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CARDINALITIES OF MODELS AND THE EXPRESSIVE POWER OF MONADIC PREDICATE LOGIC (with equality and individual constants)

1. Introduction

In this paper we improve one result from [2, p. 250–255] (cf. the last section of the paper). We also present, by a method of semantical nature, a certain description of the expressive power of first-order logic with one-place predicates and equality.

As in [2, p. 250], a *monadic* formula is a formula of first-order logic, all of whose non-logical symbols are either one-place predicate letters or name letters (resp. individual constants). Monadic formulas may contain the logical symbol '$\doteq$', i.e., the equals-sign (or the sign of identity).

A formula $\tau_1 \doteq \tau_2$ for different terms τ_1 and τ_2 is said to be a *non-tautological identity.*

We prove that for any monadic formula φ: φ is logically valid iff φ is true in every interpretation whose domain contains at most $l + 2^k \cdot n^+$ members, k being the number of predicate letters in φ, l being the number

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of individual constants in non-tautological identities which are subformulas of φ, and n^+ being the number of variables in non-tautological identities which are subformulas of φ or, when no variable occurs in non-tautological identities in φ, $n^+ = 1$.[1]

In the body of this paper we shall define each semantical notion along the line of Barwise [1] and Mendelson [3], and then prove the above fact.

2. Syntactical and semantical preliminaries

Let PL, NL and Var be fixed denumerably infinite disjoint sets; PL is the set of *one-place predicate letters*, NL is the set of *name letters*, and Var is the set of *variables*. Let $\mathsf{Term} := \mathsf{NL} \cup \mathsf{Var}$ be the set of terms and let Form be the set of formulas of monadic predicate logic (with equality '$\doteq$' and name letters), i.e., the smallest set such that:

– if $\tau \in \mathsf{Term}$ and $\mathsf{p} \in \mathsf{PL}$, then $\mathsf{p}(\tau) \in \mathsf{Form}$;
– if $\tau_1, \tau_2 \in \mathsf{Term}$, then $\tau_1 \doteq \tau_2 \in \mathsf{Form}$;
– if $\varphi, \psi \in \mathsf{Form}$, then $\neg\varphi \in \mathsf{Form}$ and $(\varphi \S \psi) \in \mathsf{Form}$, where $\S \in \{\vee, \wedge, \supset\}$;
– if $\varphi \in \mathsf{Form}$ and $\mathsf{x} \in \mathsf{Var}$, then $\forall \mathsf{x} \varphi \in \mathsf{Form}$ and $\exists \mathsf{x} \varphi \in \mathsf{Form}$.

The logical symbols $\neg$, $\vee$, $\wedge$ and $\supset$ are the propositional connectives of negation, disjunction, conjunction and material implication. Moreover, the logical symbols $\forall$ and $\exists$ are universal and existential quantifiers.

A *sentence* is a formula without any free variables.

An *interpretation* of Form is a pair $\mathcal{I} = \langle D, \imath \rangle$, where D is a non-empty set (called the *domain* of $\mathcal{I}$) and $\imath$ is a mapping from NL into D and from PL into the power set $\mathcal{P}(D)$.

Let D^{Var} be the set of functions from Var into D. All elements of D^{Var} will be called *assignments*. For any assignment s we use $s\binom{d}{\mathsf{x}}$ for the assignment s' which agrees with s except that $s'(\mathsf{x}) = d$.

Moreover, for any s and $\imath$ we define a function $t_{s,\imath} := s \cup \imath|_{\mathsf{NL}}$ of one argument, with terms as arguments and values in D.

1 For formulas without any variables this estimate can be reduced (cf. footnote 2).

Let $\mathfrak{I} = \langle D, \imath \rangle$ be an interpretation of Form. We define a relation $\mathfrak{I} \models \varphi\,[s]$ (read: the assignment s satisfies φ in $\mathfrak{I}$) for all assignments s and all formulas φ as follows:
$\mathfrak{I} \models \mathsf{p}(\tau)\,[s]$ iff $t_{s,\imath}(\tau) \in \imath(\mathsf{p})$; $\mathfrak{I} \models \tau_1 \doteq \tau_2\,[s]$ iff $t_{s,\imath}(\tau_1) = t_{s,\imath}(\tau_2)$; $\mathfrak{I} \models \neg\varphi\,[s]$ iff not $\mathfrak{I} \models \varphi\,[s]$; $\mathfrak{I} \models (\varphi \wedge \psi)\,[s]$ iff $\mathfrak{I} \models \varphi\,[s]$ and $\mathfrak{I} \models \psi\,[s]$; $\mathfrak{I} \models (\varphi \vee \psi)\,[s]$ iff either $\mathfrak{I} \models \varphi$ or $\mathfrak{I} \models \psi\,[s]$; $\mathfrak{I} \models (\varphi \supset \psi)\,[s]$ iff either not $\mathfrak{I} \models \varphi\,[s]$ or $\mathfrak{I} \models \psi\,[s]$; $\mathfrak{I} \models \forall \mathsf{x}\,\varphi\,[s]$ iff for all $d \in D$, $\mathfrak{I} \models \varphi\left[s\binom{d}{\mathsf{x}}\right]$; $\mathfrak{I} \models \exists \mathsf{x}\,\varphi\,[s]$ iff there is a $d \in D$ such that $\mathfrak{I} \models \varphi\left[s\binom{d}{\mathsf{x}}\right]$.

A formula φ is *true* in the interpretation $\mathfrak{I}$ (or $\mathfrak{I}$ is a *model* of φ; we write: $\mathfrak{I} \models \varphi$) iff for every assignment s, $\mathfrak{I} \models \varphi\,[s]$. The form "not $\mathfrak{I} \models \varphi$" will be an abbreviation for "it is not the case that $\mathfrak{I} \models \varphi$". So not $\mathfrak{I} \models \varphi$ iff for some s, not $\mathfrak{I} \models \varphi\,[s]$. If σ is a sentence, then the truth or falsity of $\mathfrak{I} \models \sigma\,[s]$ is independent of s. Thus $\mathfrak{I} \models \sigma$ iff for some (hence every) s, $\mathfrak{I} \models \sigma\,[s]$. So for any sentence σ: $\mathfrak{I} \models \neg\sigma$ iff not $\mathfrak{I} \models \sigma$.

A formula φ is called *logically valid* iff φ is true in every interpretation.

A formula φ is *satisfiable* iff there is an interpretation for which φ is satisfied by at least one assignment. A sentence σ is satisfiable iff there is a model of σ.

3. Estimation

Let φ be a given monadic formula. Let $\mathsf{p}_1, \ldots, \mathsf{p}_k$ be the k distinct one-place predicate letters (possibly $k = 0$) occurring in φ. Let $\mathsf{a}_1, \ldots, \mathsf{a}_l$ be the l distinct name letters that occur in non-tautological identities in φ (possibly $l = 0$). Moreover, let $\mathsf{x}_1, \ldots, \mathsf{x}_n$ be the n distinct variables that occur in non-tautological identities in φ (possibly $n = 0$). We put $n^+ := \max(n, 1)$.

Theorem 1. *Let $\mathfrak{I} = \langle D, \imath \rangle$ be an arbitrary interpretation of* Form. *We can construct a certain interpretation $\mathfrak{I}_\varphi = \langle D_\varphi, \imath_\varphi \rangle$, whose domain D_φ*

contains at most $l + 2^k \cdot n^+$ *members,*[2] *and which satisfies the following condition:*

$$(\dagger) \qquad \mathcal{I}_\varphi \models \varphi \quad \textit{iff} \quad \mathcal{I} \models \varphi\,.$$

Proof. We define the binary relation $\cong$ in D by:

- if $k = 0$ then we put $\cong := D \times D$
- if $k > 0$ then for $d_1, d_2 \in D$

$$d_1 \cong d_2 \quad \text{iff} \quad \text{for every } i = 1 \ldots, k : d_1 \in \imath(\mathsf{p}_i) \text{ iff } d_2 \in \imath(\mathsf{p}_i)\,.$$

Clearly, $\cong$ is an equivalence relation. We denote an equivalence class of d in D by $\|d\|$. We have: for any one-place predicate symbol p_i $(0 \leq i \leq k)$ and any $d \in D$

$$(\star) \qquad d \in \imath(\mathsf{p}_i) \quad \text{iff} \quad \|d\| \subseteq \imath(\mathsf{p}_i)\,.$$

Let $p := |D/_{\cong}|$. Obviously $0 < p \leq 2^k$. Let $\{D_1, \ldots, D_p\} := D/_{\cong}$. Obviously the sets $D_1, \ldots, D_p$ are non-empty and pairwise disjoint. Moreover, $D = \bigcup_{i=1}^{p} D_i$.

If $k > 0$ then for every $i = 1, \ldots, p$ there is a sequence $\langle \delta_{i_1}, \ldots, \delta_{i_k} \rangle$ of zeros and ones such that $D_i = \imath(\mathsf{p}_1)^{\delta_{i_1}} \cap \ldots \cap \imath(\mathsf{p}_k)^{\delta_{i_k}}$ where $\imath(\mathsf{p}_j)^1 = \imath(\mathsf{p}_j)$ and $\imath(\mathsf{p}_j)^0 = D - \imath(\mathsf{p}_j)$. If $k = 0$ then $D_1 = D$.

Let $\mathsf{A} := \{\mathsf{a}_1, \ldots, \mathsf{a}_l\}$ and $\mathsf{X} := \{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$.

For $i = 1, \ldots, p$ we choose from the set $D_i - \imath[\mathsf{A}]$ a subset $D_i^\star$ such that $|D_i^\star| = \min(|D_i - \imath[\mathsf{A}]|, n^+)$.

We put

$$D_\varphi \ := \ \imath[\mathsf{A}] \ \cup \ \bigcup_{i=1}^{p} D_i^\star\,.$$

The set D_φ contains, at most, $l + 2^k \cdot n^+$ members.

[2] For a given monadic sentence which has no variables this estimate can be reduced to $l^+ + \min(2^k - 1, q - l^+)$ where $l^+ := \max(l, 1)$ and q being the number of name letters occurring in this sentence.

A *projection* from D on D_φ is a function $\pi : D \to D_\varphi$ such that for any $d \in D$ and $i = 1, \ldots, p$:

(i) if $d \in D_i - \imath[\mathsf{A}]$ then $\pi(d) \in D_i^\star$,

(ii) if $d \in D_\varphi$ then $\pi(d) = d$.

The conditions (i) and (ii) imply:

(i′) $d \in D_i - \imath[\mathsf{A}]$ iff $\pi(d) \in D_i^\star$.

(ii′) $d \in D_i$ iff $\pi(d) \in D_i^\star \cup (D_i \cap \imath[\mathsf{A}])$.

Let **Proj** be the set of all projections from D on D_φ. Clearly, **Proj** $\neq \emptyset$.

We choose from the set **Proj** one projection $\pi^\star$. We define the mapping $\imath_\varphi$ from NL into D_φ by:[3]

$$\imath_\varphi(\mathsf{b}) := \pi^\star(\imath(\mathsf{b}))$$

and from PL into $\mathcal{P}(D_\varphi)$ by:

$$\imath_\varphi(\mathsf{p}) := D_\varphi \cap \imath(\mathsf{p})\,.$$

Obviously — by the definition of the set D_φ and by (ii) in the definition of the set **Proj** — for $i = 1, \ldots, l$ we have:

$$(\star\star) \qquad \imath_\varphi(\mathsf{a}_i) = \imath(\mathsf{a}_i)\,.$$

For any $s \in D^{\mathsf{Var}}$ let $\mathbf{Proj}^s$ be the set of all projections which are injections on the set $s[\mathsf{X}] \cup \imath[\mathsf{A}]$.

[3] If φ is a pure monadic formula (without '$\doteq$' and individual constants; $k > 0$ and $l = 0 = n$), then the given proof is "classical". In this case, we may also suppose that $D_\varphi := D/_{\cong}$ and $\imath_\varphi(\mathsf{p}) := \{\Delta \in D/_{\cong} : \Delta \subseteq \imath(\mathsf{p})\}$. Moreover, if φ is a monadic formula with individual constants and without '$\doteq$' ($k > 0$ and $l = 0 = n$), we may suppose that $D_\varphi := D/_{\cong}$, $\imath_\varphi(\mathsf{p}) := \{\Delta \in D/_{\cong} : \Delta \subseteq \imath(\mathsf{p})\}$ and $\imath_\varphi(\mathsf{b}) := \|\imath(\mathsf{b})\|$. In both cases there exists exactly one "canonical" projection $\|\cdot\| : D \ni d \mapsto \|d\| \in D_\varphi$.

Lemma 1. *For any* $s \in D^{\mathsf{Var}}$*:* $\mathbf{Proj}^s \neq \emptyset$.

Proof. Notice that for any $s \in D^{\mathsf{Var}}$ and $i = 1, \dots, m$ we have the condition (ii′) and $|D_i \cap (s[\mathsf{X}] \cup \imath[\mathsf{A}])| \leq |D_i^\star \cup (D_i \cap \imath[\mathsf{A}])|$. Actually, $D_i \cap (s[\mathsf{X}] \cup \imath[\mathsf{A}]) = (D_i \cap s[\mathsf{X}]) \cup (D_i \cap \imath[\mathsf{A}]) = ((D_i - \imath[\mathsf{A}]) \cap s[\mathsf{X}]) \cup (D_i \cap \imath[\mathsf{A}] \cap s[\mathsf{X}]) \cup (D_i \cap \imath[\mathsf{A}]) = ((D_i - \imath[\mathsf{A}]) \cap s[\mathsf{X}]) \cup (D_i \cap \imath[\mathsf{A}])$. Moreover, $|(D_i - \imath[\mathsf{A}]) \cap s[\mathsf{X}]| \leq \min(|D_i - \imath[\mathsf{A}]|, n) \leq |D_i^\star|$ and $D_i^\star \cap \imath[\mathsf{A}] = \emptyset$. Thus $|D_i \cap (s[\mathsf{X}] \cup \imath[\mathsf{A}])| \leq |D_i^\star| + |D_i \cap \imath[\mathsf{A}]| = |D_i^\star \cup (D_i \cap \imath[\mathsf{A}])|$. □

For any $s \in D^{\mathsf{Var}}$ and $\pi \in \mathbf{Proj}$ we have $\pi \circ s \in D_\varphi^{\mathsf{Var}}$. Moreover:

Lemma 2. *Suppose that* ψ *is a subformula of* φ*. Then for any* $s \in D^{\mathsf{Var}}$ *and* $\pi \in \mathbf{Proj}^s$*:*

$$\mathfrak{I}_\varphi \models \psi\,[\pi \circ s] \quad \textit{iff} \quad \mathfrak{I} \models \psi\,[s]\ .$$

Proof. Induction on ψ. If ψ is atomic, either

(a) for some $\mathsf{y} \in \mathsf{Var}$ and $i = 1, \dots, k$, $\psi = \mathsf{p}_i(\mathsf{y})$, or
(b) for some $\mathsf{b} \in \mathsf{NL}$ and $i = 1, \dots, k$, $\psi = \mathsf{p}_i(\mathsf{b})$, or
(c) for some $\mathsf{y} \in \mathsf{Var}$, $\psi = \mathsf{y} \doteq \mathsf{y}$, or
(d) for some $\mathsf{b} \in \mathsf{NL}$, $\psi = \mathsf{b} \doteq \mathsf{b}$, or
(e) for some $i, j = 1, \dots, n$, $i \neq j$, $\psi = \mathsf{x}_i \doteq \mathsf{x}_j$, or
(f) for some $i, j = 1, \dots, l$, $i \neq j$, $\psi = \mathsf{a}_i \doteq \mathsf{a}_j$, or
(g) for some $i = 1, \dots, n$ and $j = 1, \dots, l$, $\psi = \mathsf{x}_i \doteq \mathsf{a}_j$ or $\psi = \mathsf{a}_j \doteq \mathsf{x}_i$.

Let s and π be arbitrary members of D^{Var} and $\mathbf{Proj}^s$ respectively.

In the first case: $\mathfrak{I}_\varphi \models \mathsf{p}_i(\mathsf{y})\,[\pi \circ s]$ iff $\pi(s(\mathsf{y})) \in \imath_\varphi(\mathsf{p}_i)$ iff — by the definition of $\imath_\varphi$ — $\pi(s(\mathsf{y})) \in \imath(\mathsf{p}_i)$ iff — by ($\star$) and (ii′) — $s(\mathsf{y}) \in \imath(\mathsf{p}_i)$ iff $\mathfrak{I} \models \mathsf{p}_i(\mathsf{y})\,[s]$.

In the second case: $\mathfrak{I}_\varphi \models \mathsf{p}_i(\mathsf{b})\,[\pi \circ s]$ iff $\imath_\varphi(\mathsf{b}) \in \imath_\varphi(\mathsf{p}_i)$ iff — by the definition of $\imath_\varphi$ — $\pi^\star(\imath(\mathsf{b})) \in \imath(\mathsf{p}_i)$ iff — by ($\star$) and (ii′) — $\imath(\mathsf{b}) \in \imath(\mathsf{p}_i)$ iff $\mathfrak{I} \models \mathsf{p}_i(\mathsf{b})\,[s]$.

In cases (c) and (d): ψ is true in both $\mathfrak{I}_\varphi$ and $\mathfrak{I}$.

In case (e): $\mathfrak{I}_\varphi \models \mathsf{x}_i \doteq \mathsf{x}_j\,[\pi \circ s]$ iff $\pi(s(\mathsf{x}_i)) = \pi(s(\mathsf{x}_j))$ iff — by the definition of $\mathbf{Proj}^s$ — $s(\mathsf{x}_i) = s(\mathsf{x}_j)$ iff $\mathfrak{I} \models \mathsf{x}_i \doteq \mathsf{x}_j\,[s]$.

In case (f): $\mathfrak{I}_\varphi \models a_i \doteq a_j\,[\pi \circ s]$ iff $\imath_\varphi(a_i) = \imath_\varphi(a_j)$ iff — by ($\star\star$) — $\imath(a_i) = \imath(a_j)$ iff $\mathfrak{I} \models a_i \doteq a_j\,[s]$.

In case (g): $\mathfrak{I}_\varphi \models x_i \doteq a_j\,[\pi \circ s]$ iff $\pi(s(x_i)) = \imath_\varphi(a_j)$ iff — by ($\star\star$) — $\pi(s(x_i)) = \imath(a_j)$ iff — by (ii) in the definition of **Proj** — $\pi(s(x_i)) = \pi(\imath(a_j))$ iff — by the definition of $\mathbf{Proj}^s$ — $s(x_i) = \imath(a_j)$ iff $\mathfrak{I} \models x_i \doteq a_j\,[s]$. Similarly for $a_j \doteq x_i$.

If $\psi = \neg\chi$ then: $\mathfrak{I}_\varphi \models \psi\,[\pi \circ s]$ iff not $\mathfrak{I}_\varphi \models \chi\,[\pi \circ s]$ iff — by the hypothesis of the induction — not $\mathfrak{I} \models \chi\,[s]$ iff $\mathfrak{I} \models \psi\,[s]$.

The argument is similar if ψ is a conjunction or other truth-functional compound of simpler formulas.

Suppose then that $\psi = \forall y\,\chi$. Let $\mathfrak{I}_\varphi \models \forall y\,\chi\,[\pi \circ s]$, i.e. for all $e \in D_\varphi$, $\mathfrak{I}_\varphi \models \chi\left[(\pi \circ s)\binom{e}{y}\right]$. Let d be an arbitrary member of D. Then, since $\pi \circ s\binom{d}{y} = (\pi \circ s)\binom{\pi(d)}{y}$ and $\mathfrak{I}_\varphi \models \chi\left[(\pi \circ s)\binom{\pi(d)}{y}\right]$, it follows that $\mathfrak{I}_\varphi \models \chi\left[\pi \circ s\binom{d}{y}\right]$. Hence, by inductive hypothesis, $\mathfrak{I} \models \chi\left[s\binom{d}{y}\right]$. So $\mathfrak{I} \models \forall y\,\chi\,[s]$. Conversely, assume that $\mathfrak{I} \models \forall y\,\chi\,[s]$, i.e., for all $d \in D$, $\mathfrak{I} \models \chi\left[s\binom{d}{y}\right]$. Let e be an arbitrary member of D_φ. Then $\mathfrak{I} \models \chi\left[s\binom{e}{y}\right]$. Hence, by inductive hypothesis, $\mathfrak{I}_\varphi \models \chi\left[\pi \circ s\binom{e}{y}\right]$. Moreover, by (ii) in the definition of **Proj**, $\pi \circ s\binom{e}{y} = (\pi \circ s)\binom{e}{y}$. Thus $\mathfrak{I}_\varphi \models \chi\left[(\pi \circ s)\binom{e}{y}\right]$. So $\mathfrak{I}_\varphi \models \forall y\,\chi\,[(\pi \circ s)]$.

The argument is similar if ψ is an existential quantification of a simpler formula. □

Suppose now that $\mathfrak{I}_\varphi \models \varphi$. Let $s : \mathsf{Var} \to D$ be an arbitrary assignment. Then $\pi \circ s : \mathsf{Var} \to D_\varphi$ for any $\pi \in \mathbf{Proj}^s$. Thus $\mathfrak{I}_\varphi \models \varphi\,[\pi \circ s]$. Therefore $\mathfrak{I} \models \varphi\,[s]$, by Lemma 2. So $\mathfrak{I} \models \varphi$.

Assume that $\mathfrak{I} \models \varphi$. Let $s : \mathsf{Var} \to D_\varphi$ be an arbitrary assignment. Then also $s : \mathsf{Var} \to D$; so $\mathfrak{I} \models \varphi\,[s]$. Hence, by Lemma 2, $\mathfrak{I}_\varphi \models \varphi\,[\pi \circ s]$ for any $\pi \in \mathbf{Proj}^s$. Moreover, by property (ii) in the definition of **Proj**, we have $s = \pi \circ s$. Therefore $\mathfrak{I}_\varphi \models \varphi\,[s]$. Thus $\mathfrak{I}_\varphi \models \varphi$. □

4. Corollaries

Both from the "if" part of (†) and from the "only if" part of (†) in Theorem 1 we obtain:

Corollary 1. *For any $\varphi \in \mathsf{Form}$ the following conditions are equivalent:*

(i) *φ is logically valid;*

(ii) *φ is true in every interpretation, whose domain contains at most $l + 2^k \cdot n^+$ members;*

where k, l and n^+ we defined in the introduction and in section 3.

Proof. "⇒" By the definition of logically valid formulas.

"⇐" Proof by the "if" part of (†). Let $\mathfrak{I}$ be an arbitrary interpretation of Form. Then $\mathfrak{I}_\varphi \models \varphi$, by (ii).

(a) Therefore, by using the "if" part of (†), $\mathfrak{I} \models \varphi$. So φ is logically valid.

(b) Let $\mathrm{cl}(\varphi)$ be a universal closure of φ. Then $\mathfrak{I}_\varphi \models \mathrm{cl}(\varphi)$ and not $\mathfrak{I}_\varphi \models \neg\mathrm{cl}(\varphi)$, since $\mathrm{cl}(\varphi)$ is a sentence. Therefore, by using the "only if" part of (†), not $\mathfrak{I} \models \neg\mathrm{cl}(\varphi)$. Hence $\mathfrak{I} \models \mathrm{cl}(\varphi)$ and $\mathfrak{I} \models \varphi$. So φ is logically valid. □

Corollary 2. *Suppose that $\varphi \in \mathsf{Form}$, respectively:*

1) *contains no individual constants,*

2) *contains no one-place predicate letters,*

3) *contains no non-logical symbols,*

4) *does not contain the equals-sign,*

5) *does not contain the equals-sign and any individual constants.*

Then φ is logically valid iff φ is true in every interpretation whose domain contains, respectively, at most:

1) $2^k \cdot n^+$

2) $l + n^+$

3) n^+

4) 2^k

5) 2^k

members.

5. Expressive power of sentences from Form

5.1. Let σ be a given satisfiable sentence which contains no individual constants. Suppose that $p_1, \dots, p_k$ are the k (distinct) predicate letters that occur in σ ($k > 0$), and that σ has n distinct variables in non-tautological identities (possibly $n = 0$). Let $n^+ := \max(n, 1)$.

We will use the following abbreviations. Let $x_1, \dots, x_{n^+}$ be the n^+ distinct variables from Var. For $0 < j \leq n^+$ and for any sequence $\langle \delta_1, \dots, \delta_k \rangle$ of zeros and ones we say that:

$$p_1^{\delta_1} \cdots p_k^{\delta_k}(x_j) \qquad \text{abbreviates} \qquad p_1^{\delta_1}(x_j) \wedge \ldots \wedge p_k^{\delta_k}(x_j)$$

where for $i = 1, \dots, k$:

$$p_i^{\delta_i}(x_j) := \begin{cases} p_i(x_j) & \text{if } \delta_i = 1; \\ \neg p_i(x_j) & \text{if } \delta_i = 0. \end{cases}$$

Moreover

$$\mathsf{E}p_1^{\delta_1} \cdots p_k^{\delta_k} = 0 \qquad \text{abbreviates} \qquad \neg \exists x_1 \, p_1^{\delta_1} \cdots p_k^{\delta_k}(x_1)\,,$$

and for any $0 < m < n^+$:

$$\mathsf{E}p_1^{\delta_1} \cdots p_k^{\delta_k} = m \qquad \text{abbreviates}$$

$$\exists x_1 \ldots \exists x_m \Big(\bigwedge_{1 \leq j \leq m} p_1^{\delta_1} \cdots p_k^{\delta_k}(x_j) \wedge \bigwedge_{1 \leq i < j \leq m} \neg\, x_i \doteq x_j \wedge$$
$$\wedge\ \forall x_{m+1} \big(p_1^{\delta_1} \cdots p_k^{\delta_k}(x_{m+1}) \supset \bigvee_{1 \leq i \leq m} x_i \doteq x_{m+1} \big) \Big)\,.$$

The sentence "$\mathsf{E}p_1^{\delta_1} \cdots p_k^{\delta_k} = m$" says that there exist exactly m elements which are $p_1^{\delta_1} \cdots p_k^{\delta_k}$ers. Finally:

$$\mathsf{E}p_1^{\delta_1} \cdots p_k^{\delta_k} \geq n^+ \qquad \text{abbreviates}$$

$$\exists x_1 \ldots \exists x_n \Big(\bigwedge_{1 \leq j \leq n} p_1^{\delta_1} \cdots p_k^{\delta_k}(x_j) \wedge \bigwedge_{1 \leq i < j \leq n} \neg\, x_i \doteq x_j \Big)\,.$$

The sentence “$\mathsf{E}\mathsf{p}_1^{\delta_1}\cdots\mathsf{p}_k^{\delta_k} \geq n^+$” says that there exist at least n^+ elements which are $\mathsf{p}_1^{\delta_1}\cdots\mathsf{p}_k^{\delta_k}$ers.

We prove semantically that the sentence σ is equivalent to some sentence σ^{CDNF} in *canonical disjunctive normal form.* The sentence σ^{CDNF} will be a disjunction (possibly degenerate), each disjunct in which is a conjunction of the following form:

$$\bigwedge_{\delta_1,\ldots,\delta_k=0,1} \mathsf{E}\mathsf{p}_1^{\delta_1}\cdots\mathsf{p}_k^{\delta_k} = m_{\delta_1\ldots\delta_k} \;\; (\text{or } \mathsf{E}\mathsf{p}_1^{\delta_1}\cdots\mathsf{p}_k^{\delta_k} \geq n^+)$$

with 2^k cojuncts (exactly one for every sequence $\langle\delta_1,\ldots,\delta_k\rangle$) and such that $0 \leq m_{\delta_1\ldots\delta_k} < n^+$.

Let $\mathfrak{I} = \langle D, \imath\rangle$ be an interpretation of Form. Let $\imath(\mathsf{p}_i)^1 = \imath(\mathsf{p}_i)$ and $\imath(\mathsf{p}_i)^0 = D - \imath(\mathsf{p}_i)$. For every sequence $\langle\delta_1,\ldots,\delta_k\rangle$ of zeros and ones if $|\imath(\mathsf{p}_1)^{\delta_1}\cap\ldots\cap\imath(\mathsf{p}_k)^{\delta_k}| < n^+$ then let $\kappa^{\mathfrak{I}}_{\delta_1\ldots\delta_k}$ be the sentence $\mathsf{E}\mathsf{p}_1^{\delta_1}\cdots\mathsf{p}_k^{\delta_k} = |\imath(\mathsf{p}_1)^{\delta_1}\cap\ldots\cap\imath(\mathsf{p}_k)^{\delta_k}|$; otherwise let $\kappa^{\mathfrak{I}}_{\delta_1\ldots\delta_k}$ be the sentence $\mathsf{E}\mathsf{p}_1^{\delta_1}\cdots\mathsf{p}_k^{\delta_k} \geq n^+$. For $\mathfrak{I}$ we put the following conjunction:

$$\kappa^{\mathfrak{I}} := \bigwedge_{\delta_1,\ldots,\delta_k=0,1} \kappa^{\mathfrak{I}}_{\delta_1\ldots\delta_k}$$

The succession of conjuncts in $\kappa^{\mathfrak{I}}$ is assigned by the linear order $\prec$ on $\{0,1\}^k$ defined by the condition: $\langle\delta_1^1,\ldots,\delta_k^1\rangle \prec \langle\delta_1^2,\ldots,\delta_k^2\rangle$ iff $(\delta_1^1\ldots\delta_k^1)_2 < (\delta_1^2\ldots\delta_k^2)_2$. We obtain:

Lemma 3. *If $\mathfrak{I} \models \sigma$, then $\mathfrak{I} \models \kappa^{\mathfrak{I}}$.* □

We have $(n^+)^{2^k}$ mappings from $\{0,1\}^k$ into $\{0,1,\ldots,n^+\}$. Let $\mathcal{M}$ be the set of all of them. Any interpretation $\mathfrak{I}$ determines the mapping $m^{\mathfrak{I}} \in \mathcal{M}$ such that

$$m^{\mathfrak{I}}_{\delta_1\ldots\delta_k} := \begin{cases} n^+ & \text{if } |\imath(\mathsf{p}_1)^{\delta_1}\cap\ldots\cap\imath(\mathsf{p}_k)^{\delta_k}| \geq n^+ \\ |\imath(\mathsf{p}_1)^{\delta_1}\cap\ldots\cap\imath(\mathsf{p}_k)^{\delta_k}| & \text{otherwise} \end{cases}$$

where $\imath(\mathsf{p}_i)^1$ is $\imath(\mathsf{p}_i)$ while $\imath(\mathsf{p}_i)^0$ is the complement of $\imath(\mathsf{p}_i)$.

Lemma 4. *The set* $\{\kappa^{\mathfrak{I}} : \mathfrak{I} \models \sigma\}$ *is non-empty and finite.*

Proof. For any model $\mathfrak{I}$ of σ we have

$$\kappa^{\mathfrak{I}}_{\delta_1\dots\delta_k} = \begin{cases} \mathsf{E}p_1^{\delta_1}\cdots p_k^{\delta_k} = m^{\mathfrak{I}}_{\delta_1\dots\delta_k} & \text{if } m^{\mathfrak{I}}_{\delta_1\dots,\delta_k} < n^+; \\ \mathsf{E}p_1^{\delta_1}\cdots p_k^{\delta_k} \geq n^+ & \text{if } m^{\mathfrak{I}}_{\delta_1\dots\delta_k} = n^+. \end{cases}$$

Hence for all models $\mathfrak{I}_1$, $\mathfrak{I}_2$ of σ if $m^{\mathfrak{I}_1} = m^{\mathfrak{I}_2}$ then $\kappa^{\mathfrak{I}_1} = \kappa^{\mathfrak{I}_2}$. □

By Lemma 4 we can put:

$$\sigma^{\text{CDNF}} := \bigvee_{\mathfrak{I} \text{ is a model of } \sigma} \kappa^{\mathfrak{I}} .$$

The succession of disjuncts in σ^{CDNF} is assigned by the linear order $\prec^*$ on $\mathcal{M}$ defined by the condition: $m^{\mathfrak{I}} \prec^* m^{\mathfrak{J}}$ iff $(m^{\mathfrak{I}}_{\delta_1^1\dots\delta_k^1}\dots m^{\mathfrak{I}}_{\delta_1^{2^k}\dots\delta_k^{2^k}})_{n^++1} < (m^{\mathfrak{J}}_{\delta_1^1\dots\delta_k^1}\dots m^{\mathfrak{J}}_{\delta_1^{2^k}\dots\delta_k^{2^k}})_{n^++1}$ where the set $\{0,1\}^k$ is ordered linearly by $\prec$.

Theorem 2. *The sentences* σ *and* σ^{CDNF} *are equivalent.*

Proof. Suppose that $\mathfrak{I} \models \sigma$. Since by Lemma 3 we have $\mathfrak{I} \models \kappa^{\mathfrak{I}}$, $\mathfrak{I} \models \sigma^{\text{CDNF}}$, by the definition of σ^{CDNF}.

Let, on the other hand, $\mathfrak{I} = \langle D, \imath\rangle$ and $\mathfrak{I} \models \sigma^{\text{CDNF}}$. Then, by the definition of σ^{CDNF}, for some model $\mathfrak{J} = \langle E, \jmath\rangle$ of σ we obtain $\mathfrak{I} \models \kappa^{\mathfrak{J}}$.

Since $\mathfrak{I} \models \kappa^{\mathfrak{J}}_{\delta_1\dots\delta_k}$, for any $\langle \delta_1, \dots, \delta_k\rangle \in \{0,1\}^k$ we have that either

$$|\imath(p_1)^{\delta_1} \cap \ldots \cap \imath(p_k)^{\delta_k}| = |\jmath(p_1)^{\delta_1} \cap \ldots \cap \jmath(p_k)^{\delta_k}| < n^+$$

or

$$|\imath(p_1)^{\delta_1} \cap \ldots \cap \imath(p_k)^{\delta_k}| \geq n^+ \text{ and } |\jmath(p_1)^{\delta_1} \cap \ldots \cap \jmath(p_k)^{\delta_k}| \geq n^+ .$$

Let $\mathfrak{I}_\sigma = \langle D_\sigma, \imath_\sigma\rangle$ and $\mathfrak{J}_\sigma = \langle E_\sigma, \jmath_\sigma\rangle$ be interpretations obtained by the procedure given in the proof of Theorem 1. Let $p > 0$ be the number of sequences $\langle \delta_1, \dots, \delta_k\rangle$ for which $\imath(p_1)^{\delta_1} \cap \ldots \cap \imath(p_k)^{\delta_k} \neq \emptyset$. Then $D_\sigma = \bigcup_{i=1}^p D_i^\star$ and $E_\sigma = \bigcup_{i=1}^p E_i^\star$ where for $|\imath(p_1)^{\delta_1} \cap \ldots \cap \imath(p_k)^{\delta_k}| = m_{\delta_1\dots\delta_k} > 0$

we choose from the sets $\imath(\mathsf{p}_1)^{\delta_1} \cap \ldots \cap \imath(\mathsf{p}_k)^{\delta_k}$ and $\jmath(\mathsf{p}_1)^{\delta_1} \cap \ldots \cap \jmath(\mathsf{p}_k)^{\delta_k}$ respectively subsets $D_i^\star$ and $E_i^\star$ such that $|D_i^\star| = |E_i^\star| = \min(m_{\delta_1 \ldots \delta_k}, n^+)$.

There is a one-one function h with domain D_σ and range E_σ such that the restriction $h|_{D_i^\star}$ is a one-one mapping of $D_i^\star$ onto $E_i^\star$ for $i = 1, \ldots, p$. Hence for $j = 1, \ldots, k$ we obtain $h[\imath_\sigma(\mathsf{p}_j)] = \jmath_\sigma(\mathsf{p}_j)$ where $\imath_\sigma(\mathsf{p}_j) = D_\sigma \cap \imath(\mathsf{p}_j)$ and $\jmath_\sigma(\mathsf{p}_j) = E_\sigma \cap \jmath(\mathsf{p}_j)$.

By induction on the length of formulas it is easy to show that: $\mathcal{I}_\sigma \models \sigma$ iff $\mathcal{J}_\sigma \models \sigma$. Hence $\mathcal{J} \models \sigma$ iff $\mathcal{J}_\sigma \models \sigma$ iff $\mathcal{I}_\sigma \models \sigma$ iff $\mathcal{I} \models \sigma$, by Theorem 1. Since $\mathcal{J} \models \sigma$, we obtain $\mathcal{I} \models \sigma$. □

Remark. There is an effective procedure for finding σ^{CDNF}.

Actually, the sentence σ is satisfiable iff the sentence $\neg\,\sigma$ is not logically valid. Since, by Corollary 2, the second fact is decidable, so is the first fact.

For a given k and n we say that the interpretation $\mathcal{I}_1$ and $\mathcal{I}_2$ are equivalent (we write: $\mathcal{I}_1 \equiv \mathcal{I}_2$) iff $m^{\mathcal{I}_1} = m^{\mathcal{I}_2}$. Clearly, $\equiv$ is an equivalence relation. There exist $(n^+)^{2^k}$ equivalence classes.

For any interpretation $\mathcal{I}$ we have $\mathcal{I} \equiv \mathcal{I}_\sigma$ where $\mathcal{I}_\sigma$ is defined in Theorem 1 (the domain of $\mathcal{I}_\sigma$ is finite). Moreover, if $\mathcal{I}_1 \equiv \mathcal{I}_2$ then $\kappa^{\mathcal{I}_1} = \kappa^{\mathcal{I}_2}$. (By the method used in the proof of Theorem 2, we can obtain that: if $\mathcal{I}_1 \equiv \mathcal{I}_2$, then $\mathcal{I}_1 \models \sigma$ iff $\mathcal{I}_2 \models \sigma$.)

Finally, we can choose from any equivalence class of the relation $\equiv$ an interpretation $\mathcal{I}$ whose domain is finite. For $\mathcal{I}$ we have an effective method for deciding whether σ is true in $\mathcal{I}$ or not. In the first case we add $\kappa^{\mathcal{I}}$ to σ^{CDNF}. □

5.2. Let σ be a given satisfiable sentence which contains no individual constants and no one-place predicate letters, i.e., $k = 0$. Suppose that σ has $n > 0$ distinct variables in non-tautological identities.

We will use the following abbreviations. Let $\mathsf{x}_1, \ldots, \mathsf{x}_n$ be the n distinct variables from Var and for any $0 < m < n$ we establish that:

$\mathsf{E} = m$ abbreviates

$$\exists \mathsf{x}_1 \ldots \exists \mathsf{x}_m \Big(\bigwedge_{1 \le i < j \le m} \neg\, \mathsf{x}_i \doteq \mathsf{x}_j \,\wedge\, \forall \mathsf{x}_{m+1} \bigvee_{1 \le i \le m} \mathsf{x}_i \doteq x_{m+1} \Big) .$$

The sentence "$\mathsf{E} = m$" says that there exist exactly m elements.

$$\mathsf{E} \geq n \qquad \text{abbreviates} \qquad \exists x_1 \ldots \exists x_n \big(\bigwedge_{1 \leq i < j \leq n} \neg\, x_i \doteq x_j \big)\,.$$

The sentence "$\mathsf{E} \geq n$" says that there exist at least n elements.

We prove that the sentence σ is equivalent to some sentence σ^{CDNF} in *canonical disjunctive normal form.* The sentence σ^{CDNF} will be a disjunction (possibly degenerate), in which each disjunct has the following form:

$$\mathsf{E} = m \;\; (\text{or } \mathsf{E} \geq n)$$

where $0 < m < n$.

Let $\mathfrak{I} = \langle D, \imath \rangle$ be an interpretation of Form. If $|D| < n$ then let $\alpha^{\mathfrak{I}}$ be the sentence $\mathsf{E} = |D|$; otherwise let $\alpha^{\mathfrak{I}}$ be the sentence $\mathsf{E} \geq n$. We obtain:

Lemma 3′. *If $\mathfrak{I} \models \sigma$, then $\mathfrak{I} \models \alpha^{\mathfrak{I}}$.* □

Lemma 4′. *The set $\{\alpha^{\mathfrak{I}} \,:\, \mathfrak{I} \models \sigma\}$ is non-empty and finite.* □

By Lemma 4′ we can establish that:

$$\sigma^{\text{CDNF}} := \bigvee_{\mathfrak{I} \text{ is a model of } \sigma} \alpha^{\mathfrak{I}}\,.$$

The succession of disjuncts in σ^{CDNF} is the same as the natural succession in the set $\{1, 2, \ldots, n\}$.

The proof of the following theorem is similar to the proof of Theorem 2.

Theorem 2′. *The sentences σ and σ^{CDNF} are equivalent.* □

5.3. Let σ be a given satisfiable sentence from Form. Let P_σ be the set of k predicate letters (possibly $k = 0$) occurring in σ. Let B_σ be the set of q name letters occurring in subformulas $p(b)$ of σ, and let A_σ be the set of l name letters that occur in non-tautological identities in σ (possibly $q = 0$ and/or $l = 0$ and/or $B_\sigma \cap A_\sigma \neq \emptyset$). Moreover, let σ have the n variables in non-tautological identities (possibly $n = 0$).

It is possible to show that the sentence σ is equivalent to some sentence σ^{CDNF} in *canonical disjunctive normal form.* σ^{CDNF} is a disjunction (possibly degenerate), each disjunct in which is a conjunction, the conjuncts in which are of certain specified forms. Every conjunct must be either $p(b)$ or $a \doteq a'$ or the negation of them (for some $p \in P_\sigma$, $b \in B_\sigma$ and $a, a' \in A_\sigma$), or analogous sentences to those in 5.1 (resp. in 5.2 if $k = 0$).

6. Notes about the estimations in [2]

G. Boolos and R. Jeffrey (and R. Jensen) proved in [2, p. 252–254] for sentences without individual constants that:

> If σ is a monadic sentence which is satisfiable, then σ is true in some interpretation whose domain contains at most $2^k \cdot r$ members, k being the number of predicate letters and r being the number of variables in σ. (Theorem 1, p. 250)[4]

From this, one easily concludes:

> If σ is a sentence containing no non-logical symbols, then if σ is satisfiable, σ is true in some interpretation whose domain contains no more members than there are variables in σ. (Theorem 2, p. 254)

Another consequence of Theorem 1 from [2] is a result about pure monadic sentences, i.e., sentences without the equals-sign and individual constants:

> If σ is a pure monadic sentence which is satisfiable, then σ is true in some interpretation whose domain contains at most 2^k members. (Theorem 3, p. 254)

The last theorem is an immediate consequence of Theorem 1 by the fact that:

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[4] For monadic formulas containing no individual constants, we obtain from Theorem 1: *a formula φ is logically valid iff φ is true in every interpretation whose domain contains at most $2^k \cdot r$ members.*

> Any pure monadic sentence is [logically] equivalent to a pure monadic sentence [i.e. they are true in the same interpretations] containing exactly the same predicate letters and only one variable. (Theorem 4, p. 254)[5]

In exercise 25.2 Boolos and Jeffrey have written: "Show that the estimates $2^k \cdot r$, r and 2^k in Theorems 1, 2, 3 cannot be reduced". Actually the estimates in Theorems 1 and 2 cannot be reduced by the method used in the proof of Theorem 1 in [2] (cf. the relation of *exact likeness* of two finite sequences of elements of domains of interpretations). However, we have presented here a method which improves their estimations.

Observe that for the sentences containing no individual constants, a corollary of the "only if" part of (†) is:

Corollary 3. *If a sentence φ is satisfiable, then φ is also true in some interpretation whose domain contains at most $2^k \cdot n^+$ members, k being the number of predicate letters and n^+ being the number of variables in non-tautological identities which are subformulas of φ, or when non-tautological identities do not occur in φ, $n^+ = 1$.*

Corollary 3 we can obtain also from Theorem 1 in [2] and from the following analogue of Theorem 4 in [2] (which we do not prove):

For any monadic sentence σ there exists a monadic sentence σ' which has the properties:

(i) *σ and σ' are logically equivalent;*

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[5] For pure monadic formulas, a corollary of Theorem 4 is: *a formula φ is logically valid iff φ is true in every interpretation whose domain contains at most 2^k members.*

For a formula $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{a}_1, \ldots, \mathsf{a}_q)$ — containing $r \geq n^+$ variables and $q \geq l$ individual constants — we write $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{x}_{r+1}, \ldots, \mathsf{x}_{r+q})$ for the formula obtained by replacing all occurrences of a_i in $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{a}_1, \ldots, \mathsf{a}_q)$ by x_{r+i}. If $\mathsf{x}_i \neq \mathsf{x}_j$ for $i, j = r+1, \ldots, r+q$ and $i \neq j$, then: $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{a}_1, \ldots, \mathsf{a}_q)$ is logically valid iff $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{x}_{r+1}, \ldots, \mathsf{x}_{r+q})$ is logically valid. Thence, by Theorem 1, we obtain: *$\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{a}_1, \ldots, \mathsf{a}_q)$ is logically valid iff $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_r, \mathsf{x}_{r+1}, \ldots, \mathsf{x}_{r+q})$ is true in every interpretation, whose domain contains at most $2^k \cdot (q + r)$ members.* Let us notice that $2^k \cdot (q + r) \geq l + 2^k \cdot n^+$.

(ii) *σ and σ' contain exactly the same predicate letters (possibly none);*

(iii) *σ' has as many variables as σ in non-tautological identities or, when* non-tautological identities do not occur in σ, then σ' has exactly one variable.

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