariety $\mathcal{K}$ is called a *relative subvariety* of $\mathcal{K}$ if $\mathcal{K}' = \mathcal{K} \cap \mathrm{HSP}(\mathcal{M})$ for some subclass $\mathcal{M}$ of $\mathcal{K}$ (equivalently, if a quasi-equational basis for $\mathcal{K}$ together with some set of *identities* axiomatizes $\mathcal{K}'$). If $\mathcal{K}$ is the equivalent quasivariety semantics of a Hilbert system S then its relative subvarieties are exactly the equivalent quasivariety semantics of the axiomatic extensions of S. Thus, in particular, an axiomatic extension of C–H that has an LDDT has as its equivalent quasivariety semantics a relative subvariety of $\mathcal{H}_{C^*}$ with the FEP.

Theorem 5.2. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$. The following are equivalent. (T is as defined before Corollary 4.5.)*

(i) *$\mathcal{K}$ has the FEP.*

(ii) *For every $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$ there exists a formula $\psi^\tau(x, y) = \varphi^\tau(x, \vartheta_1^\tau(x, y), \ldots, \vartheta_{m(\tau)}^\tau(x, y))$, where $\varphi^\tau(x_0, \ldots, x_{m(\tau)}) \in T$ and $\vartheta_1^\tau(x, y), \ldots, \vartheta_{m(\tau)}^\tau(x, y)$ are C^*–formulas, such that $\mathcal{K} \models \tau(\vec{\S}) \approx \psi^\tau(\S_{/}, \tau(\vec{\S}))$.*

(iii) *For every $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$ there exist a positive integer k^τ and, for $i = 1, \ldots, k^\tau$, formulas $\varphi_i^\tau(x_0, \ldots, x_{m_i(\tau)}) \in T$ and binary C^*–formulas $\vartheta_{i,1}^\tau, \ldots, \vartheta_{i,m_i(\tau)}^\tau$ such that if $\psi_i^\tau(x, y) = \varphi_i^\tau(x, \vartheta_{i,1}^\tau(x, y), \ldots, \vartheta_{i,m_i(\tau)}^\tau(x, y))$ for $i = 1, \ldots, k^\tau$, then $\mathcal{K} \models \tau(\vec{\S}) \dot{-} \psi_\infty^\tau(\S_{/}, \tau(\vec{\S})) \dot{-} \ldots \dot{-} \psi_{\|^\tau}^\tau(\S_{/}, \sqcup(\vec{\S})) \approx /$.*

Proof. (i) $\Rightarrow$ (ii) Let $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$. Let $\mathbf{F}$ be the $\mathcal{K}$–free algebra on the $n+1$ free generators $\overline{x}_0, \ldots, \overline{x}_n$. We use $\overline{\tau}$ to denote $\tau(\overline{x}_0, \ldots, \overline{x}_n)$. Let $\mathbf{A}$ be the subalgebra of $\mathbf{F}$ generated by $\{\overline{x}_0, \overline{\tau}\}$. By

Corollary 4.5, $\overline{\tau} \in \langle \overline{x}_0 \rangle_{\mathbf{F}}$, so the FEP implies that $\overline{\tau} \in \langle \overline{x}_0 \rangle_{\mathbf{A}}$. By Corollary 4.5 again, there is a formula $\varphi^\tau(x_0, \dots, x_{m(\tau)}) \in T$ and $\vartheta_1^\tau(\overline{x}_0, \overline{\tau}), \dots, \vartheta_{m(\tau)}^\tau(\overline{x}_0, \overline{\tau}) \in A$ with

$$\overline{\tau} = \varphi^\tau(\overline{x}_0, \vartheta_1^\tau(\overline{x}_0, \overline{\tau}), \dots, \vartheta_{m(\tau)}^\tau(\overline{x}_0, \overline{\tau})).$$

Set $\psi^\tau(x, y) = \varphi^\tau(x, \vartheta_1^\tau(x, y), \dots, \vartheta_{m(\tau)}^\tau(x, y))$. From properties of free algebras we infer that $\mathcal{K} \models \tau(\vec{\S}) \approx \psi^\tau(\S_{\prime}, \tau(\vec{\S}))$, as required.

(ii) $\Rightarrow$ (iii) It follows from the assumptions in (ii) that

$$\mathcal{K} \models \tau(\vec{\S}) \dot{-} \psi^\tau(\S_{\prime}, \tau(\vec{\S})) \approx \prime.$$

(iii) $\Rightarrow$ (i) To show that $\mathcal{K}$ has the FEP, it suffices to show, for any subalgebra $\mathbf{B}$ of any $\mathbf{A} \in \mathcal{K}$ and any $a \in B$, that $\langle a \rangle_{\mathbf{A}} \cap B \subseteq \langle a \rangle_{\mathbf{B}}$ [4, Cor. 3.5]. Let $b \in \langle a \rangle_{\mathbf{A}} \cap B$. As $b \in \langle a \rangle_{\mathbf{A}}$, there are $\tau(x_0, \dots, x_n) \in T$ and $a_1, \dots, a_n \in A$ with $b = \tau^{\mathbf{A}}(a, a_1, \dots, a_n)$, by Corollary 4.5. By assumption, there exist $k^\tau > 0$ and

$$\psi_i^\tau(x, y) = \varphi_i^\tau(x, \vartheta_{i,1}^\tau(x, y), \dots, \vartheta_{i, m_i(\tau)}^\tau(x, y))$$

for $i = 1, \dots, k^\tau$, such that

$$b \dot{-} \psi_1^{\tau\mathbf{A}}(a, b) \dot{-} \dots \dot{-} \psi_{k^\tau}^{\tau\mathbf{A}}(a, b) = 0^{\mathbf{A}}.$$

Now $a, b \in B$, so $\vartheta_{i,j}^{\tau\mathbf{A}}(a, b) \in B$ for each i, j, whence $\psi_i^{\tau\mathbf{A}}(a, b) \in \langle a \rangle_{\mathbf{B}}$ for $i = 1, \dots, k^\tau$. It follows that $b \in \langle a \rangle_{\mathbf{B}}$, as required. ∎

Corollary 5.3. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{\mathcal{C}^*}$ that has the FEP and for each $\tau(\vec{x}) = \tau(x_0, \dots, x_n) \in T$, let $\psi^\tau(x, y)$ be as in Theorem 5.2 (ii). Then, for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,*

$$\begin{aligned} b \in \langle a \rangle_{\mathbf{A}} &\text{ if and only if } b = \psi^{\tau\,\mathbf{A}}(a, b) \quad \text{for some } \tau \in T \\ &\text{ if and only if } b \dot{-} \psi^{\tau\,\mathbf{A}}(a, b) = 0 \quad \text{for some } \tau \in T. \end{aligned}$$

Proof. If $b \in \langle a \rangle_{\mathbf{A}}$ then, by Corollary 4.5, there exist $\tau(x_0, \ldots, x_n) \in T$ and $a_1, \ldots, a_n \in A$ with $b = \tau^{\mathbf{A}}(a, a_1, \ldots, a_n)$. Then, by Theorem 5.2 (ii),

$$b = \tau^{\mathbf{A}}(a, a_1, \ldots, a_n) = \psi^{\tau\,\mathbf{A}}(a, \tau^{\mathbf{A}}(a, a_1, \ldots, a_n)) = \psi^{\tau\,\mathbf{A}}(a, b),$$

whence $b \dot{-} \psi^{\tau\,\mathbf{A}}(a, b) = 0$. Conversely, if $b \dot{-} \psi^{\tau\,\mathbf{A}}(a, b) = 0$ for some $\tau \in T$ then $b \in \langle a \rangle_{\mathbf{A}}$ because, by Corollary 4.5, $\psi^{\tau\,\mathbf{A}}(a, b) \in \langle a \rangle_{\mathbf{A}}$. ∎

We give an application of Theorem 5.2 in a very natural setting. Let κ be a nonzero ordinal and $\leq$ its natural order. If κ is closed under ordinal addition, denoted $+$, or is a successor ordinal $\kappa' + 1$ then $\boldsymbol{\kappa} = \langle \kappa; \dot{-}, \oplus, \sqcap, \sqcup, 0 \rangle \in \mathcal{H}_{\{\dot{-}, \oplus, \sqcap, \sqcup\}}$, where $\oplus$ is $+$ (or, in the case $\kappa = \kappa' + 1$, $\beta \oplus \gamma = \min\{\beta + \gamma, \kappa'\}$, for $\beta, \gamma \in \kappa$), $\dot{-}$ is *left* residuation with respect to $\oplus$ and $\leq$, and $\sqcap$, $\sqcup$ are the meet and join operations of the well ordered set $\langle \kappa; \leq \rangle$, respectively. If $1 \notin C^*$, let $\boldsymbol{\kappa}_{C^*}$ denote the C^*–reduct of $\boldsymbol{\kappa}$. From a direct consideration of algebras, it is difficult (especially when $\oplus \in C^*$) to see whether the relative subvariety of $\mathcal{H}_{\mathcal{C}^*}$ generated by $\boldsymbol{\kappa}_{C^*}$ has the FEP when $\kappa \geq \omega + 2$, but the next proposition settles this question (when $1 \notin C^*$). If $\kappa \leq \omega^\omega$ then the variety HSP($\boldsymbol{\kappa}_{C^*}$) is contained in $\mathcal{H}_{\mathcal{C}^*}$, by [28, Cor. 4.6].

Proposition 5.4. *Suppose C^* does not contain* 1 *and let κ be an ordinal greater than or equal to $\omega + 2$ for which $\boldsymbol{\kappa}_{C^*}$ exists. Then HSP($\boldsymbol{\kappa}_{C^*}$)$\cap \mathcal{H}_{\mathcal{C}^*}$ lacks the FEP (i.e., the RCEP). In particular, when $\kappa \leq \omega^\omega$, the variety HSP($\boldsymbol{\kappa}_{C^*}$) lacks the CEP.*

Proof. We show that there exists $\tau(\vec{x}) = \tau(x_0, x_1, \ldots, x_n) \in T$ such that for all $\varphi(x_0, x_1, \ldots, x_m) \in T$ and all binary C^*–formulas $\vartheta_1(x, y), \ldots, \vartheta_m(x, y)$,

$$\boldsymbol{\kappa}_{C^*} \not\models \tau(\vec{x}) \approx \varphi(x_0, \vartheta_1(x_0, \tau(\vec{x})), \ldots, \vartheta_m(x_0, \tau(\vec{x}))).$$

Set $\tau(x_0, x_1, x_2) = x_2 \dot{-} (x_2 \dot{-} (x_1 \dot{-} (x_1 \dot{-} x_0)))$. Then for any $a \in \omega$,

$$\begin{aligned}\tau^{\boldsymbol{\kappa}_{C^*}}(1,\omega+1,a) &= a \dot{-} (a \dot{-} ((\omega+1) \dot{-} ((\omega+1) \dot{-} 1))) \\ &= a \dot{-} (a \dot{-} \omega) = a. \qquad (3)\end{aligned}$$

Claim: For every $\varphi(x_0, x_1, \ldots, x_m) \in T$, there exists $k^\varphi \in \omega$ such that for all $a_1, a_2, \ldots, a_m \in \omega$, $\varphi^{\boldsymbol{\kappa}_{C^*}}(1, a_1, a_2, \ldots, a_m) \leq k^\varphi$.

This is proved by induction on the complexity of φ: If $\varphi(x_0)$ is x_0 or 0, then $\varphi^{\boldsymbol{\kappa}_{C^*}}(1) \leq 1$. Suppose $\varphi(x_0, \ldots, x_m) = x_l \dot{-} (x_l \dot{-} \varphi_1 \dot{-} \varphi_2)$, where $l \leq m$ and $\varphi_i(x_0, x_1, \ldots, x_m) \in T$ $(i = 1, 2)$, and $\varphi_i^{\boldsymbol{\kappa}_{C^*}}(1, a_1, \ldots, a_m) \leq k^{\varphi_i} \in \omega$ for all $a_1, \ldots, a_m \in \omega$. Then

$$\begin{aligned}\varphi^{\boldsymbol{\kappa}_{C^*}}(1, a_1, \ldots, a_m) &= a_l \dot{-} (a_l \dot{-} \varphi_1^{\boldsymbol{\kappa}_{C^*}}(1, a_1, \ldots, a_m) \\ &\qquad \dot{-} \varphi_2^{\boldsymbol{\kappa}_{C^*}}(1, a_1, \ldots, a_m)) \\ &\leq a_l \dot{-} (a_l \dot{-} k^{\varphi_1} \dot{-} k^{\varphi_2}) \\ &\leq k^{\varphi_1} + k^{\varphi_2} \quad (\text{since } a_l, k^{\varphi_1}, k^{\varphi_2} \in \omega),\end{aligned}$$

as required, so the Claim is true.

Let $\varphi(x_0, \ldots, x_m) \in T$ and let $\vartheta_1(x, y), \ldots, \vartheta_m(x, y)$ be binary C^*–formulas. Set $a = k^\varphi + 1$. By (3), $\vartheta_i^{\boldsymbol{\kappa}_{C^*}}(1, \tau^{\boldsymbol{\kappa}_{C^*}}(1, \omega+1, a)) = \vartheta_i^{\boldsymbol{\kappa}_{C^*}}(1, a)$ for $i = 1, \ldots, m$. Also, $1, a \in \omega$ and ω is a subuniverse of $\boldsymbol{\kappa}_{C^*}$ (since $1 \notin C^*$), so $\vartheta_i^{\boldsymbol{\kappa}_{C^*}}(1, a) \in \omega$. Thus, by the Claim and (3),

$$\begin{aligned}&\varphi^{\boldsymbol{\kappa}_{C^*}}(1, \vartheta_1^{\boldsymbol{\kappa}_{C^*}}(1, \tau^{\boldsymbol{\kappa}_{C^*}}(1, \omega+1, a)), \ldots, \vartheta_m^{\boldsymbol{\kappa}_{C^*}}(1, \tau^{\boldsymbol{\kappa}_{C^*}}(1, \omega+1, a))) \\ &= \varphi^{\boldsymbol{\kappa}_{C^*}}(1, \vartheta_1^{\boldsymbol{\kappa}_{C^*}}(1, a), \ldots, \vartheta_m^{\boldsymbol{\kappa}_{C^*}}(1, a)) \\ &\leq k^\varphi < k^\varphi + 1 = a = \tau^{\boldsymbol{\kappa}_{C^*}}(1, \omega+1, a),\end{aligned}$$

so $\boldsymbol{\kappa}_{C^*} \not\models \tau(\vec{x}) \approx \varphi(x_0, \vartheta_1(x_0, \tau(\vec{x})), \ldots, \vartheta_m(x_0, \tau(\vec{x})))$.
By Theorem 5.2 (ii), $\mathcal{H}_{C^*} \cap \mathrm{HSP}(\boldsymbol{\kappa}_{C^*})$ does not have the FEP. ∎

Let S be a Hilbert system over language $\mathcal{L}$. An *S–matrix* is a pair $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an algebra of type $\mathcal{L}$ and F an S–filter of $\mathbf{A}$. We call $\langle \mathbf{A}, F \rangle$ *reduced* if $\Omega_{\mathbf{A}} F = \{\langle a, a \rangle : a \in A\}$. The class of all [resp. all reduced] S–matrices is denoted $\mathrm{Mat}\, S$ [resp. $\mathrm{Mat}^* S$]. If S is algebraizable with equivalent quasivariety semantics $\mathcal{K}$ then the class of all algebra reducts of reduced S–matrices is precisely $\mathcal{K}$ [5, Cor. 5.3], i.e.,

$$\mathcal{K} = \{\mathbf{A} : \langle \mathbf{A}, \mathcal{F} \rangle \in \mathrm{Mat}^* \mathcal{S} \text{ for some } \mathcal{S}\text{–filter } \mathcal{F} \text{ of } \mathbf{A}\}. \tag{4}$$

Let S be an axiomatic extension of C–H and $\mathcal{K}$ the relative subvariety of $\mathcal{H}_{\mathcal{C}^*}$ that is its equivalent quasivariety semantics. For every algebra $\mathbf{A}$ of $\mathcal{K}$'s type, $\Omega_{\mathbf{A}}$ is an isomorphism from $\mathbf{Fi}^S\mathbf{A}$ to $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$; if, in addition, $\mathbf{A} \in \mathcal{K}$ then $\{0^{\mathbf{A}}\} \in \mathrm{Fi}^S\mathbf{A}$. It follows from (4) that an S–matrix $\langle \mathbf{A}, F \rangle$ is reduced if and only if $\mathbf{A} \in \mathcal{K}$ and $F = \{0^{\mathbf{A}}\}$.

For a Hilbert system S over a language $\mathcal{L}$, and a family

$$\mathcal{E} = \{\Sigma_j(x, y) : j \in J\} \tag{5}$$

of finite sets $\Sigma_j = \Sigma_j(x, y)$ of binary $\mathcal{L}$–formulas $\zeta(x, y)$, a class $\mathcal{M}$ of S–matrices is said (in [4]) to have *locally formula definable principal S–filters* (LFDPF) with *defining system* $\mathcal{E}$ if for all $\langle \mathbf{A}, F \rangle \in \mathcal{M}$ and $a, b \in A$,

$$b \in \langle F \cup \{a\} \rangle_{\mathbf{A}} \quad \text{iff} \quad \{\zeta^{\mathbf{A}}(a, b) : \zeta \in \Sigma_j\} \subseteq F \text{ for some } j \in J.$$

We drop the qualification 'locally' if it is also possible to choose $|\,J\,| = 1$, say $\mathcal{E} = \{\Sigma\}$; then we say that $\mathcal{M}$ has FDPF with *defining set* Σ.

Let $\mathcal{K}$ be the equivalent quasivariety semantics of an axiomatic extension S of C–H. Then Mat^*S has LFDPF [resp. FDPF] with defining system (5) [resp. defining set Σ] precisely when, for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,

$$b \in \langle a \rangle_{\mathbf{A}} \quad \text{if and only if} \quad \{\zeta^{\mathbf{A}}(a, b) : \zeta \in \Sigma_j\} = \{0^{\mathbf{A}}\} \text{ for some } j \in J$$

$$[\text{resp. if and only if} \quad \zeta^{\mathbf{A}}(a, b) = 0^{\mathbf{A}} \text{ for all } \zeta \in \Sigma\,].$$

In this case it is legitimate and natural to say that $\mathcal{K}$ (rather than Mat^*S) has LFDPF [resp. FDPF] with defining system (5) [resp. defining set Σ].

Corollary 5.5. *Let S be an axiomatic extension of C–H with equivalent quasivariety semantics $\mathcal{K}$. Then*

(i) Shas an LDDT if and only if $\mathcal{K}$ has the FEP.

Suppose $\mathcal{K}$ has the FEP and for each $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$, let $\psi^\tau(x, y)$ be as in Theorem 5.2 (ii). For each $\tau \in T$, let $\xi^\tau(x, y)$ be a C–formula such that $\xi^\tau(x, y)^* = \psi^\tau(x, y)$. Set $\mathcal{E} = \{\{\xi^\tau(\S, \dagger) \to \dagger\} : \tau \in \mathcal{T}\}$ and set $\mathcal{E}^* = \{\{\dagger \dot{-} \psi^\tau(\S, \dagger)\} : \tau \in \mathcal{T}\}$. Then

(ii) Both $\mathcal{K}$ and Mat$S$ have LFDPF with defining system $\mathcal{E}^*$;

(iii) $\mathcal{E}$ is a local deduction detachment system for S.

Proof. (i) is an instance of [4, Cor. 5.3]; the rest follow from Corollary 5.3 and [4, Thm. 2.4]. ∎

An alternative local deduction detachment system for S can be given in terms of the formulas ψ_i^τ whose existence is asserted by Theorem 5.2 (iii).

A quasivariety $\mathcal{K}$ has *equationally definable principal relative congruences* (EDPRC) if there is a finite set Υ of equations $\mu \approx \nu$ in four variables such that for every $\mathbf{A} \in \mathcal{K}$ and all $a, b, c, d \in A$,

$$\langle c, d\rangle \in \Theta_{\mathcal{K}}^{\mathbf{A}}(a, b) \quad \text{iff} \quad \mu^{\mathbf{A}}(a, b, c, d) = \nu^{\mathbf{A}}(a, b, c, d) \ \text{ for all } \mu \approx \nu \in \Upsilon.$$

In this case $\mathcal{K}$ has the RCEP [6, Thm. IV.3.1].

Applied to a relative subvariety $\mathcal{K}$ of $\mathcal{H}_{C^*}$, [6, Thm. IV.1.3] says that $\mathcal{K}$ has FDPF if and only if it has EDPRC.

Theorem 5.6. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{\mathcal{C}^*}$. The following conditions are equivalent:*

(i) $\mathcal{K}$ has EDPRC (i.e., $\mathcal{K}$ has FDPF).

(ii) There exists a formula $\psi(x, y) = \varphi(x, \vartheta_1(x, y), \ldots, \vartheta_m(x, y))$, where $\varphi(x_0, x_1, \ldots, x_m) \in T$ and $\vartheta_1(x, y), \ldots, \vartheta_m(x, y)$ are C^*–formulas such that for every $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$ $\mathcal{K} \models \tau(\vec{\S}) \approx \psi(\S_\prime, \tau(\vec{\S}))$.

(iii) There exist $k \in \omega$ *and formulas*
$\psi_i(x,y) = \varphi_i(x, \vartheta_{i,1}(x,y), \ldots, \vartheta_{i,m_i}(x,y))$, $i = 1, \ldots, k$, *where each* $\varphi_i(x_0, x_1, \ldots, x_{m_i}) \in T$ *and each* $\vartheta_{i,j}(x,y)$ *is a* C^**–formula, such that for every* $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$,
$\mathcal{K} \models \tau(\vec{\S}) \dot{-} \psi_\infty(\S\prime, \tau(\vec{\S})) \dot{-} \ldots \dot{-} \psi_{\|}(\S\prime, \tau(\vec{\S})) \approx \prime$.

Proof. (i) ⇒ (ii) As $\mathcal{K}$ has EDPRC, it has the RCEP, hence also the FEP. By Corollary 5.3, for any $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$, we have

$$b \in \langle a \rangle_{\mathbf{A}} \text{ if and only if } b = \psi^\tau(a,b) \text{ for some } \tau \in T, \tag{6}$$

where $\psi^\tau(x,y)$ is as in Theorem 5.2 (ii). Let $\mathcal{P}(\mathcal{T})$ be the power set of T and let $\mathcal{P}_\omega(\mathcal{T})$ the set of all finite subsets of T.

Claim: There exists $J \in \mathcal{P}_\omega(\mathcal{T})$ such that for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,

$$b \in \langle a \rangle_{\mathbf{A}} \text{ if and only if } b = \psi^\tau(a,b) \text{ for some } \tau \in J.$$

Suppose, on the contrary, that for each $J \in \mathcal{P}_\omega(\mathcal{T})$, there exist $\mathbf{A}_J \in \mathcal{K}$ and $a_J, b_J \in A_J$ with

$$b_J \in \langle a_J \rangle_{\mathbf{A}_J} \text{ but} \tag{7}$$

$$b_J \neq \psi^\tau(a_J, b_J) \text{ for all } \tau \in J. \tag{8}$$

For each $J \in \mathcal{P}_\omega(\mathcal{T})$, let $Q_J = \{K \in \mathcal{P}_\omega(\mathcal{T}) : \mathcal{J} \subseteq \mathcal{K}\}$ and

$$\mathcal{F} = \{\mathcal{X} \subseteq \mathcal{P}_\omega(\mathcal{T}) : \mathcal{Q}_\mathcal{J} \subseteq \mathcal{X} \text{ for some } \mathcal{J} \in \mathcal{P}_\omega(\mathcal{T})\}.$$

Now $\mathcal{F}$ is a filter over $\mathcal{P}_\omega(\mathcal{T})$, because $Q_J \cap Q_K = Q_{J \cup K} \in \mathcal{F}$ for all $J, K \in \mathcal{P}_\omega(\mathcal{T})$. Let $\mathcal{U}$ be an ultrafilter over $\mathcal{P}_\omega(\mathcal{T})$ with $\mathcal{F} \subseteq \mathcal{U}$, and let $\mathbf{A} = \prod_{J \in \mathcal{P}_\omega(\mathcal{T})} \mathbf{A}_J$ and $\mathbf{D} = \mathbf{A}/\mathcal{U}$. Since $\mathcal{K}$ is a quasivariety, $\mathbf{D} \in \mathcal{K}$.

Define $\overline{a}, \overline{b} \in A$ by $\overline{a}(J) = a_J$ and $\overline{b}(J) = b_J$ $(J \in \mathcal{P}_\omega(\mathcal{T}))$. Since $\mathcal{K}$ has EDPRC, it has FDPF, so there is a finite set Σ of of binary C^*–formulas

$\zeta(x,y)$ such that for any $\mathbf{B}\in\mathcal{K}$ and $c,d\in B$, we have

$$d\in\langle c\rangle_{\mathbf{B}}\ \text{ if and only if }\ \mathbf{B}\models\bigwedge_{\zeta\in\Sigma}\zeta[c,d]\approx 0.^{7} \qquad (9)$$

By (7) and (9), $\bigwedge_{\zeta\in\Sigma}\zeta[a_J,b_J]\approx 0$ is true in $\mathbf{A}_J$ for each $J\in\mathcal{P}_\omega(\mathcal{T})$, hence $\bigwedge_{\zeta\in\Sigma}\zeta[\overline{a}/\mathcal{U},\overline{\lfloor}/\mathcal{U}]\approx \prime$ is true in $\mathbf{D}$. By (9), $\overline{b}/\mathcal{U}\in\langle\overline{\dashv}/\mathcal{U}\rangle_{\mathbf{D}}$, so by (6), there is a $\tau'\in T$ with $\overline{b}/\mathcal{U}=\psi^{\tau'}(\overline{\dashv}/\mathcal{U},\overline{\lfloor}/\mathcal{U})$, i.e.,

$$U:=\{J\in\mathcal{P}_\omega(\mathcal{T}):\lfloor_{\mathcal{J}}=\psi^{\tau'}(\dashv_{\mathcal{J}},\lfloor_{\mathcal{J}})\}\in\mathcal{U}. \qquad (10)$$

Let $J_1=\{\tau'\}$ and note that $Q_{J_1}=\{K\in\mathcal{P}_\omega(\mathcal{T}):\tau'\in\mathcal{K}\}\in\mathcal{U}$, so $\emptyset\neq U\cap Q_{J_1}\in\mathcal{U}$. Let $K\in U\cap Q_{J_1}$. As $K\in U$, we have $b_K=\psi^{\tau'}(a_K,b_K)$, by (10). We also have $\tau'\in K$, since $K\in Q_{J_1}$. Then, by (8), $b_K\neq\psi^{\tau'}(a_K,b_K)$, a contradiction. This establishes the Claim.

Let $J=\{\tau_1,\ldots,\tau_k\}\subseteq T$ be as in the Claim. For $\mathbf{A}\in\mathcal{K}$ and $a,b\in A$,

$$b\in\langle a\rangle_{\mathbf{A}}\ \text{ if and only if }\ b\dot{-}\psi^{\tau_1}(a,b)\dot{-}\ldots\dot{-}\psi^{\tau_k}(a,b)=0. \qquad (11)$$

The implication from right to left follows because $\psi^{\tau_i}(a,b)\in\langle a\rangle_{\mathbf{A}}$ for $i=1,\ldots,k$, by Corollary 4.5. Define

$$\psi(x,y)=y\dot{-}(y\dot{-}\psi^{\tau_1}(x,y)\dot{-}\ldots\dot{-}\psi^{\tau_k}(x,y)).$$

If $\mathbf{G}$ is the $\mathcal{K}$–free algebra freely generated by $\overline{x},\overline{y}$ then $\psi^{\mathbf{G}}(\overline{x},\overline{y})\in\langle\overline{x}\rangle_{\mathbf{G}}$ (Lemma 4.4), so for some $\varphi(x_0,\ldots,x_m)\in T$ and binary C^*-formulas $\vartheta_1,\ldots,\vartheta_m$, we have $\psi^{\mathbf{G}}(\overline{x},\overline{y})=\varphi^{\mathbf{G}}(\overline{x},\vartheta_1(\overline{x},\overline{y}),\ldots,\vartheta_m(\overline{x},\overline{y}))$ (Corollary 4.5), i.e.,

$$\mathcal{K}\models\psi(\S,\dagger)\approx\varphi(\S,\vartheta_\infty(\S,\dagger),\ldots,\vartheta_\Updownarrow(\S,\dagger)). \qquad (12)$$

[7] This notation may be interpreted as $\zeta^{\mathbf{B}}(c,d)=0$ for all $\zeta\in\Sigma$, or, equivalently, as an assertion of the *truth* of the sentence $\bigwedge_{\zeta\in\Sigma}\zeta[c',d']\approx 0$ in the structure $\langle\mathbf{B};\langle b\!:\!b\in B\rangle\rangle$ for the first order language with equality determined by the expansion $C^*\cup\{b':b\in B\}$ of C^* by (distinct) constant symbols b' corresponding to the elements of B.

Let $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$, let $\mathbf{F}$ be the $\mathcal{K}$–free algebra freely generated by $\overline{x}_0, \ldots, \overline{x}_n$ and $\overline{\tau} = \tau^{\mathbf{F}}(\overline{x}_0, \ldots, \overline{x}_n)$. By Corollary 4.5, $\overline{\tau} \in \langle \overline{x}_0 \rangle_{\mathbf{F}}$, so by (11), $\overline{\tau} \dot{-} \psi^{\tau_1}(\overline{x}_0, \overline{\tau}) \dot{-} \ldots \dot{-} \psi^{\tau_k}(\overline{x}_0, \overline{\tau}) = 0^{\mathbf{F}}$, i.e.,

$$\mathcal{K} \models \tau(\vec{§}) \dot{-} \psi^{\tau_\infty}(§_/, \tau(\vec{§})) \dot{-} \ldots \dot{-} \psi^{\tau_\parallel}(§_/, \tau(\vec{§})) \approx /.$$

By (12),

$$\mathcal{K} \models \varphi(§_/, \vartheta_\infty(§_/, \tau(\vec{§})), \ldots, \vartheta_\Updownarrow(§_/, \tau(\vec{§}))) \approx \psi(§_/, \tau(\vec{§})) \approx \tau(\vec{§}) \dot{-} / \approx \tau(\vec{§}),$$

which proves (ii). That (ii) implies (iii) is trivial.

(iii) $\Rightarrow$ (i) We show that $\mathcal{K}$ has FDPF with defining set $\{y \dot{-} \psi_1(x, y) \dot{-} \ldots \dot{-} \psi_k(x, y)\}$. Let $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$. If

$$b \dot{-} \psi_1(a, b) \dot{-} \ldots \dot{-} \psi_k(a, b) = 0$$

then, since each $\psi_i(a, b) \in \langle a \rangle_{\mathbf{A}}$ (Corollary 4.5), it follows that $b \in \langle a \rangle_{\mathbf{A}}$. Conversely, if $b \in \langle a \rangle_{\mathbf{A}}$, then there exist a formula $\tau(x_0, \ldots, x_n) \in T$ and $a_1, \ldots, a_n \in A$ with $b = \tau(a, a_1, \ldots, a_n)$. Then

$$b \dot{-} \psi_1(a, b) \dot{-} \ldots \dot{-} \psi_k(a, b) = 0$$

is a substitution instance of the identity displayed in (iii). ∎

Corollary 5.7. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{\mathcal{C}^*}$ that has EDPRC and let $\psi(x, y)$ be as in Theorem 5.6 (ii). Then, for all $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$,*

$$b \in \langle a \rangle_{\mathbf{A}} \text{ if and only if } b = \psi^{\mathbf{A}}(a, b) \text{ if and only if } b \dot{-} \psi^{\mathbf{A}}(a, b) = 0.$$

The proof is analogous to that of Corollary 5.3.

Corollary 5.8. *Let S be the axiomatic extension of C–H whose equivalent quasivariety semantics is $\mathcal{K}$. Then*

(i) $\mathcal{K}$ has EDPRC if and only if S has a DDT.

Suppose $\mathcal{K}$ has EDPRC and let $\psi(x,y)$ be as in Theorem 5.6 (ii). Let $\xi(x,y)$ be a C–formula such that $\xi(x,y)^* = \psi(x,y)$, set $\Sigma = \{\xi(x,y) \to y\}$ and set $\Sigma^* = \{y \dot{-} \psi(x,y)\}$. Then

(ii) Both $\mathcal{K}$ and MatS have FDPF with defining set Σ^*;

(iii) Σ is a deduction detachment set for S.

Proof. (i) is an instance of [6, Thm. VI.1.3]; for the rest, use Corollary 5.7 and [4, Thm. 4.6]. ∎

Another deduction detachment set can be provided in the style of Theorem 5.6 (iii), just as in the case of the LDDT.

6. The Quasivarieties $\mathcal{H}^{\backslash}_{C^*}$

Let $\mathbf{A} \in \mathcal{H}_{C^*}$. (Recall our assumption that C^* always contains $\dot{-}$.) A nonempty subset X of A is called a *prefilter* of $\mathbf{A}$ if $b \in X$ whenever both $b \in A$ and $a, b \dot{-} a \in X$. Thus, any prefilter (in particular, any filter) of $\mathbf{A}$ is a hereditary subset of $\langle A; \leq \rangle$. For $n \in \omega$, we write $x \dot{-} ny$ for $x \dot{-} y \dot{-} \ldots \dot{-} y$ when there are n y's in the latter expression. For each $n \in \omega$, we define $\mathcal{H}^{\backslash}_{C^*}$ as the class of all algebras in $\mathcal{H}_{C^*}$ that satisfy the identity

(Z_n) $\quad x \dot{-} (x \dot{-} y) \dot{-} ny \approx 0.$

The relative subvarieties $\mathcal{K}$ of $\mathcal{H}_{C^*}$ on whose members all prefilters are (C–H–) filters turn out to be just those contained in $\mathcal{H}^{\backslash}_{C^*}$ for some $n \in \omega$, and these quasivarieties have the FEP. [8] These facts illustrate Theorem 5.2 since one can show ([28, Chap. 5]) that for every $\tau(\vec{x}) \in T$ there exists $k^\tau \in \omega$ such that $\mathcal{H}^{\backslash}_{C^*} \models \tau(\vec{§}) \dot{-} \|^{\tau}§, \approx /$. Thus, condition (iii) of Theorem 5.2 holds, with $\psi_i^\tau(x) = x$ for $i = 1, \ldots, k^\tau$. It follows that for all $\mathbf{A} \in \mathcal{H}^{\backslash}_{C^*}$ and all $a \in A$,

$$\langle a \rangle_{\mathbf{A}} = \{b \in A : \exists\, m \in \omega \text{ such that } b \dot{-} ma = 0\}. \tag{13}$$

[8] This was proved in [25, Prop. 10] for the cases $C^* = \{\dot{-}\}$ and $C^* = \{\dot{-}, \oplus\}$, but the proof applies to all C^*.

The classes $\mathcal{H}^{\setminus}_{C^*}$ are of natural interest. In particular, $\mathcal{H}^{\infty}_{\{\,\dot{-}\,\}}$ is the class of all BCK–algebras [30, Lemma 3.1], while the ordinal ω^{ω}, considered as a *right* residuated well ordered monoid (with ordinal addition) is anti-isomorphic to a member of $\mathcal{H}^{\in}_{\{\,\dot{-}\,,\oplus\}}$ (not in $\mathcal{H}^{\infty}_{\{\,\dot{-}\,,\oplus\}}$) [28, Prop. 5.9].

We shall exhibit a subvariety of $\mathcal{H}_{\{\,\dot{-}\,,\,\sqcap\,,\,\sqcup\,\}}$ with FDPF (hence also the FEP) which violates (Z_n) for all $n \in \omega$. Nevertheless, the following is true.

Theorem 6.1. *Let $\mathcal{K}$ be a relative subvariety of $\mathcal{H}_{C^*}$, where $\oplus, 1 \notin C^*$, and let $n \in \omega$. Then $\mathcal{K} \subseteq \mathcal{H}^{\setminus}_{C^*}$ if and only if $\mathcal{K}$ has the FEP and satisfies the identity $x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y$.*

Proof. If $\mathcal{K} \subseteq \mathcal{H}^{\setminus}_{C^*}$ then $\mathcal{K}$ has the FEP and the required identity is a consequence of (Z_n). For the converse, we shall consider in detail only the case $C^* = \{\,\dot{-}\,,\,\sqcap\,,\,\sqcup\,\}$. Set $\varphi(x,y) = x \dot{-} (x \dot{-} y) \dot{-} ny$. Let $\mathbf{F}$ be the $\mathcal{K}$–free algebra on two free generators $\overline{x}$, $\overline{y}$ and $\mathbf{A}$ the subalgebra of $\mathbf{F}$ generated by $\{\overline{\varphi}, \overline{y}\}$, where $\overline{\varphi} = \varphi^{\mathbf{F}}(\overline{x}, \overline{y})$. Note that $\overline{\varphi} \dot{-} \overline{y} = \overline{\varphi}$, by our assumptions. Let $f(\psi)$ be the number of occurrences of connectives in each binary C^*–formula $\psi = \psi(x_0, x_1)$ and let $\tilde{\psi} = \psi^{\mathbf{A}}(\overline{y}, \overline{\varphi}) \in A$. By induction on $f(\psi)$,

$$\tilde{\psi} \leq \overline{\varphi} \sqcup \overline{y} \text{ for every binary } C^*\text{–formula } \psi \text{ (hence for every } \tilde{\psi} \in A). \quad (14)$$

Claim: For every binary C^*–formula ψ (hence for every $\tilde{\psi} \in A$),

$$\tilde{\psi} \leq \overline{y} \quad \text{or} \quad \tilde{\psi} \geq \overline{\varphi}. \quad (15)$$

Certainly, this is true if $f(\psi) = 0$. Let $0 < m \in \omega$ and suppose (15) is true whenever $f(\psi) < m$. Consider a binary C^*–formula ψ with $f(\psi) = m$. Then $\tilde{\psi}$ is $\tilde{\psi}_1 \sqcap \tilde{\psi}_2$ or $\tilde{\psi}_1 \sqcup \tilde{\psi}_2$ or $\tilde{\psi}_1 \dot{-} \tilde{\psi}_2$ for binary C^*–formulas ψ_1, ψ_2 with $f(\psi_1), f(\psi_2) < m$. In the first two cases (15) follows easily. Suppose $\tilde{\psi}$ is $\tilde{\psi}_1 \dot{-} \tilde{\psi}_2$. If $\tilde{\psi}_1 \leq \overline{y}$ then $\tilde{\psi} \leq \overline{y}$. If $\tilde{\psi}_1 \geq \overline{\varphi}$ and $\tilde{\psi}_2 \leq \overline{y}$ then

$$\tilde{\psi} = \tilde{\psi}_1 \dot{-} \tilde{\psi}_2 \geq \overline{\varphi} \dot{-} \tilde{\psi}_2 \geq \overline{\varphi} \dot{-} \overline{y} = \overline{\varphi}.$$

If $\tilde{\psi}_1 \geq \overline{\varphi}$ and $\tilde{\psi}_2 \geq \overline{\varphi}$ then, by (14) and (L12),

$$\begin{aligned}\tilde{\psi} &= \tilde{\psi}_1 \mathbin{\dot{-}} \tilde{\psi}_2 \;\leq\; (\overline{\varphi} \sqcup \overline{y}) \mathbin{\dot{-}} \tilde{\psi}_2 \\ &= (\overline{\varphi} \mathbin{\dot{-}} \tilde{\psi}_2) \sqcup (\overline{y} \mathbin{\dot{-}} \tilde{\psi}_2) \;=\; 0^{\mathbf{F}} \sqcup (\overline{y} \mathbin{\dot{-}} \tilde{\psi}_2) \;\leq\; \overline{y}.\end{aligned}$$

This establishes the Claim.

Next, we show, using Corollary 4.5, that $\langle \overline{y} \rangle_{\mathbf{A}} = \{\tilde{\psi} \in A : \tilde{\psi} \leq \overline{y}\}$. Trivially, $0^{\mathbf{F}}, \overline{y} \leq \overline{y}$. Suppose $\tilde{\psi}_1, \tilde{\psi}_2 \in \langle \overline{y} \rangle_{\mathbf{A}}$ with $\tilde{\psi}_1, \tilde{\psi}_2 \leq \overline{y}$ and let $\tilde{\psi} \in A$. Then $\tilde{\psi} \mathbin{\dot{-}} (\tilde{\psi} \mathbin{\dot{-}} \tilde{\psi}_1 \mathbin{\dot{-}} \tilde{\psi}_2) \in \langle \overline{y} \rangle_{\mathbf{A}}$. By the Claim, $\tilde{\psi} \leq \overline{y}$ or $\tilde{\psi} \geq \overline{\varphi}$. In the first case, $\tilde{\psi} \mathbin{\dot{-}} (\tilde{\psi} \mathbin{\dot{-}} \tilde{\psi}_1 \mathbin{\dot{-}} \tilde{\psi}_2) \leq \overline{y}$; in the second,

$$\begin{aligned}&\tilde{\psi} \mathbin{\dot{-}} (\tilde{\psi} \mathbin{\dot{-}} \tilde{\psi}_1 \mathbin{\dot{-}} \tilde{\psi}_2) \;\leq\; \tilde{\psi} \mathbin{\dot{-}} (\tilde{\psi} \mathbin{\dot{-}} \overline{y} \mathbin{\dot{-}} \overline{y}) \;\leq\; \tilde{\psi} \mathbin{\dot{-}} (\overline{\varphi} \mathbin{\dot{-}} \overline{y} \mathbin{\dot{-}} \overline{y}) \\ &= \tilde{\psi} \mathbin{\dot{-}} \overline{\varphi} \;\leq\; (\overline{\varphi} \sqcup \overline{y}) \mathbin{\dot{-}} \overline{\varphi} \quad \text{(by (14))} \\ &\leq\; \overline{y} \quad \text{(by (L13)), as required.}\end{aligned}$$

Evidently, $\overline{\varphi} \in \langle \overline{y} \rangle_{\mathbf{F}}$. By the FEP, $\overline{\varphi} \in \langle \overline{y} \rangle_{\mathbf{A}}$, so $\overline{\varphi} \leq \overline{y}$. Thus, $\overline{\varphi} \mathbin{\dot{-}} \overline{y} = 0^{\mathbf{F}}$. It follows that $\overline{\varphi} = 0^{\mathbf{F}}$, since $\overline{\varphi} \mathbin{\dot{-}} \overline{y} = \overline{\varphi}$. Thus, $\mathcal{K} \models \varphi(\S, \dagger) \approx /$.

For $C^* = \{\mathbin{\dot{-}}\}$, this argument may be modified, the essential fact being that $\mathcal{H}_{\{\dot{-}\}}$ satisfies $\psi(x_0, x_1) \leq x_0$ or $\psi(x_0, x_1) \leq x_1$ for any binary C^*–formula ψ. The result for intermediate values of C^* follows from these two cases. ■

Any locally finite (quasi)variety $\mathcal{K}$ with the (R)CEP has 'definable principal (relative) congruences' in the sense of [4]: see [21] (and [1] for varieties). When $\mathcal{K}$ is also (relatively) congruence distributive, $\mathcal{K}$ has EDP(R)C by [4, Cor. 4.7]. Thus, in particular, every locally finite relative subvariety of $\mathcal{H}_{C^*}$ with the FEP (e.g., each $\mathcal{H}^{\backslash}_{C^*}$) must have FDPF (see Proposition 4.2). The next result clarifies and strengthens this observation.

Corollary 6.2. *Let $\mathcal{K}$ be a locally finite relative subvariety of $\mathcal{H}_{C^*}$, where $\oplus, 1 \notin C^*$, such that $\mathcal{K}$ has the FEP. Then there exist $n, m \in \omega$ such that $\mathcal{K} \subseteq \mathcal{H}^{\backslash}_{C^*}$ and $\mathcal{K}$ has FDPF with defining set $\{x \mathbin{\dot{-}} my\}$.*

Proof. Since $\mathcal{K}$ is locally finite, the $\mathcal{K}$–free algebra $\mathbf{F}$ on two free generators $\overline{x}, \overline{y}$ is a finite algebra in which $\{\overline{x} \mathbin{\dot{-}} (\overline{x} \mathbin{\dot{-}} \overline{y}) \mathbin{\dot{-}} n\overline{y} : n \in \omega\}$ is a descending chain. Thus, $\mathcal{K} \models \S \dot{-} (\S \dot{-} \dagger) \dot{-} \backslash\dagger \approx \S \dot{-} (\S \dot{-} \dagger) \dot{-} (\backslash + \infty)\dagger$ for some $n \in \omega$. Now the first part of the corollary follows from the previous theorem.

By a similar argument, since $\mathcal{K}$ is locally finite, it satisfies an identity $x \dot{-} my \approx x \dot{-} (m+1)y$ for some $m \in \omega$. By (13), for all $\mathbf{A} \in \mathcal{K}$ and all $a \in A$, $\langle a \rangle_{\mathbf{A}} = \{b \in A : b \dot{-} ma = 0\}$. ∎

Example 6.3. We exhibit an algebra $\mathbf{A}$ such that $\mathrm{HSP}(\mathbf{A})$ is a subvariety of $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ with the FEP not satisfying (Z_n) for any $n \in \omega$. Let $A = \{0\} \cup \{a_i : i \in \omega\}$, where $\{a_i : i \in \omega\}$ is a one-to-one sequence whose range excludes 0. Let $\leq$ be the linear order of A shown in Figure 6. For $i, j \in \omega$, define $0 \dot{-} 0 = 0 \dot{-} a_i = 0$, $a_i \dot{-} 0 = a_i$ and

$$a_i \dot{-} a_j := \begin{cases} 0 & \text{if } j \leq i \\ a_{i+2} & \text{if } j = i+1 \\ a_{i+1} & \text{if } j \geq i+2. \end{cases}$$

Let $\mathbf{A} = \langle A; \dot{-}, \sqcap, \sqcup, 0\rangle$, where $\sqcap$ and $\sqcup$ are, respectively, the meet and join operations of $\langle A; \leq\rangle$.

Then $\mathbf{A} \in \mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$ but $\mathbf{A} \notin \mathcal{H}^{\backslash}_{\{\dot{-},\sqcap,\sqcup\}}$ for any $n \in \omega$, since $a_0 \dot{-} (a_0 \dot{-} a_{n+3}) \dot{-} na_{n+3} = a_2 \dot{-} na_{n+3} = a_{n+2} \neq 0$. More strongly, $\mathbf{A}$ violates $x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y$ for all $n \in \omega$. For example, $a_0 \dot{-} (a_0 \dot{-} a_{n+3}) \dot{-} (n+1)a_{n+3} = a_{n+2} \dot{-} a_{n+3} = a_{n+4}$. One can show that $\mathbf{A}$ satisfies

$$x \dot{-} 2(x \dot{-} y) \dot{-} (y \dot{-} x) \approx y \dot{-} 2(y \dot{-} x) \dot{-} (x \dot{-} y),$$

so if $\mathcal{V} = \mathrm{HSP}(\mathbf{A})$ then $\mathcal{V} \subseteq \mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$: see [25, Prop. 15] (or modify the argument of Proposition 4.1). The syntactic form of the formula

$$\psi(x,y) = y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} (y \dot{-} (y \dot{-} x \dot{-} x)))))$$

is as prescribed in Theorem 5.6 (ii). Let $\tau(\vec{x}) = \tau(x_0, \ldots, x_n) \in T$. We claim that $\mathcal{V} \models \tau(\vec{\S}) \approx \psi(\S\prime, \tau(\vec{\S}))$, whence $\mathcal{V}$ has FDPF. Now $\mathcal{H}_{\{\dot{-},\sqcap,\sqcup\}}$

satisfies $\tau(0, x_1, \ldots, x_n) \approx 0$, hence also $\psi(0, \tau(0, x_1, \ldots, x_n)) \approx \psi(0, 0) \approx 0 \approx \tau(0, x_1, \ldots, x_n)$. For any $i, j \in \omega$,

$$a_i \dot{-} (a_i \dot{-} a_j \dot{-} (a_i \dot{-} (a_i \dot{-} a_j \dot{-} (a_i \dot{-} (a_i \dot{-} a_j \dot{-} a_j))))) = a_i. \quad (16)$$

Thus, for all $b_1, \ldots, b_n \in A$, we have

$$\psi(a_i, \tau(a_i, b_1, \ldots, b_n)) = \tau(a_i, b_1, \ldots, b_n)$$

regardless of the value of $\tau(a_i, b_1, \ldots, b_n)$, establishing our claim.

This example shows that we cannot drop from Theorem 6.1 the condition that $\mathcal{K}$ satisfy $x \dot{-} (x \dot{-} y) \dot{-} ny \approx x \dot{-} (x \dot{-} y) \dot{-} (n+1)y$. (Of course, $\mathcal{V}$ is not locally finite.)

Figure 6. **Figure 7.**

Example 6.4. We show now that Corollary 6.2 would fail if we allowed $\oplus \in C^*$ or $1 \in C^*$. Let $\langle B; \leq \rangle$ be the four-element poset depicted in Figure 7. For $s, t \in B$, define $0 \oplus s = s \oplus 0 = s$; $a \oplus a = a; a \oplus b = b$ and $s \oplus t = 1$ otherwise. The ordered monoid $\langle B; \oplus, 0; \leq \rangle$ is residuated on the left as follows: $s \dot{-} t = 0$ for $s, t \in B$ with $s \leq t$; $b \dot{-} a = 1 \dot{-} a = 1 \dot{-} b = b$. This fact and the linearity of $\leq$ ensure that $\mathbf{B} := \langle B; \dot{-}, \oplus, \sqcap, \sqcup, 0, 1 \rangle$ is in $\mathcal{H}$, where $\sqcap$ and $\sqcup$ are, respectively, the meet and join operations of $\langle B, \leq \rangle$. For each C^*, let $\mathbf{B}_{C^*}$ be the C^*–reduct of $\mathbf{B}$.[9] By [25, Example 2], $\mathbf{B}_{\{\dot{-}\}}$

[9] The $\langle \dot{-}, \sqcap, \sqcup, 0 \rangle$–reduct of $\mathbf{B}$ is isomorphic to the corresponding subreduct on $\{0, 1, \omega, \omega+1\}$ of the ordinal ω^2 considered as a *left* residuated well ordered monoid with ordinal addition as $\oplus$.

generates a subvariety of $\mathcal{H}_{\{\,\dot{-}\,\}}$, whence $\mathbf{B}_{C^*}$ generates a subvariety, say $\mathcal{B}_{\mathcal{C}^*}$ of $\mathcal{H}_{\mathcal{C}^*}$. Note that $1 \dot{-} (1 \dot{-} a) \dot{-} na = b \neq 0$ for each $n \in \omega$. If C^* contains $\oplus$ then $\mathcal{B}_{\mathcal{C}^*}$ has FDPF because the C^*–formula

$$\psi(x, y) = y \dot{-} (y \dot{-} x \dot{-} ((y \oplus x) \dot{-} ((y \oplus x) \dot{-} x)))$$

has the syntactic form prescribed in Theorem 5.6 (ii) and it may be shown that $\mathbf{B}_{C^*} \models \tau(\vec{x}) \approx \psi(x_0, \tau(\vec{x}))$ for every $\tau(\vec{x}) \in T$. When $1 \in C^*$, one may show similarly that $\mathcal{B}_{\mathcal{C}^*}$ has FDPF, using the C^*–formula

$$\psi'(x, y) = y \dot{-} (y \dot{-} x \dot{-} (1 \dot{-} (1 \dot{-} x))).$$

6.1. Finitely Subdirectly Irreducible Algebras in $\mathcal{H}^{\setminus}_{\mathcal{C}^*}$. Given an algebra $\mathbf{A}$ in a relatively congruence distributive quasivariety $\mathcal{K}$, the nonidentity congruences of $\mathbf{A}$ are closed under finite intersections if and only if the same is true of the nonidentity $\mathcal{K}$–congruences of $\mathbf{A}$: see [13] and, for stronger results, [21] and [19]. In this case we call $\mathbf{A}$ *finitely subdirectly irreducible* and the class of all such algebras in $\mathcal{K}$ is denoted $\mathcal{K}_{\mathrm{FSI}}$, while $\mathcal{K}_{\mathrm{SI}}$ is the class of subdirectly irreducible algebras in $\mathcal{K}$.

For any $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$, the requirement that the nonidentity $\mathcal{H}_{\mathcal{C}^*}$–congruences of $\mathbf{A}$ be closed under *arbitrary* intersections, i.e., that $\mathbf{A}$ be 'relatively (or $\mathcal{H}_{\mathcal{C}^*}$–) subdirectly irreducible', amounts just to the (ordinary) subdirect irreducibility of $\mathbf{A}$ [29], [28, Prop. 4.21] (provided, as always, that $\dot{-} \in C^*$). Thus, the subdirect [resp. finite subdirect] irreducibility of $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$ means just that the nonzero *filters* of $\mathbf{A}$ are closed under arbitrary [resp. finite] intersections.

If $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$ and $C^* \subseteq \{\,\dot{-}\,, \sqcap\,\}$ then, since $\mathbf{A} \models x \dot{-} y \leq x$, any hereditary subset of $\langle A; \leq\rangle$ is a subuniverse of $\mathbf{A}$. If $a \in A$, we write $(a]$ for the principal *order* ideal $\{b \in A : b \leq a\}$ of $\langle A; \leq\rangle$ generated by a.

Proposition 6.5. *For each $n \in \omega$, an algebra $\mathbf{A} \in \mathcal{H}^{\setminus}_{\mathcal{C}^*}$ is finitely subdirectly irreducible if and only if $0^{\mathbf{A}}$ is meet irreducible in $\langle A, \leq\rangle$.*

Proof. By the above remarks, algebras in $\mathcal{H}_{\mathcal{C}^*}$ are finitely subdirectly irreducible if and only if their $\langle\,\dot{-}\,, 0\rangle$–reducts are. We may therefore assume without loss of generality that $C^* = \{\,\dot{-}\,\}$.

(⇒) Let $\mathbf{A} \in (\mathcal{H}^{\setminus}_{C^*})_{\mathrm{FSI}}$. Suppose that $0 \neq a, b \in A$ and that 0 is the only common lower bound of a and b in $\langle A; \leq \rangle$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ whose universe is $(a] \cup (b]$. Since $(a] \cap (b] = \{0\}$, $(a]$ and $(b]$ are prefilters (hence filters) of $\mathbf{B}$. Now $\mathcal{H}^{\setminus}_{C^*}$ has the FEP, so $\langle a \rangle_{\mathbf{A}} \cap B = \langle a \rangle_{\mathbf{B}} = (a]$ and $\langle b \rangle_{\mathbf{A}} \cap B = \langle b \rangle_{\mathbf{B}} = (b]$, whence $\langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}} \cap B = (a] \cap (b] = \{0\}$. Let $0 \neq c \in \langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}}$. By (13), there exist $k, l \in \omega$ with

$$c \dot{-} ka = 0 = c \dot{-} lb. \tag{17}$$

Choose k, l minimal such that (17) is true and note that $k, l > 0$. Then

$$0 \neq c \dot{-} (k-1)a \leq a \quad \text{and} \quad 0 \neq c \dot{-} (l-1)b \leq b, \tag{18}$$

hence $c \dot{-} (k-1)a \dot{-} (l-1)b \leq a, b$. By assumption, therefore,

$$c \dot{-} (k-1)a \dot{-} (l-1)b = 0.$$

Thus, $c \dot{-} (k-1)a \in B \cap \langle b \rangle_{\mathbf{A}}$ and, by (17), $c \dot{-} (k-1)a \in \langle a \rangle_{\mathbf{A}}$, so $c \dot{-} (k-1)a \in \langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}} \cap B$, i.e., $c \dot{-} (k-1)a = 0$, contradicting (18). It follows that $\langle a \rangle_{\mathbf{A}} \cap \langle b \rangle_{\mathbf{A}} = \{0\}$. Then, by finite subdirect irreducibility, $a = 0$ or $b = 0$.

(⇐) Let 0 be meet irreducible in $\langle A; \leq \rangle$ and let F and G be filters of $\mathbf{A}$ with $F \cap G = \{0\}$. Suppose $0 \neq a \in F$. Let $b \in G$. Any common lower bound of a and b is in $F \cap G$, so the meet of a and b in $\langle A; \leq \rangle$ is 0, whence $b = 0$. Thus, $G = \{0\}$ and $\mathbf{A}$ is finitely subdirectly irreducible. ∎

The above proposition does not generalize from $\mathcal{H}^{n}_{C^*}$ to $\mathcal{H}_{C^*}$. For example, in [30] we construct a finite subdirectly irreducible algebra $\mathbf{P}_5 \in \mathcal{H}_{\{\dot{-}\}}$ with $\mathrm{HSP}(\mathbf{P}_5) \subseteq \mathcal{H}_{\{\dot{-}\}}$ such that $0^{\mathbf{P}_5}$ is not meet irreducible in $\langle P_5; \leq \rangle$.

Consider the following quasi-identity:

$$z \dot{-} (x \dot{-} y) \approx 0 \quad \text{and} \quad z \dot{-} (y \dot{-} x) \approx 0 \;\Rightarrow\; z \approx 0 \tag{19}$$

If $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$ satisfies (19), then for all $a, b \in A$, 0 is the only common lower bound of $a \dot{-} b$ and $b \dot{-} a$ in $\langle A; \leq \rangle$. So, when $\sqcap \in C^*$, (19) is equivalent to

$$(x \dot{-} y) \sqcap (y \dot{-} x) \approx 0. \tag{20}$$

Moreover, when $\sqcap \in C^*$, the $\langle \dot{-}, \sqcap, 0 \rangle$–identity (20) is equivalent to the $\langle \dot{-}, 0 \rangle$–identity

$$z \dot{-} (z \dot{-} (x \dot{-} y)) \dot{-} (z \dot{-} (y \dot{-} x)) \approx 0. \tag{21}$$

For, by (Y_0), $\mathcal{H}_{\{\dot{-}, \sqcap\}}$ satisfies

$$z \dot{-} ((x \dot{-} y) \sqcap (y \dot{-} x)) \dot{-} (z \dot{-} (x \dot{-} y)) \dot{-} (z \dot{-} (y \dot{-} x)) \approx 0.$$

Thus, over $\mathcal{H}_{\{\dot{-}, \sqcap\}}$, (20) implies (21). Conversely, over $\mathcal{H}_{\{\dot{-}, \sqcap\}}$, we may derive (20) from (21) if we set $z = (x \dot{-} y) \sqcap (y \dot{-} x)$.

Corollary 6.6. *For $\mathbf{A} \in \mathcal{H}_{\mathcal{C}^*}$, consider the following conditions:*

(i) A is linearly ordered;

(ii) A satisfies (21);

(iii) A satisfies (19);

(iv) A satisfies (20).

Generally, (i)$\Rightarrow$(ii)$\Rightarrow$(iii). For any $n \in \omega$, if $\mathbf{A} \in (\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{FSI}$ then conditions (i), (ii) and (iii) are equivalent. If, in addition, $\sqcap \in C^*$ then all four conditions are equivalent.

Proof. (i) $\Rightarrow$ (ii) Under any interpretation of x, y, z in A, one of $x \dot{-} y$ and $y \dot{-} x$ takes the value 0, hence (21) holds in $\mathbf{A}$. That (ii) $\Rightarrow$ (iii) is clear.

Suppose now that $\mathbf{A} \in (\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{FSI}}$, where $n \in \omega$.

(iii) $\Rightarrow$ (i) By Proposition 6.5, 0 is meet irreducible in $\langle A; \leq \rangle$. By (19), if $a, b \in A$ then $a \dot{-} b = 0$ or $b \dot{-} a = 0$, i.e., $a \leq b$ or $b \leq a$. ∎

The case of Corollary 6.6 in which $C^* = \{\dot{-}\}$, $\mathbf{A}$ is subdirectly irreducible and $n = 1$ (so $\mathbf{A}$ is a BCK–algebra) was discovered by M. Pałasinski [23]. The next result therefore generalizes a known property of BCK–algebras.

Corollary 6.7. *For $n \in \omega$, an algebra $\mathbf{A} \in \mathcal{H}^{\backslash}_{\mathcal{C}^*}$ satisfies (21) if and only if it is a subdirect product of linearly ordered algebras in $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$ (equivalently, in $(\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{SI}$).*

Proof. Since $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$ is a relative subvariety of $\mathcal{H}_{\mathcal{C}^*}$, $\mathbf{A}$ is a subdirect product of $\mathcal{H}_{\mathcal{C}^*}$–subdirectly irreducible algebras $\mathbf{A}_i \in \mathrm{H}(\mathbf{A}) \cap \mathcal{H}^{\backslash}_{\mathcal{C}^*}$, $i \in I$. Then $\mathbf{A}$ satisfies (21) if and only if each $\mathbf{A}_i$ does. Also, relatively subdirectly irreducible members of $\mathcal{H}_{\mathcal{C}^*}$ are subdirectly irreducible, hence finitely subdirectly irreducible. By Corollary 6.6, $\mathbf{A}$ satisfies (21) if and only if each $\mathbf{A}_i$ is linearly ordered. ∎

For $\mathcal{M} \subseteq \mathcal{H}_{C^*}$, let $\mathcal{M}_{\mathrm{L}}$ be the class of all linearly ordered members of $\mathcal{M}$.

Corollary 6.8. *For each $n \in \omega$, the quasivariety generated by $(\mathcal{H}^{\backslash}_{\mathcal{C}^*})_L$ is a relative subvariety of $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$. It is axiomatized, relative to $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$, by (21) (equivalently, by (20) if $\sqcap \in C^*$).*

Proof. Let $\mathcal{R}^{\backslash}_{\mathcal{C}^*} = \mathcal{H}^{\backslash}_{\mathcal{C}^*} \cap \mathrm{HSP}((\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}})$ (the relative subvariety of $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$ generated by $(\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}}$). Then $\mathrm{Q}((\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}}) \subseteq \mathcal{R}^{\backslash}_{\mathcal{C}^*}$. If $\mathbf{A} \in \mathcal{R}^{\backslash}_{\mathcal{C}^*}$ is $\mathcal{R}^{\backslash}_{\mathcal{C}^*}$–subdirectly irreducible then it is $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$–subdirectly irreducible, so it is a finitely subdirectly irreducible algebra in $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$ and satisfies (21). By Corollary 6.6, $\mathbf{A} \in (\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}}$. Now $\mathcal{R}^{\backslash}_{\mathcal{C}^*}$ is the closure under subdirect products of the class of all such $\mathbf{A}$, so $\mathcal{R}^{\backslash}_{\mathcal{C}^*} \subseteq \mathrm{Q}((\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}})$, as required. By Corollary 6.7, any $\mathbf{A} \in \mathcal{H}^{n}_{\mathcal{C}^*}$ satisfying (21) is a subdirect product of linearly ordered members of $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$, so $\mathbf{A} \in \mathrm{Q}((\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}})$. Thus, $\mathrm{Q}((\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}})$ is axiomatized, relative to $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$, by (21). ∎

Algebras in $\mathcal{H}_{\mathcal{C}^*}$ are called *representable* if they are subdirect products of linearly ordered algebras in $\mathcal{H}_{\mathcal{C}^*}$. Thus, $\mathcal{R}^{\backslash}_{\mathcal{C}^*} = \mathrm{Q}((\mathcal{H}^{\backslash}_{\mathcal{C}^*})_{\mathrm{L}})$ is the class of representable algebras in $\mathcal{H}^{\backslash}_{\mathcal{C}^*}$. Even for $n = 1$, this is not a variety [32].

6.2. Equationally Definable Principal Meets. A quasivariety $\mathcal{K}$ has *equationally definable principal meets* (EDPM) if there exists a finite set Σ

of pairs $\langle\varphi,\psi\rangle$ of 4-ary formulas such that, for all $\mathbf{A}\in\mathcal{K}$ and $a,b,c,d\in A$,

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(a,b)\cap\Theta^{\mathbf{A}}_{\mathcal{K}}(c,d)=\Theta^{\mathbf{A}}_{\mathcal{K}}(\{\langle\varphi^{\mathbf{A}}(a,b,c,d),\psi^{\mathbf{A}}(a,b,c,d)\rangle:\langle\varphi,\psi\rangle\in\Sigma\}).$$

In this case we call Σ a *set of principal intersection equations* for $\mathcal{K}$.

Theorem 6.9. [9, Thm. 2.3, Cor. 2.4] *For a quasivariety $\mathcal{K}$ of algebras the following are equivalent:*

(i) *$\mathcal{K}$ has EDPM;*

(ii) *$\mathcal{K}$ is relatively congruence distributive and $\mathcal{K}_{FSI}$ is a universal class* [10]*;*

(iii) *$\mathcal{K}$ is relatively congruence distributive and for any $\mathbf{A}\in\mathcal{K}$, the finitely generated $\mathcal{K}$–congruences of $\mathbf{A}$ form a sublattice of $\mathbf{Con}_{\mathcal{K}}\,\mathbf{A}$;*

(iv) *for some finite set $\Sigma=\{\langle\varphi_i,\psi_i\rangle:i<n\}$ of pairs of 4-ary formulas,*

$$\mathcal{K}_{FSI}\models\forall x\,\forall y\,\forall z\,\forall w$$

$$[(\bigwedge_{i<n}\varphi_i(x,y,z,w)\approx\psi_i(x,y,z,w))\ \Leftrightarrow\ (x\approx y\quad\textit{or}\ z\approx w)].$$

Moreover, a set Σ of pairs of binary formulas satisfies (iv) if and only if it is a set of principal intersection equations for $\mathcal{K}$.

The definition of EDPM may be simplified for certain quasivarieties. For a quasivariety $\mathcal{K}$ with constant formula 0, the *assertional logic of* $\mathcal{K}$ (at 0) is the (finitary and structural) Hilbert system $S=S(\mathcal{K},0)$ defined by

$$\Gamma\vdash_S\varphi\quad\text{iff}\quad\{\gamma(\vec{x})\approx 0:\gamma\in\Gamma\}\models_{\mathcal{K}}\varphi(\vec{x})\approx 0.$$

In contexts where such $\mathcal{K}$ (and no other Hilbert system) has been specified, we use $\langle\cdot\rangle_{\mathbf{A}}$ to denote $S(\mathcal{K},0)$–filter generation in an algebra $\mathbf{A}$. We say that $\mathcal{K}$ is 0–*regular* if it is relatively point regular at 0, as defined before

[10] i.e. it is axiomatized by a set of universal sentences of the first order language (with equality) of $\mathcal{K}$.

Lemma 3.1. It is well known that $\mathcal{K}$ is 0–regular if and only if there are binary formulas $\Delta_j(x, y)$, $j < m \in \omega$, such that

$$\mathcal{K} \models (\bigwedge_{j<m} \Delta_j(x, y) \approx 0) \Leftrightarrow x \approx y. \tag{22}$$

The 0–regularity of $\mathcal{K}$ is also equivalent to the requirement that $\mathcal{K}$ be the equivalent quasivariety semantics of its assertional logic at 0 [7, Thm. 5.2]. In this case we call Δ_j, $j < m$, *equivalence formulas for* $\mathcal{K}$. (They are indeed equivalence formulas for $S(\mathcal{K}, 0)$ and $\mathcal{K}$.)

Lemma 6.10. *If* $\mathbf{A}$ *belongs to a 0–regular quasivariety* $\mathcal{K}$ *and* $Y, Z, W \subseteq A$ *then* $\langle Y\rangle_{\mathbf{A}} = 0^{\mathbf{A}}/\Theta^{\mathbf{A}}_{\mathcal{K}}(Y \times \{0^{\mathbf{A}}\})$. *Consequently,*

$$\Theta^{\mathbf{A}}_{\mathcal{K}}(Y \times \{0^{\mathbf{A}}\}) \cap \Theta^{\mathbf{A}}_{\mathcal{K}}(Z \times \{0^{\mathbf{A}}\}) = \Theta^{\mathbf{A}}_{\mathcal{K}}(W \times \{0^{\mathbf{A}}\})$$
$$\textit{iff } \langle Y\rangle_{\mathbf{A}} \cap \langle Z\rangle_{\mathbf{A}} = \langle W\rangle_{\mathbf{A}}.$$

Proof. Let $S = S(\mathcal{K}, 0)$ and $\eta = \Theta^{\mathbf{A}}_{\mathcal{K}}(Y \times \{0^{\mathbf{A}}\})$. Certainly, $\langle Y\rangle_{\mathbf{A}} \subseteq 0^{\mathbf{A}}/\eta$, as the latter is an S–filter of $\mathbf{A}$ containing Y. Conversely, let $\langle b, 0^{\mathbf{A}}\rangle \in \eta$. By a characterization of relative congruence generation in [7, Lemma 4.2], there exist a quasi-identity

$$\bigwedge_{i<k} \varphi_i(\vec{x}) \approx \psi_i(\vec{x}) \quad \text{and} \quad \bigwedge_{i<k} \varphi'_i(\vec{x}) \approx \psi'_i(\vec{x}) \;\Rightarrow\; \varphi(\vec{x}) \approx \psi(\vec{x})$$

satisfied by $\mathcal{K}$ and elements $\vec{a} \in A$ such that $\langle \varphi^{\mathbf{A}}(\vec{a}), \psi^{\mathbf{A}}(\vec{a})\rangle = \langle b, 0^{\mathbf{A}}\rangle$ and for each $i < k$, $\langle \varphi^{\mathbf{A}}_i(\vec{a}), \psi^{\mathbf{A}}_i(\vec{a})\rangle \in Y \times \{0^{\mathbf{A}}\}$ and $\varphi'^{\mathbf{A}}_i(\vec{a}) = \psi'^{\mathbf{A}}_i(\vec{a})$. Now

$$\{\varphi_i : i < k\} \cup \{\psi_i : i < k\} \cup \{\Delta_j(\varphi'_i, \psi'_i) : \; i < k; \, j < m\} \cup \{\psi\} \vdash_S \varphi,$$

so $b = \varphi^{\mathbf{A}}(\vec{a}) \in \langle Y \cup \{0^{\mathbf{A}}\}\rangle_{\mathbf{A}} = \langle Y\rangle_{\mathbf{A}}$. The second assertion follows from the first by 0–regularity. ∎

Corollary 6.11. *Let* $\mathcal{K}$ *be a 0–regular quasivariety with equivalence formulas* Δ_j, $j < m$. *For any finite set* Σ_0 *of binary formulas, the following conditions are equivalent:*

(i) for any $\mathbf{A} \in \mathcal{K}$ *and* $a, b \in A$, $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \langle\{\zeta^{\mathbf{A}}(a, b) : \zeta \in \Sigma_0\}\rangle_{\mathbf{A}}$;

(ii) $\mathcal{K}_{FSI} \models \forall x \forall y \; [(\bigwedge_{\zeta \in \Sigma_0} \zeta(x,y) \approx 0) \;\Leftrightarrow\; (x \approx 0 \;\; or \; y \approx 0)]$.

Moreover, $\mathcal{K}$ *has EDPM iff some finite set* Σ_0 *of binary formulas meets these conditions. In this case* $\Sigma = \{\langle \zeta(\Delta_j(x,y), \Delta_k(z,w)), 0\rangle : \zeta \in \Sigma_0; j,k < m\}$ *is a set of principal intersection equations for* $\mathcal{K}$ *and for any* $S(\mathcal{K},0)$*–filter* F *of any* $\mathbf{A} \in \mathcal{K}$*, if* $a,b \in A$ *and* $\zeta^{\mathbf{A}}(a,b) \in F$ *for all* $\zeta \in \Sigma_0$ *then*

$$F = \langle F \cup \{a\}\rangle_{\mathbf{A}} \cap \langle F \cup \{b\}\rangle_{\mathbf{A}}.$$

Proof. Let $S = S(\mathcal{K},0)$. That (i) implies (ii) is obvious. Assume (ii), define Σ as in the corollary's statement and let $\mathbf{A} \in \mathcal{K}_{\mathrm{FSI}}$ and $a,b,c,d \in A$. By (ii), $\zeta^{\mathbf{A}}(\Delta_j^{\mathbf{A}}(a,b), \Delta_k^{\mathbf{A}}(c,d)) = 0^{\mathbf{A}}$ for all $j,k < m$ and all $\zeta \in \Sigma_0$ if and only if for all $j,k < m$, $\Delta_j^{\mathbf{A}}(a,b) = 0^{\mathbf{A}}$ or $\Delta_k^{\mathbf{A}}(c,d) = 0^{\mathbf{A}}$, if and only if $\Delta_j^{\mathbf{A}}(a,b) = 0^{\mathbf{A}}$ for all $j < m$ or $\Delta_k^{\mathbf{A}}(c,d) = 0^{\mathbf{A}}$ for all $k < m$, if and only if $a = b$ or $c = d$ (by (22)). This proves the claim about Σ. By (ii) and Theorem 6.9, $\mathcal{K}_{\mathrm{FSI}}$ satisfies

$$\forall x \forall y \; [(\textstyle\bigwedge_{\zeta \in \Sigma_0} \zeta(x,y) \approx 0) \;\Leftrightarrow\; (\bigwedge_{\zeta \in \Sigma_0;\, j,\, k<m} \zeta(\Delta_j(x,0), \Delta_k(y,0)) \approx 0)];$$

therefore, so does $\mathcal{K}$. By definition of S, for any $\mathbf{A} \in \mathcal{K}$ and $a,b \in A$,

$$\langle\{\zeta^{\mathbf{A}}(a,b) : \zeta \in \Sigma_0\}\rangle_{\mathbf{A}} = \langle\{\zeta^{\mathbf{A}}(\Delta_j^{\mathbf{A}}(a,0^{\mathbf{A}}), \Delta_k^{\mathbf{A}}(b,0^{\mathbf{A}})) : \zeta \in \Sigma_0\}\rangle_{\mathbf{A}}.$$

By Lemma 6.10 and since Σ is a set of principal intersection equations, this S–filter is just $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}}$. Thus, (ii) implies (i).

If $\mathcal{K}$ has EDPM with an unspecified set Σ of principal intersection equations then

$$\Sigma_0 := \{\Delta_j(\varphi(x,0,y,0), \psi(x,0,y,0)) : j < m, \langle\varphi,\psi\rangle \in \Sigma\}$$

satisfies condition (ii), by Theorem 6.9 and (22). The last assertion follows from the distributivity of $\mathbf{Con}_{\mathcal{K}}\mathbf{A}$ (hence of $\mathbf{Fi}^S\mathbf{A}$) for any $\mathbf{A} \in \mathcal{K}$: if $a,b \in A$ and $F \in \mathrm{Fi}^S\mathbf{A}$ then $\langle F \cup \{a\}\rangle_{\mathbf{A}} \cap \langle F \cup \{b\}\rangle_{\mathbf{A}} = \langle F \cup (\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}})\rangle_{\mathbf{A}} = \langle F \cup \langle\{\zeta^{\mathbf{A}}(a,b) : \zeta \in \Sigma_0\}\rangle_{\mathbf{A}}\rangle_{\mathbf{A}} = F$. ∎

A (not necessarily 0–regular) quasivariety $\mathcal{K}$ with a constant formula 0 satisfying condition (i) above may be said to have *formula definable principal meets* with Σ_0 as a *set of principal intersection formulas.* We shall use Corollary 6.11 to identify some subquasivarieties of $\mathcal{H}^{\backslash}_{C^*}$ with EDPM.

By Proposition 6.5, if $\sqcap \in C^*$ then $(\mathcal{H}^{\backslash}_{C^*})_{\mathrm{FSI}}$ is a universal class axiomatized, relative to $\mathcal{H}^{\backslash}_{C^*}$, by the sentence $\forall x\,\forall y\,(x \sqcap y \approx 0 \;\Leftrightarrow\; (x \approx 0 \;\text{ or }\; y \approx 0))$. From Corollary 6.11, we deduce:

Proposition 6.12. *If $n \in \omega$ and $\sqcap \in C^*$ then $\mathcal{H}^{\backslash}_{C^*}$ has EDPM. In this case if $\mathbf{A} \in \mathcal{H}^{\backslash}_{C^*}$ and $a, b \in A$ then $\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \langle a \sqcap b\rangle_{\mathbf{A}}$.*

Since $\mathcal{H}^{\infty}_{\{\,\dot{-}\,,\,\sqcap\,\}}$ is the variety of 'lower BCK-semilattices', this proposition generalizes some results from [24]. One easily sees (e.g., using Hilbert algebras) that the class $(\mathcal{H}^{\infty}_{\{\,\dot{-}\,\}})_{\mathrm{FSI}}$ is not universal.

Example 6.13. Let $\leq$ and $\sqcap$ be the partial order and the meet operation of the nonlinearly ordered lattice on a four-element set D with incomparable elements a, b and least [resp. greatest] element 0 [resp. 1]. For $s, t \in D$, define $s \mathbin{\dot{-}} 0 = s$; $s \mathbin{\dot{-}} t = 0$ if $s \leq t$; $1 \mathbin{\dot{-}} b = a$ and $s \mathbin{\dot{-}} t = s$, otherwise. Then $\mathbf{D} := \langle D; \dot{-}\,, \sqcap\,, 0\rangle \in \mathcal{H}_{\{\,\dot{-}\,,\,\sqcap\,\}}$ is a finite subdirectly irreducible algebra whose unique nonzero proper filter is $\{0, a\}$, but its subalgebra on $\{0, a, b\}$ is not (finitely) subdirectly irreducible. Thus, $(\mathcal{H}_{\{\,\dot{-}\,,\,\sqcap\,\}})_{\mathrm{FSI}}$ is not universal, so $\mathcal{H}_{\{\,\dot{-}\,,\,\sqcap\,\}}$ lacks EDPM. Notice that $\mathbf{D}$ satisfies (20), hence also (19). In Corollaries 6.7 and 6.8, therefore, we cannot replace $\mathcal{H}^n_{C^*}$ by $\mathcal{H}_{C^*}$. The reader may also verify that $\mathbf{D}$ has a homomorphic image not in $\mathcal{H}_{\{\,\dot{-}\,,\,\sqcap\,\}}$ and that the $\langle\,\dot{-}\,,\,\sqcap\,,\,\sqcup\,,0\rangle$–expansion of $\mathbf{D}$ is not in $\mathcal{H}_{\{\,\dot{-}\,,\,\sqcap\,,\,\sqcup\,\}}$.

Obviously, any linearly ordered algebra in $\mathcal{H}_{C^*}$ is finitely subdirectly irreducible. For any subquasivariety $\mathcal{K}$ of $\mathcal{H}_{C^*}$, the class $\mathcal{K}_{\mathrm{L}}$ is axiomatized by the axioms of $\mathcal{K}$ together with the universal sentence

$$\forall x\,\forall y\,(x \mathbin{\dot{-}} y \approx 0 \;\text{ or }\; y \mathbin{\dot{-}} x \approx 0), \tag{23}$$

so $\mathcal{K}_{\mathrm{L}}$ is a universal class, i.e., it is closed under the operators I, S and $\mathrm{P_U}$. By [9, Lemma 1.5], for any class $\mathcal{M}$ of similar algebras, every nontrivial relatively finitely subdirectly irreducible member of $\mathrm{Q}(\mathcal{M})$ belongs to $\mathrm{ISP_U}(\mathcal{M})$. Thus, if a subquasivariety $\mathcal{K}$ of $\mathcal{H}_{C^*}$ is generated (as a quasivariety) by its linearly ordered members then $\mathcal{K}_{\mathrm{FSI}} = \mathcal{K}_{\mathrm{L}}$. Over $\mathcal{H}_{C^*}$, (23) clearly entails

$$\forall x\,\forall y\,[(x \dot{-} (x \dot{-} y) \approx 0 \quad\text{and}\quad y \dot{-} (y \dot{-} x) \approx 0) \;\Leftrightarrow\; (x \approx 0 \quad\text{or}\quad y \approx 0)].$$

From this and Corollary 6.11, we obtain:

Proposition 6.14. *Any quasivariety $\mathcal{K}$ of representable algebras in $\mathcal{H}_{C^*}$ (in particular, each $\mathcal{R}^{\setminus}_{\mathcal{C}^*}$) has EDPM and if $\mathbf{A} \in \mathcal{K}$ and $a, b \in A$ then*

$$\langle a\rangle_{\mathbf{A}} \cap \langle b\rangle_{\mathbf{A}} = \langle\{a \dot{-} (a \dot{-} b),\, b \dot{-} (b \dot{-} a)\}\rangle_{\mathbf{A}}.$$

Not every subvariety of $\mathcal{H}_{C^*}$ with EDPM lies within one of the classes $\mathcal{H}^n_{C^*}$: let $\mathbf{A}_{\{\,\dot{-}\,\}}$ be the $\langle\,\dot{-}\,, 0\rangle$–reduct of the algebra $\mathbf{A}$ of Example 6.3, which violates (Z_n) for all $n \in \omega$, and let $\mathcal{V}_{\{\,\dot{-}\,\}} = \mathrm{HSP}(\mathbf{A}_{\{\,\dot{-}\,\}})$. By [30, Lemma 6.2], $(\mathcal{V}_{\{\,\dot{-}\,\}})_{\mathrm{FSI}} = \mathrm{ISP_U}(\mathbf{A}_{\{\,\dot{-}\,\}})$ consists of linearly ordered algebras and satisfies

$$\forall x\,\forall y\,(x \dot{-} y \approx x \;\Leftrightarrow\; (x \approx 0 \quad\text{or}\quad y \approx 0)).$$

It follows that $\mathcal{V}_{\{\,\dot{-}\,\}} = \mathrm{Q}(\mathbf{A}_{\{\,\dot{-}\,\}})$. By the previous result, $\mathcal{V}_{\{\,\dot{-}\,\}}$ has EDPM, as has $\mathcal{V} = \mathrm{HSP}(\mathbf{A})$ ($= Q(\mathbf{A})$), and $\{x \dot{-} (x \dot{-} y)\}$ is a set of principal intersection formulas for $\mathcal{V}_{\{\,\dot{-}\,\}}$. This set may be refined for $\mathcal{V}$ to $\{x \sqcap y\}$, as $\mathcal{V}_{\mathrm{FSI}}$ clearly satisfies $\forall x\,\forall y\,(x \sqcap y \approx 0 \;\Leftrightarrow\; (x \approx 0 \;\text{ or } y \approx 0))$.

Similarly, by Proposition 6.14, the varieties $\mathcal{B}_{\mathcal{C}^*}$ of Example 6.4 (which are in no $\mathcal{H}^n_{\mathcal{C}^*}$) have EDPM. Since $\mathcal{B}_{\mathcal{C}^*}$ is finitely generated, its principal intersection formulas and the methods of [3] can be employed to construct a finite equational basis for $\mathcal{B}_{\mathcal{C}^*}$. An application of Corollary 6.2 to $\mathcal{B}_{\{\,\dot{-}\,\}}$ shows that subvarieties of $\mathcal{H}_{C^*}$ with EDPM need not have the CEP.

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Department of Mathematics and Applied Mathematics
University of Natal
King George V Avenue, Durban 4001
South Africa
&
Department of Mathematics and Applied Mathematics
University of Natal
King George V Avenue, Durban 4001
South Africa

e-mail: alten@scifs1.und.ac.za
&
raftery@scifs1.und.ac.za