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# THEOREMS OF PEANO ARITHMETIC ARE BURIDAN-VOLPIN RECURSIVELY SATISFIABLE

Abstract. The notion of recursive satisfaction is extended from prenex $\forall\exists$ arithmetic sentences to any first-order arithmetic sentence by allowing the scope of a negative (existential) quantifier to depend on positive (universal) quantifiers which may lie within its scope.

## 1. Introduction

An immediate application of the techniques Gentzen used in his 1938 consistency proof "Neue Fassung der Widerspruchsfreiheitsbeweises für die reiner Zahlentheorie" (translated in Gentzen [1, pp 252-286] as "New Version of the Consistency Proof for Elementary Number Theory") was to show that if $(x)(Ey)F(x, y)$ is an $\forall\exists$-theorem of Peano arithmetic (first-order arithmetic) there is an ($\epsilon_0$-) recursive function $f$ so that for every number $n$, $F(\overline{n}, \overline{f(n)})$ is true (here $\overline{n}$ is the term denoting $n$ in the system.) Because $(Ey)(x)G$ logically implies, $(x)(Ey)G$ we can immediately extend

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this to conclude that if $(x_1)(Ey_1)\cdots(x_k)(Ey_k)F(x_1,y_1,\ldots,x_k,y_k)$ is any theorem of Peano arithmetic there are ($\epsilon_0$-) recursive functions $f_1,\ldots,f_k$ so that $F(\overline{n_1},\overline{f_1(n_1,\ldots,n_k)},\ldots,\overline{n_k},\overline{f_k(n_1,\ldots,n_k)})$ is true for any natural numbers $n_1,\ldots,n_k$.

In this paper we establish a modest generalization of this result. Given *any* first-order arithmetic theorem $\mathcal{T}$ we will show that finite ranges for the negative quantifiers can be determined via ($\epsilon_0$-) recursive functions from finite ranges for the positive quantifiers, and that if the quantifiers of $\mathcal{T}$ are expanded over these ranges, a true arithmetic formula results.The present proof is of some independent interest because it shows that to describe an existential quantifier as being functionally dependent on the universal quantifiers preceding it in the prefix (as one does , for example, with Skolem functions) is inappropriate in the case of arithmetic *theorems.* In this situation, an existential quantifier is functionally dependent on those quantifiers with which it has become "referentially involved" as a result of the inference steps of the derivation [1], and these may include quantifiers which lie within its scope in the final (rather arbitrary) linear form which the theorem finally achieves.

Our results are obtained by making slight modifications to the methods of Gentzen's proof (with which we will assume the reader is familiar.) It will be shown that any derivation in a (sequent version of) first-order arithmetic can be transformed (reduced) to a derivation where cuts occur only on quantifier-free subformulas of the endsequent. However, this "quantifier-normal" derivation may include instances of the "bounded quantifier inferences"

$$\begin{array}{cc}
\textbf{B}\forall\textbf{R} & \textbf{BEL} \\
\dfrac{\Gamma \Longrightarrow \Delta, F(\overline{n_1}) \wedge \cdots \wedge F(\overline{n_k})}{\Gamma \Longrightarrow \Delta, (x)F(x)} & \dfrac{F(\overline{n_1}) \vee \cdots \vee F(\overline{n_k}), \Gamma \Longrightarrow \Delta}{(Ex)F(x), \Gamma \Longrightarrow \Delta}
\end{array}$$

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[1] These reference relations are studied in Isles [3]

The existence of such a derivation for a theorem $\mathcal{T}$ of arithmetic is obtained by means of certain transformations on derivations ("reduction steps"). These steps define ($\epsilon_0$ -) recursive functions which, when applied to given finite ranges for the positive (universal) quantifiers in $\mathcal{T}$ yield finite ranges for the negative (existential) quantifiers of $\mathcal{T}$. When the quantifiers of $\mathcal{T}$ are expanded over these ranges the result is a true quantifier-free formula[2].

## 2. Rewriting Gentzen's Proof: Basic Definitions

We introduce the usual definitions for a sequent calculus of first-order arithmetic and define certain structural features of its derivations. Many of the definitions are meant to highlight the connection between different *occurrences* of a quantifier in a derivation.

**Definition 1.** Terms, formulas, and sequents of P (Peano arithmetic.)

1. The language of P is specified by giving logical connectives: $\wedge$, $\vee$, $\neg$, existential quantifier ($E$ ), and universal quantifier ( ); a constant symbol 0; one 1-place function symbol | (successor); two 3-place relation letters $A$ (addition), and $M$ (multiplication); and equality =. In addition there are denumerably many

   *variables:* $x, y, \cdots, x_1, y_1, \cdots$ (*"bound variables"*)

   and

   *parameters:* $a, b, \cdots, a_1, b_1, \cdots$ (*"free variables"*)

2. The set of "pseudo-terms" is defined in the usual way. "Terms" are pseudo terms which contain no variables. Notice that the only terms of the system are of the form 0, $a$ (parameter), $0|\cdots|$, or $a|\cdots|$ with n, $0 \leq n$, occurrences of |. $\overline{n}$ is the abbreviation for "numerals" which are terms of the form $0\underbrace{|\cdots|}_{n}$, $1 \leq n$ ($\overline{0} = 0$.)

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[2] This is a special case of the "Buridan-Volpin" interpretations studied in Isles [3]

3. The set of "pseudo-formulas" is defined in the usual manner with $(Ex)F$ and $(x)F$ being the notation for existentially and universally quantified formulas. "Formulas" are pseudo-formulas where all variables are bound. Note: A formula may have parameters.
4. The "degree" of a (pseudo-)formula is the number of logical connectives occurring within it.
5. Two formulas are "equiform" if they differ only in the naming of their bound variables.
6. $F[x/t]$ (or $F(t)$) denotes the result of replacing $x$ at all of its free occurrences in the pseudo-formula $F$ by the term $t$.
7. The "(pseudo-)subformulas" of $F$ are defined in the usual way.
8. Let $\Gamma$, $\Delta$ be sequences of finitely many (perhaps 0) formulas. The configuration $\Gamma \Longrightarrow \Delta$ is called a "sequent". $\Gamma$ is called its "antecedent" and $\Delta$ its "consequent."

**Definition 2.** Inference rules and derivations. Derivations are trees of sequents built up inductively from initial sequents using the following inference rules. Except for cuts, the rules are divided into *Left*-rules (indicated by –**L**) and *Right*-rules (indicated by –**R**). $\Gamma$, $\Delta$, $\Theta$, $\Sigma$ are sequences of formulas, $F$ and $G$ are formulas, and $D_i$ is a derivation.

1. Initial sequents are of the form $F \Longrightarrow F$. It is convenient (and no restriction) to take $F$ as quantifier-free.
2. Structural inferences.

   (a) Permutations. $$\overset{\textbf{PL}}{\dfrac{\Gamma, F, G, \Delta \Longrightarrow \Theta}{\Gamma, G, F, \Delta \Longrightarrow \Theta}} \quad \overset{\textbf{PR}}{\dfrac{\Gamma \Longrightarrow \Delta, F, G, \Theta}{\Gamma \Longrightarrow \Delta, G, F, \Theta}}$$

   (b) Weakenings. $$\overset{\textbf{WL}}{\dfrac{\Gamma \Longrightarrow \Delta}{F, \Gamma \Longrightarrow \Delta}} \quad \overset{\textbf{WR}}{\dfrac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, F}}$$

   No parameter of $F$ is an eigenparameter (see section 5a of this definition) of the derivation to that point.

(c) Contractions.
$$\begin{array}{cc} \mathbf{CL} & \mathbf{CR} \\ \dfrac{F_1, F_2, \Gamma \Longrightarrow \Delta}{F, \Gamma \Longrightarrow \Delta} & \dfrac{\Gamma \Longrightarrow \Delta, F_1, F_2}{\Gamma \Longrightarrow \Delta, F} \end{array}$$

Here $F_1$, $F_2$, and $F$ are equiform.

3. Cuts.
$$\dfrac{\overset{D_1}{\Gamma \Longrightarrow \Delta, F_1} \quad \overset{D_2}{F_2, \Theta \Longrightarrow \Sigma}}{\Gamma, \Theta \Longrightarrow \Delta, \Sigma}$$

The equiform formulas $F_1$, $F_2$ are the "cut formulas." No parameter of $D_1$ is an eigenparameter of $D_2$ and conversely. The "cut degree" is the degree of its cut formulas.

4. Propositional inferences.

(a) Negation inferences.
$$\begin{array}{cc} \neg\mathbf{L} & \neg\mathbf{R} \\ \dfrac{\Gamma \Longrightarrow \Delta, F}{\neg F, \Gamma \Longrightarrow \Delta} & \dfrac{F, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \neg F, \Delta} \end{array}$$

(b) Conjunction inferences.

i. Left inferences.
$$\begin{array}{cc} \wedge\mathbf{L_1} & \wedge\mathbf{L_2} \\ \dfrac{G, \Gamma \Longrightarrow \Delta}{G \wedge F, \Gamma \Longrightarrow \Delta} & \dfrac{G, \Gamma \Longrightarrow \Delta}{F \wedge G, \Gamma \Longrightarrow \Delta} \end{array}$$

ii. Right inference ($\wedge$**R**).
$$\dfrac{\overset{D_1}{\Gamma \Longrightarrow \Delta, F} \quad \overset{D_2}{\Theta \Longrightarrow \Sigma, G}}{\Gamma, \Theta \Longrightarrow \Delta, \Sigma, F \wedge G}$$

(c) Disjunction inferences.

i. Left inference ($\vee$**L**.)
$$\dfrac{\overset{D_1}{F, \Gamma \Longrightarrow \Delta} \quad \overset{D_2}{G, \Theta \Longrightarrow \Sigma}}{F \vee G, \Gamma, \Theta \Longrightarrow \Delta, \Sigma}$$

ii. Right inferences.
$$\begin{array}{cc} \vee\mathbf{R_1} & \vee\mathbf{R_2} \\ \dfrac{\Gamma \Longrightarrow \Delta, G}{\Gamma \Longrightarrow \Delta, G \vee F} & \dfrac{\Gamma \Longrightarrow \Delta, G}{\Gamma \Longrightarrow \Delta, F \vee G} \end{array}$$

In $\wedge$**L** and $\vee$**R** no parameter of $F$ is an eigenparameter of the derivation to that point. In $\wedge$**R** and $\vee$**L** no eigenparameter of $D_1$ is an eigenparameter of $D_2$ (and conversely.)

5. Quantifier inferences.

(a) Logical quantifier inferences.

i. Existential inferences.
$$\textbf{EL}\quad \frac{F[x/a], \Gamma \Longrightarrow \Delta}{(Ex)F, \Gamma \Longrightarrow \Delta} \qquad \textbf{ER}\quad \frac{\Gamma \Longrightarrow \Delta, F[x/t]}{\Gamma \Longrightarrow \Delta, (Ex)F}$$

ii. Universal inferences.
$$\forall\textbf{L}\quad \frac{F[x/t], \Gamma \Longrightarrow \Delta}{(x)F, \Gamma \Longrightarrow \Delta} \qquad \forall\textbf{R}\quad \frac{\Gamma \Longrightarrow \Delta, F[x/a]}{\Gamma \Longrightarrow \Delta, (x)F}$$

In $\forall$**R** and **EL**, $a$ is a parameter called the "eigenparameter" of the inference. It does not occur in the sequent which is the conclusion of the inference.

(b) Bounded quantifier inferences.

$$\textbf{BEL}\quad \frac{F_1(\overline{n_1}) \vee \cdots \vee F_k(\overline{n_k}), \Gamma \Longrightarrow \Delta}{(Ex)F, \Gamma \Longrightarrow \Delta} \qquad \textbf{B}\forall\textbf{R}\quad \frac{\Gamma \Longrightarrow \Delta, F_1(\overline{n_1}) \wedge \cdots \wedge F_k(\overline{n_k})}{\Gamma \Longrightarrow \Delta, (x)F}$$

Here $F_1(x)$, $F_2(x)$, and $F$ are equiform. $\{n_1, \ldots, n_k\}$ is the "range" of the inference.

6. Induction inference. $\dfrac{F_1(a), \Gamma \Longrightarrow \Delta, F_2(a|)}{F_1[a/0], \Gamma \Longrightarrow \Delta, F_3[a/t]}$ The parameter $a$ does not occur in the concluding sequent. The formulas $F_1(a)$, $F_2(a)$, and $F_3(a)$ are all equiform. The "degree of the inference" is the degree of $F_i(a)$.
7. The upper sequent in the preceding inferences is called the "premise sequent" and the lower sequent is the "concluding sequent." The distinguished formula(s) ($F$, $G$, $F \vee G$, $F_3(t)$, etc.) in the concluding sequent (except for cut) is(are) called the "principal formula(s)" of the inference. The distinguished formula(s) of the premise sequent(s) is (are) called the "secondary formula(s)." All other formulas, that is, those in $\Gamma$, $\Delta$, $\Theta$, $\Sigma$ are called "side formulas." Inferences given in clauses 4 and 5 are "logical inferences."
8. A "derivation in P" is a tree of sequents which may contain any of the above inferences except for bounded quantifier inferences. "A derivation in PB" may also have bounded quantifier inferences. In indicating derivations the notation $\genfrac{}{}{2pt}{}{\Gamma \Longrightarrow \Delta}{\Theta \Longrightarrow \Sigma}$ will indicate that a sequence of structural inferences occurs between $\Gamma \Longrightarrow \Delta$ and $\Theta \Longrightarrow \Sigma$.

9. Let $\Gamma \Longrightarrow \Delta$ be the last sequent of a derivation in P. Then $\Delta$ is called a "theorem of Peano arithmetic"if $\Gamma = \langle A_1, \dots, A_k \rangle$ where $A_i$ is a formula equiform to one in the following list ($\mathcal{F}$ stands for either $A$ or $M$.)
   (a) $(x)\neg[x| = 0]$
   (b) $(x)(y)[\neg(x = y) \vee (x| = y|)]$
   (c) $(x)(y)[\neg(x| = y|) \vee (x = y)]$
   (d) $(x)(y)(z)[\neg(x = y) \vee [\neg(x = z) \vee (y = z)]]$
   (e) $(x)(y)(z)(w)[\neg(x = y) \vee [\neg\mathcal{F}(x, z, w) \vee \mathcal{F}(y, z, w)]]$
   (f) $(x)(y)(z)(w)[\neg(x = y) \vee [\neg\mathcal{F}(z, x, w) \vee \mathcal{F}(z, y, w)]]$
   (g) $(x)(y)(z)(w)[\neg(x = y) \vee [\neg\mathcal{F}(z, w, x) \vee \mathcal{F}(z, w, y)]]$
   (h) $(x)A(x, \overline{1}, x|)$
   (i) $(x)(y)(z)[\neg A(x, y, z) \vee A(x, y|, z|)]$
   (j) $(x)M(x, \overline{1}, x)$
   (k) $(x)(y)(z)(w)[\neg M(x, y, z) \vee [\neg A(z, x, w) \vee M(x, y|, w)]]$

*From this point on, unless explicitly mentioned, "derivation" will mean a derivation in PB.* Because of the restrictions on parameters given in the inference rules, any derivation will have the "pure variable property", that is, an eigenparameter will not occur anywhere in a derivation except above the quantifier inference ($\forall$**R** or **EL**) with which it is associated.

**Definition 3.** A derivation is called "quantifier-normal" or "Q-normal" if the only cut-formulas in it are quantifier-free.

**Definition 4.** Following Gentzen, we define a "path" beginning at a sequent $S_1$ in a derivation as a sequence of sequents which we must run through in passing from sequent $S_1$ to the last sequent (or "endsequent") E. If $\langle S_1, \dots, E \rangle$ is a path then $S_i$ is "above" $S_j$ (and $S_j$ is"below" $S_i$) if $i < j$.

**Definition 5.** We extend Gentzen's notion of "clustered" formulas slightly.

1. Secondary formulas in the premise sequent are "clustered with" principal formulas in the concluding sequent (and conversely.) Side formulas

in the premise sequent are "clustered with" formulas in the corresponding location in the concluding sequent. Notice that a cut-formula is not clustered with any formula below it nor is the principal formula of a weakening clustered with any formula above it.

2. Let $F_i$ be a formula in a sequent $S_i$ where $\langle S_1, \cdots, S_i, \cdots, S_k \rangle$ is a path and $F_i$, $F_{i+1}$, $1 \leq i < k$ are clustered. Then $\langle F_1, \cdots, F_k \rangle$ is called a "formula thread" containing $F_i$ which "impinges on" $S_i$.
3. If $F$ is a formula at a given location in a derivation, there will, in general, be many threads containing $F$. A "maximal thread" $\langle F_1, \cdots, F_k \rangle$ containing $F$ is a thread where $F_1$ is not clustered with any formula above it, nor is $F_k$ clustered with any formula below it. *Notice that $F_1$ is either in an initial sequent or is the principal formula of a weakening and that $F_k$ is either a cut formula or is in the endsequent.*
4. The "formula cluster associated with" a particular occurrence of a formula $F$ in a derivation is the tree consisting of threads terminating in $F$ and beginning with either a formula in an initial sequent or the principal formula of a weakening.

This notion of formula cluster is more general than that in Gentzen. Because he only considers formula threads which impinge on structural inferences, all members of one of his threads are equiform. In this definition, a formula may be clustered with a subformula of another. Notice that as one ascends a path, a thread in that path may bifurcate at a contraction. Of course, of two given occurrences of a formula in a derivation, one may belong to a given formula thread and the other may not.

**Definition 6.** Quantifier threads.

1. Let $\langle F_1, \cdots, F_k \rangle$ be a formula thread. If $(Qx_i)$ is an occurrence of a quantifier in $F_i$, it is said to be "clustered with" the quantifier in the corresponding location in $F_{i+1}$ (and conversely.). A sequence $\langle (Qx_1), \ldots, (Qx_n) \rangle$ is called a "quantifier thread" if the formulas $F_i$, $1 \leq i \leq n$, to which $(Qx_i)$ belong constitute a formula thread and $(Qx_i)$ is clustered with $(Qx_{i+1})$.

2. A quantifier thread $\langle (Qx_1), \ldots, (Qx_n) \rangle$ is said to "impinge on"an occurrence of a formula $G$ if the occurrence of$(Qx_i)$ is in $G$. It "impinges on" an occurrence of a sequent $S$ if it impinges on a formula in $S$.
3. A "maximal quantifier thread" $\langle (Qx_1), \ldots, (Qx_n) \rangle$ containing a particular occurrence of $(Qx_i)$ is one where $(Qx_1)$ is not clustered with any quantifier above it, nor is $(Qx_n)$ clustered with any below it. *Notice that* $(Qx_1)$ *will either be in the principal formula of a weakening, conjunction left, or disjunction right, or will be the quantifier introduced in the principal formula of a quantifier inference.* $(Qx_n)$ *will either be in the endsequent or will be contained in a cut formula.*
4. The "quantifier cluster" associated with an occurrence of $(Qx)$ in a derivation is the tree of quantifier threads terminating in $(Qx)$ at that occurrence and beginning in a weakening or with the principal formula of a logical inference.

**Example:** In the following (partial) derivation, the left-hand quantifier thread of the quantifier occurrence of $(x)(D(x) \vee (Ey)H(y))$ in the endsequent begins at a $\forall \mathbf{R}$ and the right-hand quantifier thread of the same occurrence begins at **WR**. The left-hand quantifier thread of the quantifier occurrence of $(Ey)H(y)$ in the endsequent begins at a $\vee \mathbf{R_1}$ and the right-hand quantifier thread also begins at **WR**.

$$\dfrac{\dfrac{\dfrac{\dfrac{G \Longrightarrow D(a)}{G \Longrightarrow (D(a) \vee (Ey)H(y))}}{G \Longrightarrow (x)(D(x) \vee (Ey)H(y))} \qquad \dfrac{\dfrac{\neg G \Longrightarrow J}{\neg G \Longrightarrow J, (x)(D(x) \vee (Ey)H(y))}}{\neg G \Longrightarrow (x)(D(x) \vee (Ey)H(y)), J}}{G \vee \neg G \Longrightarrow (x)(D(x) \vee (Ey)H(y)), (x)(D(x) \vee (Ey)H(y)), J}}{G \vee \neg G \Longrightarrow (x)(D(x) \vee (Ey)H(y)), J}$$

## 3. Ordinal Assignments and the Reduction Process

Gentzen's ordinal assignments can be used. The key observation is that his notion of "level" in a derivation depends only on the degree of cut-formulas (and inductions.) But the occurrence of a formula (or subformula) in a cut-formula means that that occurrence of the formula (or subformula) will not have a "corresponding occurrence " in the endformula (although an equiform formula or subformula may well occur in the endformula.) We make this precise in the next definition.

**Definition 7.** Let $D$ be a derivation.

1. Let $\langle F_1, \cdots, F_k \rangle$ be the unique formula thread which begins with $F_1$ and terminates in either the endsequent or a cut-formula. $F_1$ is called an "essential formula occurrence" in $D$ (or an "essential formula" of $D$) if $F_k$ belongs to the endsequent. $F_1$ is called an "inessential formula occurrence" in $D$ (or an "inessential formula" of $D$) if $F_k$ is a subformula of a cut-formula.
2. An initial sequent $A \Longrightarrow A$ is called "inessential" if both occurrences of $A$ are inessential. Otherwise, it is essential.
3. An inference is called "essential" if its principal formula is essential. Otherwise, it is "inessential".

*Unless otherwise indicated, from this point on all bounded quantifier inferences in any derivation we consider will be essential.*

**Definition 8.** Let $\langle S_1, \cdots, E \rangle$ be a path in a derivation $D$ with endsequent $E$. The "level of $S_1$" at that occurrence in $D$ is the highest level of any cut or induction inference whose lower sequent is below $S_1$ (and the level of $S_1$ is 0 if there are no such inferences.) Notice that if $S_1$ is a premise sequent to a cut then the level of $S_1$ is greater than or equal to its cut degree.

The assignment of ordinals below $\epsilon_0$ to derivations is defined as in Gentzen [1 p.279] and will not be repeated here.

**Definition 9.** Let $D$ be a derivation. The "ending" of $D$ consists of the tree of sequents that are encountered as one ascends any path from the endsequent until one reaches either an initial sequent or the concluding sequent of an inessential logical inference (the principal formula of which therefore lies on a formula thread which terminates in a cut-formula.)

Notice that the top sequents of an ending consist of initial sequents and the concluding sequents of inessential logical inferences ("inessential logical top-sequents".) Below these are located cuts, structural inferences, induction inferences, essential propositional and quantifier inferences, and the endsequent.

In his paper, Gentzen describes a sequence of transformations on derivations (so-called "reduction steps".) After a finite number of such reductions, the original derivation is transformed to one with a certain structure. Before applying these reductions, we must now add a preparatory step which removes all essential occurrences of the logical quantifier inferences **EL** or $\forall$**R** from the derivation.

**Definition 10.** Positive and negative occurrences of quantifiers.

1. Occurrences of a quantifier $(Qx)$ at a "positive" or "negative location" in a formula $F$.

   (a) If $F = (Qx_i)G$ and $x = x_i$, then $(Qx)$ occurs at a positive location in $F$. If $x \neq x_i$, then if $(Qx)$ occurs at a positive (negative) location in $G$, it occurs at a positive (negative) location in $F$.

   (b) If $F = \neg F_1, F_2 \vee F_3, F_2 \wedge F_3$ then if the occurrence of $(Qx)$ is at a positive (negative) location in $F_1$, it is at a negative (positive) location in $F$; if the occurrence is at a positive (negative) location in $F_2$ or $F_3$, it is at a positive (negative) location in $F$.
2. If an occurrence of $(x)$ (or $(Ex)$) is at a positive (or negative) location in $F$, that is said to be a "positive occurrence". If an occurrence of $(x)$ (or $(Ex)$) is at a negative (or positive) location in $F$, that is said to be a "negative occurrence".
3. If $(Qx)$ has a positive (negative) occurrence in $\Delta$ or a negative (positive) occurrence in $\Gamma$, then it has a "positive (negative) occurrence in

$\Gamma \Longrightarrow \Delta$". (Notice that a given quantifier may have both positive and negative occurrences in a formula, sequent, or derivation [3].)

Let $\langle S_1, \cdots, S_k \rangle$ be a path in a derivation $D$ and $\langle (Qx_1). \cdots, (Qx_k) \rangle$ be a quantifier thread where each occurrence of $(Qx_i)$ is in $S_i$. Then if one $(Qx_{i_0})$ has a positive (respectively negative) occurrence in $S_{i_0}$ every other $(Qx_j)$ has a positive (respectively negative) occurrence in $S_j$, $1 \leq j \leq k$. Hence if this is a maximal thread containing $(Qx_{i_0})$ the quantifier occurrence $(Qx_1)$ must either lie in the principal formula of a weakening, $\wedge$**L**, or $\vee$**R**, or be the introduced quantifier in the principal formula of a quantifier inference $\forall$**R**, **EL**, **B**$\forall$**R**, or **BEL** (respectively $\forall$**L** or **ER**.) In particular, each maximal quantifier-thread of the quantifier cluster of any positive quantifier in the endsequent $E$ of a derivation in P must begin either with the principal formula of a weakening, $\wedge$**L**, or $\vee$**R**, or with the quantifier introduced in quantifier inference $\forall$**R** (if the quantifier is $(x)$) or **EL** (if it is $(Ex)$.) Conversely, the quantifier introduced in either of these two logical quantifier inferences at any location in $D$ must be the initial quantifier occurrence in a quantifier thread which terminates at a positive quantifier occurrence in $E$ *if these inferences are essential.*

*The Reduction Steps* Let a derivation $D$ in P be given. In his paper, Gentzen defines four reduction steps or derivation rewritings which, in the present context, can be described as follows:

1. Removal of induction inferences from the ending.
2. Removal of inessential weakenings from the ending.
3. Removal of inessential initial sequents from the ending.
4. Removal from the ending of inessential propositional and logical quantifier inferences whose concluding sequent is a top sequent of the ending.

However, before these reduction transformations (particularly the induction reduction) are carried out, one must first remove various parameter occur-

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[3] This ambiguity can be avoided by using rectified formulas. See Isles [3].

rences from the derivation and eliminate all essential **EL** or $\forall$**R** inferences (this last is the main addition to Gentzen's procedure.)

*The preparatory transformation* First replace any parameter in the endsequent by a numeral and carry back the substitution throughout $D$. Assign to each positive occurrence of a quantifier $(Qx)$ of $E$ a finite set of natural numbers $r(x) = U = \{n_1, \ldots, n_k\}$ (with possibly a different set of numbers for each occurrence of $(Qx)$). Consider an *uppermost* occurrence of an *essential* $\forall$**R** or **EL** inference in the derivation. For example,

$$\begin{array}{c} D(a) \\ \Theta \Longrightarrow \Sigma, G(a) \\ \hline \Theta \Longrightarrow \Sigma, (y)G \end{array}$$

The eigenvariable $a$ does not occur below the concluding sequent. The unique quantifier thread which begins with $(y)$ and ends in $E$, terminates at a positive quantifier occurrence $(y^*)$ of $E$. Suppose $r(y^*) = \{n_1, \ldots, n_k\}$. Substitute $\overline{n_i}$ for $a$ in derivation $D(a)$ to obtain k derivations $D_i[a/\overline{n_i}]$ , $1 \leq i \leq k$. We can assume (and this will be true of similar moves in the other reduction steps) that no parameter of $D_i(\overline{n_i})$ is an eigenparameter of $D_j(\overline{n_j})$, $i \neq j$ and conversely. The sequent $\Theta \Longrightarrow \Sigma, (x)G$

is then recovered as follows:

$$\begin{array}{c} D_1(\overline{n_1}) \qquad\qquad\qquad D_2(\overline{n_2}) \\ \Theta_1 \Longrightarrow \Sigma_1, G_1(\overline{n_1}) \qquad \Theta_2, \Longrightarrow \Sigma_2, G_2(\overline{n_2}) \\ \hline \Theta_1, \Theta_2 \Longrightarrow \Sigma_1, \Sigma_2, G_1(\overline{n_1}) \wedge G_2(\overline{n_2}) \\ \hline\hline \Theta \Longrightarrow \Sigma, G_1(\overline{n_1}) \wedge G_2(\overline{n_2}) \\ \vdots \\ \Theta \Longrightarrow \Sigma, G_1(\overline{n_1}) \wedge \cdots \wedge G_k(\overline{n_k}) \\ \hline \Theta \Longrightarrow \Sigma, (x)G \end{array}$$

The last inference step is an instance of **B**$\forall$**R**. Because this was an *uppermost* essential logical quantifier inference, the result of this step is to reduce the total number of essential $\forall$**R** or **EL** inferences in the derivation by one. Thus all such essential inferences can be removed from the derivation.

Once this preparatory step has been carried out, the reduction steps can be carried out in a manner almost identical to that described in Gentzen ([2 pp. 264–274].) Each step has the effect of either lowering the derivation ordinal or of leaving the derivation ordinal unchanged while reducing the size of the derivation. It should be pointed out, that when, during the induction reduction, one deals with an induction inference in the ending

$$\frac{F_1(a), \Gamma \Longrightarrow \Delta, F_2(a|)}{F_1(0), \Gamma \Longrightarrow \Delta, F_3(t)}$$

the term $t$ may contain a parameter $b$: $t(b) = b| \cdots |$. However, in the derivation below the concluding sequent $b$ *will not be the eigenvariable of any quantifier inference* because all occurrences of essential logical quantifier inferences that involve eigenvariables have been removed from the derivation and no occurrences of inessential inferences ∀**R** or **EL** lie below this sequent. Consequently a numeral, for example 0, can be substituted for $b$ throughout the derivation without disrupting any inference and without affecting the endsequent. If $t(0) = \overline{m}$, the induction inference can then be replaced by $m$ cuts.

## 4. BV Recursive Satisfaction

**Theorem 1.** *Let $D$ be derivation in $P$ of a sequent $E$ where $E$ has no parameters. Given an assignment of finite ranges to each positive quantifier in $E$, a finite number of reduction steps transforms $D$ to a derivation $D_N$ in PB of $E$. $D_N$ has no parameters and includes only*

1. *essential initial sequents,*
2. *structural inferences (with no inessential weakenings),*
3. *essential propositional inferences,*
4. *essential* ∀**L** *or* **ER** *inferences, or*
5. *essential* **B∀R** *or* **BEL** *inferences.*

*Furthermore, the only cut-formulas occurring in $D_N$ are quantifier-free subformulas of formulas in $E$.*

**Proof.** Each reduction step can be applied only finitely often. The preparatory step removes all essential ∀**R** or **EL** inferences from the derivation and none of the subsequent reductions reintroduces such inferences.

After the preparatory step, one carries out reductions 1 through 3 each time doing so until the particular reduction step can no longer be applied. At this point, one removes an inessential propositional or quantificational top-sequent from the ending (step 4). This will, in general, bring more induction inferences into the ending and one must return to step 1 again. Eventually, none of the reductions can be applied. At that point, the ending of the derivation is the whole derivation. There are no inessential propositional inferences, no inessential quantificational inferences, no essential $\forall$**R** or **EL** inferences, and no inessential initial sequents or weakenings. As there are no eigenparameters, all parameters can be replaced by a numeral. If a cut occurs, any formula thread which terminates at one of the cut formulas $G$ must consist of equiform formulas and must originate at an essential initial sequent $G \Longrightarrow G$. Thus $G$ is a subformula of some formula in $E$. ■

One of the effects of the preparatory step is to connect every positive quantifier occurrence in $E$ via a quantifier-thread with zero or more bounded quantifier inferences *all with the same range.* Inspection of the reconstructions which occur during the subsequent reduction steps shows that although some of these threads may be eliminated or multiplied, the bounded quantifier inferences to which a given positive quantifier occurrence in $E$ is connected *will still all have the same range.* Therefore in the normal derivation $D_N$ the following are true:

1. Each quantifier occurrence in $D_N$ lies on a quantifier-thread which connects it to a quantifier occurrence in $E$.
2. Each positive quantifier occurrence in $E$ is the terminus of quantifier-threads which begins either at the principal formula of a weakening, $\wedge$**L**, $\vee$**R**, or at the introduced quantifier of one or more occurrences of a bounded quantifier inference (**B**$\forall$**R** if $(Qx) = (x)$ and **BEL** if $(Qx) = (Ex)$) *all of which have the same ranges.*
3. Each negative quantifier occurrence in $E$ is the terminus of quantifier-threads which begin either at the principal formula of a weakening,

$\wedge$**L**, $\vee$**R** or at the introduced quantifier of one or more occurrences of an $\forall$**L** inference (if $(Qy) = (y)$) or an **ER** inference (if $(Qy) = (Ey)$).

**Definition 11.** Let $D_N$ be the derivation of $E$ described in Theorem 1. Suppose $E$ has positive quantifier occurrences $(Qx_i)$ $1 \leq i \leq k$, negative quantifier occurrences $(Qy_j)$ $1 \leq j \leq d$, and that ranges $r(x_i) = U_i$ have been assigned to each positive quantifier occurrence. If $(Qy_j) = (y_j)$ look at all instances of $\forall$**L** inferences (or **ER** inferences if $(Qy_j) = (Ey_j)$) which initiate the quantifier cluster of $(Qy_j)$. Suppose these infererence occurrences are $\dfrac{F_l(t_l), \Gamma_l \Longrightarrow \Delta_l}{(z_l)F_l, \Gamma_l \Longrightarrow \Delta_l}$ $1 \leq l \leq s$. Notice that all of the terms $t_l$ are numerals. Define $r(y_j) = S_j = \{t_1, \ldots, t_s\}$ (or $r(y_j) = \{0\}$ if there are no such inferences.)

**Definition 12.** Suppose $F$ is a formula all of whose quantifier occurrences have been assigned finite ranges. Then $F^*$, the "Buridan-Volpin (BV) propositional expansion of $F$" over these ranges, is defined by expanding each quantifier occurrence over its range. More precisely:

1. If $F$ is quantifier-free, $F^* = F$.
2. If $F = F_1 \wedge F_2, F_1 \vee F_2, \neg F_1$ then $F^* = F_1^* \wedge F_2^*, F_1^* \vee F_2^*, \neg(F_1^*)$.
3. If $r(x) = \{\overline{n_1}, \ldots, \overline{n_k}\}$ and $F = (x)G$ then $G^*[x/\overline{n_1}] \wedge \cdots \wedge G^*[x/\overline{n_k}] = F^*$; if $F = (Ex)G$ then $F^* = G^*[x/\overline{n_1}] \vee \cdots \vee G^*[x/\overline{n_k}]$.
4. If $D$ is a derivation, then $D^*$ is the derivation obtained by replacing each formula in $D$ by its BV expansion.

**Corollary 1.** *Let $D_N$ be the quantifier-normal derivation of Theorem 1 with an assignment of ranges as prescribed in Definition 11. $D_N^*$ is a propositional derivation of $E^*$ in the quantifier-free portion of $P$.*

**Proof.** Apply * first to the initial sequents and then downwards along a path towards $E$. All propositional and structural inferences remain undisturbed. A contraction $\dfrac{\Gamma \Longrightarrow \Delta, F_1, F_2}{\Gamma \Longrightarrow \Delta, F}$ becomes $\dfrac{\Gamma \Longrightarrow \Delta, F_1^*, F_2^*}{\Gamma \Longrightarrow \Delta, F^*}$ which is a contraction because the formulas $F_1^*, F_2^*, F^*$ are identical. This is so because in $D_N$ they are equiform and clustered quantifiers in $F_i$ and $F$ are connected by quantifier-threads to the *same* quantifier occurrence in $E$. For the same reason, the *-transformation of a **B∀R** or **BEL** inference remains a legitimate inference (and, in fact. becomes superfluous.) An inference $\dfrac{\Gamma \Longrightarrow \Delta, F(t)}{\Gamma \Longrightarrow \Delta, (Ex)F}$ becomes

$$\frac{\Gamma^* \Longrightarrow \Delta^*, F^*(t)}{\Gamma^* \Longrightarrow \Delta^*, F^*(t_1) \vee \cdots \vee F^*(t) \vee \cdots \vee F^*(t_k)}$$

and this inference can be reconstructed propositionally. ∎

**Definition 13.** A (closed) formula $F$ of P with positive quantifier occurrences $(Qx_i)$ $1 \leq i \leq k$ and negative quantifier occurrences $(Qy_j)$ $1 \leq j \leq r$ is called "Buridan- Volpin (BV) recursively satisfiable" if there are ($\epsilon_0$-) recursive functions $\Phi_1, \ldots .\Phi_r$ so that for any assignment of finite sets $U_i$ to $(x_i)$, $1 \leq i \leq k$, and $\Phi_j(U_1, \ldots, U_k)$ to $(Qy_j)$, $1 \leq j \leq r$, the BV propositional expansion of $F$ over these ranges is true.

**Theorem 2.** *Let $\Delta$ be a theorem of Peano arithmetic. Then $\Delta$ is BV recursively satisfiable.*

**Proof.** Let $D$ be a derivation of $\Gamma \Longrightarrow \Delta$ in P where the formulas in $\Gamma$ are arithmetic axioms. The only positive quantifier occurrences in $\Gamma \Longrightarrow \Delta$ are in formulas of $\Delta$. Given an assignment of finite ranges $\{U_i\}$ to these, $D$ can be transformed to the quantifier-normal derivation $D_N$ (Theorem 1.) The transformation defines ($\epsilon_0$-) recursive functions $\{\Phi_j\}$ and ranges $\{\Phi_j(U_1, \ldots, U_n)\}$ for the negative occurrences of quantifiers in $D_N$. The BV propositional expansion of $D_N$ over these ranges, $D_N^*$, will be a propositional derivation of $\Gamma^* \Longrightarrow \Delta^*$. But if $A$ is in $\Gamma$, then $A^*$ is a true arithmetic formula. Hence $\Delta^*$ is true. (Notice that this conclusion holds if $A$ is any true $\Pi_1^0$ formula.) ∎

**Corollary 2.** *If $(x_1)(Ey_1) \cdots (x_k)(Ey_k)F(x_1, y_1, \ldots, x_k, y_k)$ is a theorem of Peano arithmetic, there are recursive functions $\phi_i$ $1 \leq i \leq k$ so that for any values $n_1, \ldots, n_k$, $F(\overline{n_1}, \overline{\phi_1(n_1 \ldots .n_k)}, \ldots, \overline{n_k}, \overline{\phi_k(n_1 \ldots, n_k)})$ is true.*(cf [5] pp 34-35).

**Proof.** Let $\phi_i(n_1, \ldots .n_k) = \Phi_i(\{n_1\}, \ldots, \{n_k\})$. ∎

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