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THE UNIQUENESS OF THE DECOMPOSITION OF DISTRIBUTIVE LATTICES INTO SUMS OF BOOLEAN LATTICES.

A b s t r a c t We prove that any decomposition of any finite distributive lattice into (the Wroński sum of) Boolean algebras contains all maximal Boolean fragments of the lattice. The maximal elements of the decomposition are thus uniquely determined. We also exhibit a practical method of finding them.

Let $\mathcal{A} =< A, \leq_A>$ and $\mathcal{B} =< B, \leq_B>$ be lattices such that $A \cap B$ is a filter in $\mathcal{A}$ and an ideal in $\mathcal{B}$, and the orderings $\leq_A$ and $\leq_B$ coincide on $A \cap B$. Then $\leq_A \cup \leq_B \cup (\leq_A \circ \leq_B)$, is a lattice ordering on $A \cup B$ and the resulting lattice, called a sum of $\mathcal{A}$ and $\mathcal{B}$, is denoted by $\mathcal{A} \oplus \mathcal{B}$. The sum operation was introduced by Wroński [5], and its special case with $A \cap B = \{\mathbf{1}_A\} = \{\mathbf{0}_B\}$ by Troelstra [4]. In particular, if $\mathcal{B}$ is a two–element Boolean algebra then $\mathcal{A} \oplus \mathcal{B}$ is the same as $\mathcal{A}\oplus$, where $\oplus$ is the Jaśkowski operation of adding to $\mathcal{A}$ the top element (so called "mast"), see [2].

Kotas, Wojtylak [3] proved that the closure of the class of all finite

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Boolean algebras with respect to the sum operation is the class of all finite distributive lattices. It means that for every finite distributive lattice $\mathcal{D}$ there is a finite family $\{\mathcal{B}_i\}_{i\in T}$ of Boolean algebras such that $\mathcal{D}$ is a sum of that family. In this case we shall write $\mathcal{D} = \oplus\{\mathcal{B}_i\}_{i\in T}$. As the sum operation is nonassociative and noncommutative, this notation does not give us any clue about the ordering in which the summation should be performed. As a matter of fact, there is no uniqueness of doing it. Despite this, we are going to prove that all maximal elements of the decomposition are uniquely determined.

Let $\mathcal{D}$ be a lattice. A sublattice $\mathcal{D}_1$ of $\mathcal{D}$ is said to be a fragment of $\mathcal{D}$, and we shall write $\mathcal{D}_1 \sqsubseteq \mathcal{D}$ in this case, if

$$a \leq c \leq b \quad \text{and} \quad a, b \in D_1 \;\Rightarrow\; c \in D_1, \qquad \text{for every } a, b, c \in D.$$

If, additionally, $\mathcal{D}_1$ is a Boolean lattice, then $\mathcal{D}_1$ is said to be a Boolean fragment of $\mathcal{D}$. Any filter (ideal) of $\mathcal{D}$ is its fragment and, if $\mathcal{D}$ is finite, then any fragment of $\mathcal{D}$ is its interval, that is a set of the form $\{x \in D : a \leq x \leq b\}$ for some a and b. Clearly, $\sqsubseteq$ is transitive and $D_i \sqsubseteq D_1 \oplus D_2$ for $i = 1, 2$.

Theorem 1. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distributive lattices and $\mathcal{B}$ be a Boolean algebra. If $\mathcal{B} \sqsubseteq \mathcal{D}_1 \oplus \mathcal{D}_2$ then $\mathcal{B} \sqsubseteq \mathcal{D}_1$ or $\mathcal{B} \sqsubseteq \mathcal{D}_2$.*

Proof. Let $\mathcal{B} \sqsubseteq \mathcal{D}_1 \oplus \mathcal{D}_2$ and assume that there are elements $a \in B \setminus D_2$ and $b \in B \setminus D_1$. Let $\mathbf{0}_B$ be the zero of $\mathcal{B}$ and $\mathbf{1}_B$ be the unit of $\mathcal{B}$. Since $\mathbf{0}_B \leq a$ and $a \in D_1$, then $\mathbf{0}_B \in D_1 \setminus D_2$. Similarly, we conclude that $\mathbf{1}_B \in D_2 \setminus D_1$. Since $\mathbf{0}_B \leq \mathbf{1}_B$, there is an element $x \in D_1 \cap D_2 \cap B$ by the definition of the ordering in $\mathcal{D}_1 \oplus \mathcal{D}_2$. Let $y \in B$ be the complement of x in $\mathcal{B}$. If $y \in D_1$ then $\mathbf{1}_B = x \vee y \in D_1$, which is impossible. Otherwise $y \in D_2$, so $\mathbf{0}_B = x \wedge y \in D_2$, which also contradicts our assumptions. Thus, $\mathcal{B} \subseteq \mathcal{D}_1$ or $\mathcal{B} \subseteq \mathcal{D}_2$, and it yields $\mathcal{B} \sqsubseteq \mathcal{D}_1$ or $\mathcal{B} \sqsubseteq \mathcal{D}_2$ as $\mathcal{B} \sqsubseteq \mathcal{D}_1 \oplus \mathcal{D}_2$. ∎

Since every finite distributive lattice $\mathcal{D}$ is a sum $\oplus$ of Boolean lattices $\mathcal{B}_i$, then D is also the set-theoretical sum $\bigcup B_i$ of its Boolean fragments.

The components of the sum are not uniquely determined as it is possible that $B_i \subseteq B_j$ for some i, j. We call a family $\{\mathcal{B}_i\}$ of Boolean lattices a scarce decomposition of $\mathcal{D}$ iff $\mathcal{D} = \bigoplus \mathcal{B}_i$ and $\mathcal{B}_i \subseteq \mathcal{B}_j$ does not hold for any $i \neq j$. As the immediate Corollaries of Theorem 1 we get:

Corollary 1. *If a lattice $\mathcal{D}$ is a sum $\oplus$ of a family $\{\mathcal{B}_i\}$ of Boolean lattices, then the family contains all maximal Boolean fragments of $\mathcal{D}$.*

Corollary 2. *There is at most one scarce decomposition of any finite distributive lattice $\mathcal{D}$ and the decomposition consists of all maximal Boolean fragments of $\mathcal{D}$.*

It may happen, however, that a finite distributive lattice $\mathcal{D}$ does not have the scarce decomposition. More specifically, it is sometimes impossible to get $\mathcal{D}$ as a sum $\bigoplus$ of its maximal Boolean fragments without taking subalgebras of maximal fragments or repeating them in the sum operations.

Example 1. Let $F_D(3)$ be a free distributive lattice on three generators. The lattice has five maximal Boolean fragments:

but it is impossible to get $F_D(3)$ as their sum $\oplus$. It means that $F_D(3)$ is not a sum $\mathcal{A}_1 \oplus \mathcal{A}_2$, where each of $\mathcal{A}_i$ is a sum of disjoint subsets of

$\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5\}$. Indeed, if $\mathcal{A}_1 = \mathcal{B}_1$, then any sum of the rest does not form a distributive lattice. We deal with the other possibilities in a similar way. If we decompose $F_D(3)$ into a sum of proper fragments $\mathcal{A}_1 \oplus \mathcal{A}_2$, then either the lower lattice will contain a proper subalgebras of $\mathcal{B}_1$ or the upper lattice will contain a subalgebra of $\mathcal{B}_5$.

Now, we are going to show a practical method of finding all maximal Boolean fragments of a given distributive lattice $\mathcal{D}$. First let us identify the "top" maximal Boolean fragment of $\mathcal{D}$.

Theorem 2. *The filter ∇ generated by all coatoms of any finite distributive lattice $\mathcal{D}$ is a maximal Boolean fragment of $\mathcal{D}$.*

Proof. Let A be the set of all coatoms of $\mathcal{D}$. It is clear that $\nabla \sqsubseteq D$ with the unit element $\mathbf{1}$ of $\mathcal{D}$ as the greatest and $\Theta = \bigwedge A$ as the least element.

Let $x \in \nabla$. Then $x = \bigwedge\{a \in A : x \leq a\}$. Let $-x = \bigwedge\{a \in A : x \wedge a < x\}$. Clearly, $-x \in \nabla$, and $x \wedge -x = \bigwedge A = \Theta$, and

$$x \vee -x = \bigwedge\{x \vee a;\ a \in A \text{ and } a < x \vee a\} = \mathbf{1}$$

Thus $-x$ is the complement of x in ∇ and hence ∇ is a Boolean algebra.

We need to prove that ∇ is a maximal Boolean fragment of $\mathcal{D}$. Let $\nabla \sqsubseteq \mathcal{B}$, where $\mathcal{B}$ is a Boolean fragment of $\mathcal{D}$. Then for every $x \in B$ we have

$$\Theta \vee x \leq \Theta \vee \bigwedge\{a \in A :\ x \leq a\} = \bigwedge\{a \in A :\ x \leq a\},$$

so $x \vee \Theta = \mathbf{1}$ iff $x = \mathbf{1}$. Hence Θ is the zero of $\mathcal{B}$ which means that $\nabla = B$. ■

Let $\mathcal{D}$ be a finite distributive lattice and $b \in D$. An element $a \in D$ is said to be a b-coatom iff a is a coatom in the fragment $\{x \in D :\ x \leq b\}$ of $\mathcal{D}$. We denote by $\nabla(b)$ the Boolean fragment of $\mathcal{D}$ determined by all b-coatoms in $\mathcal{D}$. In other words, $\nabla(b)$ is the "top" Boolean fragment of $\{x \in D :\ x \leq b\}$. Dually, the "bottom" Boolean fragment Δ of $\mathcal{D}$ is the ideal generated by all atoms. By $\Delta(b)$ we denote the Boolean fragment of $\mathcal{D}$ determined by all b-atoms, that is atoms in the fragment $\{x \in D :\ b \leq x\}$ of $\mathcal{D}$.

Theorem 3. *Let $\mathcal{B}$ be a maximal Boolean fragment of a finite distributive lattice $\mathcal{D}$ and let b be a maximal element of $\mathcal{B}$ such that the set of all b-coatoms is not contained in B. Then $b < \mathbf{1}_B$ and $\nabla(b)$ is a maximal Boolean fragment of $\mathcal{D}$.*

Proof. First, note that if some 1_B-coatoms were not elements of B (i.e. $b = 1_B$), then $\mathcal{B}$ would be a proper sublattice of $\nabla(b)$.

Let $b \in B$ and $\Theta = \bigwedge\{y : y \text{ is a } b\text{-coatom}\}$. We can easily prove that $\nabla(b)$ is a maximal Boolean fragment of $\mathcal{D}$ iff all Θ-atoms are below b (i.e. $\nabla(b) = \Delta(\Theta)$).[1]

One implication follows from the (dual counterpart) of the above theorem. If $\nabla(b)$ is a maximal Boolean fragment of $\mathcal{D}$, then it is also a maximal Boolean fragment of $\{x \in D : x \geq \Theta\}$ and hence $\nabla(b) = \Delta(\Theta)$. On the other hand, suppose that all Θ-atoms are below b and $\nabla(b)$ is a sublattice of some Boolean fragment $\mathcal{B}_1$ of $\mathcal{D}$. Let $-\Theta$ be the complement of Θ in $\mathcal{B}_1$. If $-\Theta \wedge b < b$, then $-\Theta \wedge b \leq y_0$ for some b-coatom y_0 and hence

$$b = (\Theta \vee -\Theta) \wedge (b \vee \Theta) = (-\Theta \wedge b) \vee \Theta \leq y_0 \vee \Theta = y_0$$

which is impossible. Hence $-\Theta \wedge b = b$ and it means that $\Theta \leq b \leq -\Theta$. Thus, Θ is the zero element in $\mathcal{B}_1$. Since b must be then the unit element there, we get $\nabla(b) = \mathcal{B}_1$.

Now, let us assume that $\mathcal{B}$ is a maximal Boolean fragment of $\mathcal{D}$ and b is a maximal element of $\mathcal{B}$ such that the set of all b-coatoms is not contained in B. We need to show that $\nabla(b)$ is a maximal Boolean fragment of $\mathcal{D}$. Suppose, to the contrary, that

$$a \wedge b = \Theta, \qquad \text{for some } \Theta\text{-atom } a.$$

Let us prove that $a \wedge 1_B = \Theta$. Suppose that $a \leq 1_B$. According to the choice of b, there is a b-coatom y_0 such that $y_0 \notin B$. Since

$$0_B \leq b \leq a \vee b \leq 1_B,$$

[1] We wish to thank an anonymous refree for his suggestions concerning the presentation of our argument.

we get $a \vee b \in B$. If $a \vee y_0 \in B$, then

$$y_0 = y_0 \vee \Theta = (b \wedge y_0) \vee (a \wedge b) = (a \vee y_0) \wedge b \in B$$

which is impossible. Thus, $a \vee y_0 \notin B$ which means, in particular, that

$$a \vee y_0 < a \vee b.$$

But $b < a \vee b$ and hence (by the choice of the element b) the element $a \vee y_0$ cannot be a $(a \vee b)$-coatom. So

$$a \vee y_0 < x_0 < a \vee b, \quad \text{for some } x_0.$$

Then, we get

$$y_0 = \Theta \vee y_0 = (a \wedge b) \vee (y_0 \wedge b) = (a \vee y_0) \wedge b \leq x_0 \wedge b \leq (a \vee b) \wedge b = b$$

and hence $x_0 \wedge b = b$ or $y_0 = x_0 \wedge b$ as y_0 is a b-coatom. But $b \leq x_0$ yields $a \wedge b \leq x_0$ which is not the case and, on the other hand, $y_0 = x_0 \wedge b$ yields

$$a \vee y_0 = a \vee (x_0 \wedge b) = (a \vee x_0) \wedge (a \vee b) = x_0 \wedge (a \vee b) = x_0$$

which is not the case, either. Thus, our assumption that $a \leq 1_B$ leads us to a contradition. We conclude therefore that $a \wedge 1_B = \Theta$ and hence

$$a \wedge x = \Theta, \quad \text{for every } x \in B.$$

Let us take $B_a = \{x \in D : \ 0_B \vee a \leq x \leq 1_B \vee a\}$ and let $g : B \to B_a$ and $h : B_a \to B$ be the mappings defined by

$$g(x) = x \vee a \qquad\qquad h(y) = y \wedge 1_B.$$

It is clear that h and g are lattice homomorphisms. Since $h(g(x)) = x$ for every $x \in B$ and $g(h(y)) = y$ for every $y \in B_a$, we conclude that B_a is a Boolean fragment of $\mathcal{D}$ isomorphic to $\mathcal{B}$. Since $\mathcal{B}$ is a proper sublattice of $B \cup B_a$, to complete our argument it suffices to notice that $B \cup B_a$ is a fragment of $\mathcal{D}$ isomorphic to the product $B \times \{\Theta, a\}$. Since any product of Boolean lattices is a Boolean lattice, then $\mathcal{B}$ is not a maximal Boolean fragment of $\mathcal{D}$ which contradicts our assumptions.

The isomorphism $f : B \times \{\Theta, a\} \to B \cup B_a$ is defined by $f(x, y) = x \vee y$. We have to show that

$$x_1 \leq x_2 \text{ and } y_1 \leq y_2 \qquad \text{iff} \qquad x_1 \vee y_1 \leq x_2 \vee y_2$$

Let us note that the implication $(\Rightarrow)$ is obvious. To prove the converse we consider four possibilities. The implication is obvious if $y_1 = y_2 = \Theta$. If $y_1 = y_2 = a$, we get

$$x_1 = (x_1 \vee y_1) \wedge 1_B \leq (x_2 \vee y_2) \wedge 1_B = x_2.$$

If $y_1 = a$ and $y_2 = \Theta$, then $x_2 \vee y_2 = x_2$ and hence $x_1 \vee a \leq x_2$ cannot happen as it yields $a \leq x_2 \in B$. If $y_1 = \Theta$ and $y_2 = a$, then $x_1 \leq x_2 \vee a$ gives us $x_1 \leq (x_2 \vee a) \wedge 1_B = x_2$.

There remains to show that $B \cup B_a$ is a fragment of $\mathcal{D}$. Clearly, B and B_a are fragments. Suppose that $x_1 \leq y \leq x_2 \vee a$ for some $x_1, x_2 \in B$ and $y \in D$ (note that $x_1 \vee a \leq y \leq x_2$ cannot happen for any $x_1, x_2 \in B$). Then

$$x_1 = x_1 \wedge 1_B \leq y \wedge 1_B \leq (x_2 \vee a) \wedge 1_B = x_2 \wedge 1_B = x_2 \in B$$

If $a \leq y$, then $x_1 \vee a \leq y = (y \wedge 1_B) \vee a \leq x_2 \vee a$ and hence $y \in B_a$. If, on the other hand, $a \wedge y = \Theta$, then $x_1 \leq y = y \wedge (1_B \vee a) = y \wedge 1_B \leq x_2$ and hence $y \in B$. Thus, we have shown that $y \in B \cup B_a$ in any case and this means that $B \cup B_a$ is a fragment of $\mathcal{D}$. ∎

The fragment $\nabla(b)$ will be called a bottom companion of $\mathcal{B}$ if b is a maximal element of $\mathcal{B}$ such that the set of all b-coatoms is not contained in $\mathcal{B}$. Of course, every maximal Boolean fragment $\mathcal{B}$ of a finite distributive lattice $\mathcal{D}$ (except the bottom one) has at least one bottom companion. In a similar way we can define, for every maximal Boolean fragment $\mathcal{B}$ an upper companion $\Delta(b)$ by choosing a minimal element b of B such that the set of b-atoms is not contained in B. Let us show by use of an example that none of the assumptions of the above theorem can be dropped.

Example 2. Consider the lattice

If $B = \{l, d\}$ then d is a maximal element of B such that d-coatoms are not in B. But $\nabla(d) = \{d, f\}$ is not a maximal Boolean fragment of the whole lattice. Moreover, if one takes $B = \{l, d, r, m\} = \nabla$, then f (which is an d-coatom) is not in B but $\nabla(d)$ is not a bottom companion of B. We have $d < m$ and $\nabla(m)$ is the only bottom companion of B.

The above two Theorems give us a procedure of decomposing any finite distributive lattice onto its maximal Boolean fragments. We start with the "top" fragment (by Theorem 2) and find, for any given maximal fragment of $\mathcal{D}$, its bottom companions till we get the "bottom" algebra. This decomposition is uniquely determined. However, some fragments of $\mathcal{D}$ can be obtained several times in the procedure.

Example 3. If we consider the lattice $F_D(3)$ from Example 1, we get

$$\nabla = \mathcal{B}_1, \ \ \nabla(a) = \mathcal{B}_2, \ \ \nabla(b) = \mathcal{B}_3, \ \ \nabla(c) = \mathcal{B}_4, \ \ \mathcal{B}_5 = \nabla(d) = \Delta;$$

where

$$a = (x \vee y) \wedge (x \vee z), \ \ b = (x \vee y) \wedge (y \vee z), \ \ c = (x \vee z) \wedge (y \vee z),$$

$$d = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z);$$

The problem of describing finite distributive lattices with a scarce decomposition into Boolean fragments remains open.

References

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