CARDINALITIES OF MODELS AND THE EXPRESSIVE POWER OF MONADIC PREDICATE LOGIC
(with equality and individual constants)

1. Introduction

In this paper we improve one result from [2, p. 250–255] (cf. the last section of the paper). We also present, by a method of semantical nature, a certain description of the expressive power of first-order logic with one-place predicates and equality.

As in [2, p. 250], a monadic formula is a formula of first-order logic, all of whose non-logical symbols are either one-place predicate letters or name letters (resp. individual constants). Monadic formulas may contain the logical symbol ‘\(=\)’, i.e., the equals-sign (or the sign of identity).

A formula \(\tau_1 \equiv \tau_2\) for different terms \(\tau_1\) and \(\tau_2\) is said to be a non-tautological identity.

We prove that for any monadic formula \(\varphi\): \(\varphi\) is logically valid iff \(\varphi\) is true in every interpretation whose domain contains at most \(l + 2^k \cdot n^+\) members, \(k\) being the number of predicate letters in \(\varphi\), \(l\) being the number

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of individual constants in non-tautological identities which are subformulas of \( \varphi \), and \( n^+ \) being the number of variables in non-tautological identities which are subformulas of \( \varphi \) or, when no variable occurs in non-tautological identities in \( \varphi \), \( n^+ = 1.1 \)

In the body of this paper we shall define each semantical notion along the line of Barwise [1] and Mendelson [3], and then prove the above fact.

2. Syntactical and semantical preliminaries

Let \( \text{PL}, \text{NL} \) and \( \text{Var} \) be fixed denumerably infinite disjoint sets; \( \text{PL} \) is the set of one-place predicate letters, \( \text{NL} \) is the set of name letters, and \( \text{Var} \) is the set of variables. Let \( \text{Term} := \text{NL} \cup \text{Var} \) be the set of terms and let \( \text{Form} \) be the set of formulas of monadic predicate logic (with equality ‘=’ and name letters), i.e., the smallest set such that:

- if \( \tau \in \text{Term} \) and \( p \in \text{PL} \), then \( p(\tau) \in \text{Form} \);
- if \( \tau_1, \tau_2 \in \text{Term} \), then \( \tau_1 = \tau_2 \in \text{Form} \);
- if \( \varphi, \psi \in \text{Form} \), then \( \neg \varphi \in \text{Form} \) and \( (\varphi \psi) \in \text{Form} \), where \( \psi \in \{ \vee, \wedge, [] \} \);
- if \( \varphi \in \text{Form} \) and \( x \in \text{Var} \), then \( \forall x \varphi \in \text{Form} \) and \( \exists x \varphi \in \text{Form} \).

The logical symbols \( \neg, \vee, \wedge, [ ] \) are the propositional connectives of negation, disjunction, conjunction and material implication. Moreover, the logical symbols \( \forall \) and \( \exists \) are universal and existential quantifiers.

A sentence is a formula without any free variables.

An interpretation of \( \text{Form} \) is a pair \( I = (D, i) \), where \( D \) is a non-empty set (called the domain of \( I \)) and \( i \) is a mapping from \( \text{NL} \) into \( D \) and from \( \text{PL} \) into the power set \( \mathcal{P}(D) \).

Let \( D^{\text{Var}} \) be the set of functions from \( \text{Var} \) into \( D \). All elements of \( D^{\text{Var}} \) will be called assignments. For any assignment \( s \) we use \( s^{(x)} \) for the assignment \( s' \) which agrees with \( s \) except that \( s'(x) = d \).

Moreover, for any \( s \) and \( i \) we define a function \( t_{s,i} := s \cup i|_{\text{NL}} \) of one argument, with terms as arguments and values in \( D \).

\footnote{For formulas without any variables this estimate can be reduced (cf. footnote 2).}
Let $I = \langle D, i \rangle$ be an interpretation of Form. We define a relation $I \models \varphi [s]$ (read: the assignment $s$ satisfies $\varphi$ in $I$) for all assignments $s$ and all formulas $\varphi$ as follows:

$I \models p(\tau) [s]$ iff $t_{s,1}(\tau) \in i(p); I \models \tau_1 \equiv \tau_2 [s]$ iff $t_{s,1}(\tau_1) = t_{s,1}(\tau_2); I \models \neg \varphi [s]$

iff not $I \models \varphi [s]; I \models (\varphi \land \psi) [s]$ iff $I \models \varphi [s]$ and $I \models \psi [s]; I \models (\varphi \lor \psi) [s]$ iff either $I \models \varphi$ or $I \models \psi [s]; I \models (\varphi \supset \psi) [s]$ iff either not $I \models \varphi [s]$ or $I \models \psi [s]; I \models \forall x \varphi [s]$ iff for all $d \in D, I \models \varphi \left[ s(d) \right]; I \models \exists x \varphi [s]$ iff there is a $d \in D$ such that $I \models \varphi \left[ s(d) \right]$.

A formula $\varphi$ is true in the interpretation $I$ (or $I$ is a model of $\varphi$; we write: $I \models \varphi$) iff for every assignment $s$, $I \models \varphi [s]$. The form “not $I \models \varphi$” will be an abbreviation for “it is not the case that $I \models \varphi$”. So not $I \models \varphi$ iff for some $s$, not $I \models \varphi [s]$. If $\sigma$ is a sentence, then the truth or falsity of $I \models \sigma [s]$ is independent of $s$. Thus $I \models \sigma$ iff for some (hence every) $s$, $I \models \sigma [s]$. So for any sentence $\sigma$: $I \models \neg \sigma$ iff not $I \models \sigma$.

A formula $\varphi$ is called logically valid iff $\varphi$ is true in every interpretation.

A formula $\varphi$ is satisfiable iff there is an interpretation for which $\varphi$ is satisfied by at least one assignment. A sentence $\sigma$ is satisfiable iff there is a model of $\sigma$.

3. Estimation

Let $\varphi$ be a given monadic formula. Let $p_1, \ldots, p_k$ be the $k$ distinct one-place predicate letters (possibly $k = 0$) occurring in $\varphi$. Let $a_1, \ldots, a_l$ be the $l$ distinct name letters that occur in non-tautological identities in $\varphi$ (possibly $l = 0$). Moreover, let $x_1, \ldots, x_n$ be the $n$ distinct variables that occur in non-tautological identities in $\varphi$ (possibly $n = 0$). We put $n^+ := \max(n, 1)$.

Theorem 1. Let $I = \langle D, i \rangle$ be an arbitrary interpretation of Form. We can construct a certain interpretation $I_\varphi = \langle D_\varphi, t_\varphi \rangle$, whose domain $D_\varphi$
contains at most \( l + 2^k \cdot n^+ \) members,\(^2\) and which satisfies the following condition:

\[
\mathcal{J}_\varphi \models \varphi \quad \text{iff} \quad \mathcal{J} \models \varphi .
\]

**Proof.** We define the binary relation \( \equiv \) in \( D \) by:

- if \( k = 0 \) then we put \( \equiv := D \times D \)
- if \( k > 0 \) then for \( d_1, d_2 \in D \)

\[
d_1 \equiv d_2 \quad \text{iff} \quad \text{for every } i = 1, \ldots, k : d_1 \in \iota(p_i) \text{ iff } d_2 \in \iota(p_i) .
\]

Clearly, \( \equiv \) is an equivalence relation. We denote an equivalence class of \( d \) in \( D \) by \( \|d\| \). We have: for any one-place predicate symbol \( p_i \) (\( 0 \leq i \leq k \)) and any \( d \in D \)

\[
d \in \iota(p_i) \quad \text{iff} \quad \|d\| \subseteq \iota(p_i) .
\]

Let \( p := |D/\equiv| \). Obviously \( 0 < p \leq 2^k \). Let \( \{D_1, \ldots, D_p\} := D/\equiv \). Obviously the sets \( D_1, \ldots, D_p \) are non-empty and pairwise disjoint. Moreover, \( D = \bigcup_{i=1}^{p} D_i \).

If \( k > 0 \) then for every \( i = 1, \ldots, p \) there is a sequence \( \langle \delta_{i_1}, \ldots, \delta_{i_k} \rangle \) of zeros and ones such that \( D_i = \iota(p_1)^{\delta_{i_1}} \cap \ldots \cap \iota(p_k)^{\delta_{i_k}} \) where \( \iota(p_j)^1 = \iota(p_j) \) and \( \iota(p_j)^0 = D - \iota(p_j) \). If \( k = 0 \) then \( D_1 = D \).

Let \( A := \{a_1, \ldots, a_l\} \) and \( X := \{x_1, \ldots, x_n\} \).

For \( i = 1, \ldots, p \) we choose from the set \( D_i - \iota[A] \) a subset \( D_i^* \) such that \( |D_i^*| = \min(|D_i - \iota[A]|, n^+) \).

We put

\[
D_\varphi := \iota[A] \cup \bigcup_{i=1}^{p} D_i^* .
\]

The set \( D_\varphi \) contains, at most, \( l + 2^k \cdot n^+ \) members.

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\(^2\) For a given monadic sentence which has no variables this estimate can be reduced to \( l^+ + \min(2^k - 1, q - l^+) \) where \( l^+ := \max(l, 1) \) and \( q \) being the number of name letters occurring in this sentence.
A projection from $D$ on $D_\varphi$ is a function $\pi : D \to D_\varphi$ such that for any $d \in D$ and $i = 1, \ldots, p$:

(i) if $d \in D_i - i[A]$ then $\pi(d) \in D^*_i$,
(ii) if $d \in D_\varphi$ then $\pi(d) = d$.

The conditions (i) and (ii) imply:

(i') $d \in D_i - i[A]$ iff $\pi(d) \in D^*_i$.
(ii') $d \in D_i$ iff $\pi(d) \in D^*_i \cup (D_i \cap i[A])$.

Let $\text{Proj}$ be the set of all projections from $D$ on $D_\varphi$. Clearly, $\text{Proj} \neq \emptyset$.

We choose from the set $\text{Proj}$ one projection $\pi^*$. We define the mapping $i_\varphi$ from $\text{NL}$ into $D_\varphi$ by:\footnote{If $\varphi$ is a pure monadic formula (without `\&` and individual constants; $k > 0$ and $l = 0 = n$), then the given proof is “classical”. In this case, we may also suppose that $D_\varphi := D/\equiv$ and $i_\varphi(p) := \{\Delta \in D/\equiv : \Delta \subseteq i(p)\}$. Moreover, if $\varphi$ is a monadic formula with individual constants and without `\&` ($k > 0$ and $l = 0 = n$), we may suppose that $D_\varphi := D/\equiv$, $i_\varphi(p) := \{\Delta \in D/\equiv : \Delta \subseteq i(p)\}$ and $i_\varphi(b) := \|i(b)\|$. In both cases there exists exactly one “canonical” projection $\|\cdot\| : D \ni d \mapsto \|d\| \in D_\varphi$.}$

$$i_\varphi(b) := \pi^*(i(b))$$

and from $\text{PL}$ into $\mathcal{P}(D_\varphi)$ by:

$$i_\varphi(p) := D_\varphi \cap i(p).$$

Obviously — by the definition of the set $D_\varphi$ and by (ii) in the definition of the set $\text{Proj}$ — for $i = 1, \ldots, l$ we have:

$$i_\varphi(a_i) = i(a_i).$$

For any $s \in D^{\mathsf{Var}}$ let $\text{Proj}^s$ be the set of all projections which are injections on the set $s[X] \cup i[A]$. \footnote{If $\varphi$ is a pure monadic formula (without `\&` and individual constants; $k > 0$ and $l = 0 = n$), then the given proof is “classical”. In this case, we may also suppose that $D_\varphi := D/\equiv$ and $i_\varphi(p) := \{\Delta \in D/\equiv : \Delta \subseteq i(p)\}$. Moreover, if $\varphi$ is a monadic formula with individual constants and without `\&` ($k > 0$ and $l = 0 = n$), we may suppose that $D_\varphi := D/\equiv$, $i_\varphi(p) := \{\Delta \in D/\equiv : \Delta \subseteq i(p)\}$ and $i_\varphi(b) := \|i(b)\|$. In both cases there exists exactly one “canonical” projection $\|\cdot\| : D \ni d \mapsto \|d\| \in D_\varphi$.}
Lemma 1. For any $s \in D^\Var$: $\Proj^s \neq \emptyset$.

Proof. Notice that for any $s \in D^\Var$ and $i = 1, \ldots, m$ we have the condition (ii') and $|D_i \cap (s[X] \cup i[A])| \leq |D_i^* \cup (D_i \cap i[A])|$. Actually, $D_i \cap (s[X] \cup i[A]) = (D_i \cap s[X]) \cup (D_i \cap i[A]) = ((D_i - i[A]) \cap s[X]) \cup (D_i \cap i[A])$ $\cup (D_i - i[A]) \cap s[X]) \cup (D_i \cap i[A])$. Moreover, $|D_i - i[A]| \cap s[X]| \leq \min(|D_i - i[A]|, n) \leq |D_i^*|$ and $D_i^* \cap i[A] = \emptyset$. Thus $|D_i \cap (s[X] \cup i[A])| \leq |D_i^*| + |D_i \cap i[A]| = |D_i^* \cup (D_i \cap i[A])|$. $\square$

For any $s \in D^\Var$ and $\pi \in \Proj$ we have $\pi \circ s \in D^\Var$. Moreover:

Lemma 2. Suppose that $\psi$ is a subformula of $\varphi$. Then for any $s \in D^\Var$ and $\pi \in \Proj^\varphi$:

$$J_\varphi \models \psi[\pi \circ s] \iff J \models \psi[s].$$

Proof. Induction on $\psi$. If $\psi$ is atomic, either

(a) for some $y \in \Var$ and $i = 1, \ldots, k$, $\psi = p_i(y)$, or

(b) for some $b \in \NL$ and $i = 1, \ldots, k$, $\psi = p_i(b)$, or

(c) for some $y \in \Var$, $\psi = y \equiv y$, or

(d) for some $b \in \NL$, $\psi = b \equiv b$, or

(e) for some $i, j = 1, \ldots, n, i \neq j$, $\psi = x_i \equiv x_j$, or

(f) for some $i, j = 1, \ldots, n, i \neq j$, $\psi = a_i \equiv a_j$, or

(g) for some $i = 1, \ldots, n$ and $j = 1, \ldots, l, \psi = x_i \equiv a_j$ or $\psi = a_j \equiv x_i$.

Let $s$ and $\pi$ be arbitrary members of $D^\Var$ and $\Proj^\varphi$ respectively.

In the first case: $J_\varphi \models p_i(y)[\pi \circ s]$ iff $\pi(s(y)) \in \tau_\varphi(p_i)$ iff — by the definition of $\tau_\varphi$ — $\pi(s(y)) \in \tau(p_i)$ iff — by ($\ast$) and (ii') — $s(y) \in \tau(p_i)$ iff $J \models p_i(y)[s]$.

In the second case: $J_\varphi \models p_i(b)[\pi \circ s]$ iff $\tau_\varphi(b) \in \tau_\varphi(p_i)$ iff — by the definition of $\tau_\varphi$ — $\tau^\ast(\tau(b)) \in \tau(p_i)$ iff — by ($\ast$) and (ii') — $\tau(b) \in \tau(p_i)$ iff $J \models p_i(b)[s]$.

In cases (c) and (d): $\psi$ is true in both $J_\varphi$ and $J$.

In case (e): $J_\varphi \models x_i \equiv x_j[\pi \circ s]$ iff $\pi(s(x_i)) = \pi(s(x_j))$ iff — by the definition of $\Proj^\varphi$ — $s(x_i) = s(x_j)$ iff $J \models x_i \equiv x_j[s]$. 
In case (f): $J_\varphi \models a_i \models a_j [\pi \circ s]$ iff $\nu_\varphi (a_i) = \nu_\varphi (a_j)$ iff $\models a_i \models a_j [s]$.

In case (g): $J_\varphi \models x_i \models a_j [\pi \circ s]$ iff $\pi (x_i) = \nu_\varphi (a_j)$ iff $\models x_i \models a_j [s]$.

Conversely, assume that $\models x_i \models a_j [\pi \circ s]$ and let $\models e \models a_j [\pi \circ s]$.

Then $\models x_i \models a_j [\pi \circ s]$.

Thus $\models x_i \models a_j [\pi \circ s]$.

Therefore $\models x_i \models a_j [\pi \circ s]$.

So $\models x_i \models a_j [\pi \circ s]$.

The argument is similar if $\psi$ is a conjunction or other truth-functional compound of simpler formulas.

Suppose then that $\psi = \forall y \chi$. Let $J_\varphi \models \forall y \chi [\pi \circ s]$, i.e. for all $d \in D_\varphi$, $J_\varphi \models \chi [\pi \circ s]$. Let $d$ be an arbitrary member of $D$. Then,

$\pi \circ s = (\pi \circ s)'$ and $J_\varphi \models \chi [\pi \circ s]'$.

Hence, by inductive hypothesis, $J_\varphi \models \chi [s]$. So $J_\varphi \models \forall y \chi [s]$. Conversely, assume that $J_\varphi \models \forall y \chi [s]$, i.e., for all $d \in D$, $J_\varphi \models \chi [s]$.

Let $e$ be an arbitrary member of $D_\varphi$. Then $J_\varphi \models \chi [s]$.

Hence, by inductive hypothesis, $J_\varphi \models \chi [s]$. Moreover, by (ii) in the definition of $\text{Proj}$,

$\pi \circ s = (\pi \circ s)'$. Thus $J_\varphi \models \chi [s]$. So $J_\varphi \models \forall y \chi [\pi \circ s]$.

The argument is similar if $\psi$ is an existential quantification of a simpler formula.

Suppose now that $J_\varphi \models \varphi$. Let $s : \text{Var} \rightarrow D$ be an arbitrary assignment.

Then $\pi \circ s : \text{Var} \rightarrow D_\varphi$ for any $\pi \in \text{Proj}^*$. Thus $J_\varphi \models \varphi [\pi \circ s]$. Therefore $J_\varphi \models \varphi [s]$, by Lemma 2. So $J_\varphi \models \varphi$.

Assume that $J_\varphi \models \varphi$. Let $s : \text{Var} \rightarrow D$ be an arbitrary assignment.

Then also $s : \text{Var} \rightarrow D$; so $J_\varphi \models \varphi [s]$. Hence, by Lemma 2, $J_\varphi \models \varphi [\pi \circ s]$ for any $\pi \in \text{Proj}^*$. Moreover, by property (ii) in the definition of $\text{Proj}$, we have $s = \pi \circ s$. Therefore $J_\varphi \models \varphi$. Thus $J_\varphi \models \varphi$.

4. Corollaries
Both from the “if” part of (†) and from the “only if” part of (†) in Theorem 1 we obtain:

**Corollary 1.** For any $\varphi \in \text{Form}$ the following conditions are equivalent:

(i) $\varphi$ is logically valid;
(ii) $\varphi$ is true in every interpretation, whose domain contains at most $l + 2^k \cdot n^+$ members;

where $k$, $l$ and $n^+$ we defined in the introduction and in section 3.

**Proof.** “$\Rightarrow$” By the definition of logically valid formulas.

“$\Leftarrow$” Proof by the “if” part of (†). Let $I$ be an arbitrary interpretation of Form. Then $I_\varphi \models \varphi$, by (ii).

(a) Therefore, by using the “if” part of (†), $I \models \varphi$. So $\varphi$ is logically valid.

(b) Let $\text{cl}(\varphi)$ be a universal closure of $\varphi$. Then $I_\varphi \models \text{cl}(\varphi)$ and not $I_\varphi \models \neg \text{cl}(\varphi)$, since $\text{cl}(\varphi)$ is a sentence. Therefore, by using the “only if” part of (†), not $I \models \neg \text{cl}(\varphi)$. Hence $I \models \text{cl}(\varphi)$ and $I \models \varphi$. So $\varphi$ is logically valid. \qed

**Corollary 2.** Suppose that $\varphi \in \text{Form}$, respectively:

1) contains no individual constants,
2) contains no one-place predicate letters,
3) contains no non-logical symbols,
4) does not contain the equals-sign,
5) does not contain the equals-sign and any individual constants.

Then $\varphi$ is logically valid iff $\varphi$ is true in every interpretation whose domain contains, respectively, at most:

1) $2^k \cdot n^+$
2) $l + n^+$
3) $n^+$
4) $2^k$
5) $2^k$

members.

5. Expressive power of sentences from Form

5.1. Let $\sigma$ be a given satisfiable sentence which contains no individual constants. Suppose that $p_1, \ldots, p_k$ are the $k$ (distinct) predicate letters that occur in $\sigma$ ($k > 0$), and that $\sigma$ has $n$ distinct variables in non-tautological identities (possibly $n = 0$). Let $n^+ := \max(n, 1)$.

We will use the following abbreviations. Let $x_1, \ldots, x_{n^+}$ be the $n^+$ distinct variables from $\text{Var}$. For $0 < j \leq n^+$ and for any sequence $\langle \delta_1, \ldots, \delta_k \rangle$ of zeros and ones we say that:

$p_1^{\delta_1} \cdots p_k^{\delta_k}(x_j)$ abbreviates $p_1^{\delta_1}(x_j) \land \cdots \land p_k^{\delta_k}(x_j)$

where for $i = 1, \ldots, k$: $p_i^{\delta_i}(x_j) := \begin{cases} p_i(x_j) & \text{if } \delta_i = 1; \\ \neg p_i(x_j) & \text{if } \delta_i = 0. \end{cases}$

Moreover

$\exists p_1^{\delta_1} \cdots p_k^{\delta_k} = 0$ abbreviates $\exists p_1^{\delta_1} \cdots p_k^{\delta_k}(x_1)$,

and for any $0 < m < n^+$:

$\exists p_1^{\delta_1} \cdots p_k^{\delta_k} = m$ abbreviates

$\exists x_1 \cdots \exists x_m (\bigwedge_{1 \leq j \leq m} p_1^{\delta_1} \cdots p_k^{\delta_k}(x_j) \land \bigwedge_{1 \leq i < j \leq m} \neg x_i \equiv x_j \land \forall x_{m+1} (p_1^{\delta_1} \cdots p_k^{\delta_k}(x_{m+1}) \supset \bigvee_{1 \leq i \leq m} x_i = x_{m+1})$.

The sentence “$\exists p_1^{\delta_1} \cdots p_k^{\delta_k} = m$” says that there exist exactly $m$ elements which are $p_1^{\delta_1} \cdots p_k^{\delta_k}$ers. Finally:

$\exists p_1^{\delta_1} \cdots p_k^{\delta_k} \geq n^+$ abbreviates

$\exists x_1 \cdots \exists x_n (\bigwedge_{1 \leq j \leq n} p_1^{\delta_1} \cdots p_k^{\delta_k}(x_j) \land \bigwedge_{1 \leq i < j \leq n} \neg x_i \equiv x_j)$.
The sentence \( \text{"EP}^1 \cdots p^k \geq n^+ \)" says that there exist at least \( n^+ \) elements which are \( p^1 \cdots p^k \) s.

We prove semantically that the sentence \( \sigma \) is equivalent to some sen-
tence \( \sigma_{\text{CDNF}} \) in \textit{canonical disjunctive normal form}. The sentence \( \sigma_{\text{CDNF}} \) will be a disjunction (possibly degenerate), each disjunct in which is a conjunc-
tion of the following form:

\[
\bigwedge_{\delta_1, \ldots, \delta_k = 0, 1} \text{EP}_{\delta_1} \cdots p^k_{\delta_k} = m_{\delta_1 \cdots \delta_k} \quad \text{(or EP}^1 \cdots p^k \geq n^+)\]

with \( 2^k \) conjuncts (exactly one for every sequence \( \langle \delta_1, \ldots, \delta_k \rangle \)) and such that
\( 0 \leq m_{\delta_1 \cdots \delta_k} < n^+ \).

Let \( J = \langle D, i \rangle \) be an interpretation of \textit{Form}. Let \( i(p_1)^1 = i(p_i) \) and \( i(p_i)^0 = D - i(p_i) \). For every sequence \( \langle \delta_1, \ldots, \delta_k \rangle \) of zeros and ones if
\( |i(p_1)^{\delta_1} \cap \ldots \cap i(p_k)^{\delta_k}| < n^+ \) then let \( \kappa_{\delta_1 \cdots \delta_k}^J \) be the sentence \( \text{EP}^1 \cdots p^k_{\delta_k} = |i(p_1)^{\delta_1} \cap \ldots \cap i(p_k)^{\delta_k}|; \) otherwise let \( \kappa_{\delta_1 \cdots \delta_k}^J \) be the sentence \( \text{EP}^1 \cdots p^k_{\delta_k} \geq n^+ \). For \( J \) we put the following conjunction:

\[
\kappa^J := \bigwedge_{\delta_1, \ldots, \delta_k = 0, 1} \kappa_{\delta_1 \cdots \delta_k}^J
\]

The succession of conjuncts in \( \kappa^J \) is assigned by the linear order \( \prec \) on \( \{0, 1\}^k \) defined by the condition: \( \langle \delta_1^1, \ldots, \delta_k^1 \rangle \prec \langle \delta_1^2, \ldots, \delta_k^2 \rangle \) iff \( \langle \delta_1 \ldots \delta_k \rangle_2 < \langle \delta_1^2 \ldots \delta_k^2 \rangle_2 \). We obtain:

**Lemma 3.** If \( J \models \sigma \), then \( J \models \kappa^J \). \( \square \)

We have \( (n^+)^2^k \) mappings from \( \{0, 1\}^k \) into \( \{0, 1, \ldots, n^+\} \). Let \( M \) be the set of all of them. Any interpretation \( J \) determines the mapping \( m^J \in M \) such that

\[
m^J_{\delta_1 \cdots \delta_k} := \begin{cases} n^+ & \text{if } |i(p_1)^{\delta_1} \cap \ldots \cap i(p_k)^{\delta_k}| \geq n^+ \\ |i(p_1)^{\delta_1} \cap \ldots \cap i(p_k)^{\delta_k}| & \text{otherwise} \end{cases}
\]

where \( i(p_i)^1 \) is \( i(p_i) \) while \( i(p_i)^0 \) is the complement of \( i(p_i) \).
Lemma 4. The set \( \{ \kappa^j : \mathcal{I} \models \sigma \} \) is non-empty and finite.

Proof. For any model \( \mathcal{I} \) of \( \sigma \) we have

\[
\kappa^j_{\delta_1 \ldots \delta_k} = \begin{cases} 
E \rho_1^{\delta_1} \cdots \rho_k^{\delta_k} = m^j_{\delta_1 \ldots \delta_k} & \text{if } m^j_{\delta_1 \ldots \delta_k} < n^+; \\
E \rho_1^{\delta_1} \cdots \rho_k^{\delta_k} \geq n^+ & \text{if } m^j_{\delta_1 \ldots \delta_k} = n^+.
\end{cases}
\]

Hence for all models \( \mathcal{I}_1, \mathcal{I}_2 \) of \( \sigma \) if \( m^{j_1} = m^{j_2} \) then \( \kappa^{j_1} = \kappa^{j_2} \).

By Lemma 4 we can put:

\[
\sigma^{\text{CDNF}} := \bigvee_{\mathcal{I} \text{ is a model of } \sigma} \kappa^j.
\]

The succession of disjuncts in \( \sigma^{\text{CDNF}} \) is assigned by the linear order \( \prec^* \) on \( \mathcal{M} \) defined by the condition: \( m^j \prec^* m^g \) iff \( (m^j_{\delta_1 \ldots \delta_k} \cdots m^g_{\delta_1 \ldots \delta_k})_{n^+ + 1} < (m^j_{\delta_1 \ldots \delta_k} \cdots m^g_{\delta_1 \ldots \delta_k})_{n^+ + 1} \) where the set \( \{0, 1\}^k \) is ordered linearly by \( \prec \).

Theorem 2. The sentences \( \sigma \) and \( \sigma^{\text{CDNF}} \) are equivalent.

Proof. Suppose that \( \mathcal{I} \models \sigma \). Since by Lemma 3 we have \( \mathcal{I} \models \kappa^j \), \( \mathcal{I} \models \sigma^{\text{CDNF}} \), by the definition of \( \sigma^{\text{CDNF}} \).

Let, on the other hand, \( \mathcal{I} = (D, \iota) \) and \( \mathcal{I} \models \sigma^{\text{CDNF}} \). Then, by the definition of \( \sigma^{\text{CDNF}} \), for some model \( \mathcal{J} = (E, j) \) of \( \sigma \) we obtain \( \mathcal{J} \models \kappa^j \).

Since \( \mathcal{J} \models \kappa^j_{\delta_1 \ldots \delta_k} \), for any \( \langle \delta_1, \ldots, \delta_k \rangle \in \{0, 1\}^k \) we have that either

\[
|\iota(p_1)^{\delta_1} \cap \ldots \cap \iota(p_k)^{\delta_k}| = |\jmath(p_1)^{\delta_1} \cap \ldots \cap \jmath(p_k)^{\delta_k}| < n^+
\]

or

\[
|\iota(p_1)^{\delta_1} \cap \ldots \cap \iota(p_k)^{\delta_k}| \geq n^+ \text{ and } |\jmath(p_1)^{\delta_1} \cap \ldots \cap \jmath(p_k)^{\delta_k}| \geq n^+.
\]

Let \( \mathcal{I}_\sigma = (D_\sigma, \iota_\sigma) \) and \( \mathcal{J}_\sigma = (E_\sigma, j_\sigma) \) be interpretations obtained by the procedure given in the proof of Theorem 1. Let \( p > 0 \) be the number of sequences \( \langle \delta_1, \ldots, \delta_k \rangle \) for which \( \iota(p_1)^{\delta_1} \cap \ldots \cap \iota(p_k)^{\delta_k} \neq \emptyset \). Then \( D_\sigma = \bigcup_{i=1}^p D_i^* \) and \( E_\sigma = \bigcup_{i=1}^p E_i^* \) where for \( \iota(p_1)^{\delta_1} \cap \ldots \cap \iota(p_k)^{\delta_k} = m_{\delta_1 \ldots \delta_k} > 0 \)
we choose from the sets \( \nu(p_1)^{\delta_1} \cap \ldots \cap \nu(p_k)^{\delta_k} \) and \( \nu(p_1)^{\delta_1} \cap \ldots \cap \nu(p_k)^{\delta_k} \) respectively subsets \( D_i^\star \) and \( E_i^\star \) such that \( |D_i^\star| = |E_i^\star| = \min(m_{\delta_i}, n^+). \)

There is a one-one function \( h \) with domain \( D_\sigma \) and range \( E_\sigma \) such that the restriction \( h|_{D_i^\star} \) is a one-one mapping of \( D_i^\star \) onto \( E_i^\star \) for \( i = 1, \ldots, p. \)

Hence for \( j = 1, \ldots, k \) we obtain \( h[\nu(p_j)] = j_\sigma(p_j) \) where \( \nu(p_j) = D_\sigma \cap \nu(p_j) \) and \( j_\sigma(p_j) = E_\sigma \cap j(p_j). \)

By induction on the length of formulas it is easy to show that: \( J_\sigma \models \sigma \) iff \( J_\sigma \models \sigma \) iff \( J_\sigma \models \sigma \) iff \( J \models \sigma \), by Theorem 1.

Since \( J \models \sigma \), we obtain \( J \models \sigma \).

\[ \square \]

**Remark.** There is an effective procedure for finding \( \sigma_{\text{CDNF}} \).

Actually, the sentence \( \sigma \) is satisfiable iff the sentence \( \neg \sigma \) is not logically valid. Since, by Corollary 2, the second fact is decidable, so is the first fact.

For a given \( k \) and \( n \) we say that the interpretation \( J_1 \) and \( J_2 \) are equivalent (we write: \( J_1 \equiv J_2 \)) iff \( m_{J_1}^J = m_{J_2}^J \). Clearly, \( \equiv \) is an equivalence relation. There exist \( (n^+)2^k \) equivalence classes.

For any interpretation \( J \) we have \( J \equiv J_\sigma \) where \( J_\sigma \) is defined in Theorem 1 (the domain of \( J_\sigma \) is finite). Moreover, if \( J_1 \equiv J_2 \) then \( \kappa_{J_1}^J = \kappa_{J_2}^J. \) (By the method used in the proof of Theorem 2, we can obtain that: if \( J_1 \equiv J_2 \), then \( J_1 \models \sigma \) iff \( J_2 \models \sigma \).)

Finally, we can choose from any equivalence class of the relation \( \equiv \) an interpretation \( J \) whose domain is finite. For \( J \) we have an effective method for deciding whether \( \sigma \) is true in \( J \) or not. In the first case we add \( \kappa_J^J \) to \( \sigma_{\text{CDNF}} \).

\[ \square \]

5.2. Let \( \sigma \) be a given satisfiable sentence which contains no individual constants and no one-place predicate letters, i.e., \( k = 0. \) Suppose that \( \sigma \) has \( n > 0 \) distinct variables in non-tautological identities.

We will use the following abbreviations. Let \( x_1, \ldots, x_n \) be the \( n \) distinct variables from \( \text{Var} \) and for any \( 0 < m < n \) we establish that:

\[ E = m \]

abbreviates

\[ \exists x_1 \ldots \exists x_n ( \bigwedge_{1 \leq i < j \leq m} \neg x_i \models x_j \land \forall x_{m+1} \bigvee_{1 \leq i \leq m} x_i \models x_{m+1}). \]
The sentence “$E = m$” says that there exist exactly $m$ elements.

$$E \geq n \quad \text{abbreviates} \quad \exists x_1 \ldots \exists x_n (\bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j).$$

The sentence “$E \geq n$” says that there exist at least $n$ elements.

We prove that the sentence $\sigma$ is equivalent to some sentence $\sigma_{\text{CDNF}}$ in canonical disjunctive normal form. The sentence $\sigma_{\text{CDNF}}$ will be a disjunction (possibly degenerate), in which each disjunct has the following form:

$$E = m \ (\text{or} \ E \geq n)$$

where $0 < m < n$.

Let $I = \langle D, i \rangle$ be an interpretation of Form. If $|D| < n$ then let $\alpha^3$ be the sentence $E = |D|$; otherwise let $\alpha^3$ be the sentence $E \geq n$. We obtain:

**Lemma 3’.** If $I \models \sigma$, then $I \models \alpha^3$. \hfill $\square$

**Lemma 4’.** The set $\{\alpha^3 : I \models \sigma\}$ is non-empty and finite. \hfill $\square$

By Lemma 4’ we can establish that:

$$\sigma_{\text{CDNF}} := \bigvee_{J \text{ is a model of } \sigma} \alpha^3.$$ 

The succession of disjuncts in $\sigma_{\text{CDNF}}$ is the same as the natural succession in the set $\{1, 2, \ldots, n\}$.

The proof of the following theorem is similar to the proof of Theorem 2.

**Theorem 2’.** The sentences $\sigma$ and $\sigma_{\text{CDNF}}$ are equivalent. \hfill $\square$

5.3. Let $\sigma$ be a given satisfiable sentence from Form. Let $P_\sigma$ be the set of $k$ predicate letters (possibly $k = 0$) occurring in $\sigma$. Let $B_\sigma$ be the set of $q$ name letters occurring in subformulas $p(b)$ of $\sigma$, and let $A_\sigma$ be the set of $l$ name letters that occur in non-tautological identities in $\sigma$ (possibly $q = 0$ and/or $l = 0$ and/or $B_\sigma \cap A_\sigma \neq \emptyset$). Moreover, let $\sigma$ have the $n$ variables in non-tautological identities (possibly $n = 0$).
It is possible to show that the sentence $\sigma$ is equivalent to some sentence $\sigma^{\text{CDNF}}$ in canonical disjunctive normal form. $\sigma^{\text{CDNF}}$ is a disjunction (possibly degenerate), each disjunct in which is a conjunction, the conjuncts in which are of certain specified forms. Every conjunct must be either $p(b)$ or $a \equiv a'$ or the negation of them (for some $p \in P_\sigma$, $b \in B_\sigma$ and $a, a' \in A_\sigma$), or analogous sentences to those in 5.1 (resp. in 5.2 if $k = 0$).

6. Notes about the estimations in [2]

G. Boolos and R. Jeffrey (and R. Jensen) proved in [2, p. 252–254] for sentences without individual constants that:

If $\sigma$ is a monadic sentence which is satisfiable, then $\sigma$ is true in some interpretation whose domain contains at most $2^k \cdot r$ members, $k$ being the number of predicate letters and $r$ being the number of variables in $\sigma$. (Theorem 1, p. 250)

From this, one easily concludes:

If $\sigma$ is a sentence containing no non-logical symbols, then if $\sigma$ is satisfiable, $\sigma$ is true in some interpretation whose domain contains no more members than there are variables in $\sigma$. (Theorem 2, p. 254)

Another consequence of Theorem 1 from [2] is a result about pure monadic sentences, i.e., sentences without the equals-sign and individual constants:

If $\sigma$ is a pure monadic sentence which is satisfiable, then $\sigma$ is true in some interpretation whose domain contains at most $2^k$ members. (Theorem 3, p. 254)

The last theorem is an immediate consequence of Theorem 1 by the fact that:

\footnote{For monadic formulas containing no individual constants, we obtain from Theorem 1: a formula $\varphi$ is logically valid iff $\varphi$ is true in every interpretation whose domain contains at most $2^k \cdot r$ members.}
Any pure monadic sentence is [logically] equivalent to a pure monadic sentence [i.e. they are true in the same interpretations] containing exactly the same predicate letters and only one variable. (Theorem 4, p. 254)

In exercise 25.2 Boolos and Jeffrey have written: “Show that the estimates \(2^k \cdot r, r\) and \(2^k\) in Theorems 1, 2, 3 cannot be reduced”. Actually the estimates in Theorems 1 and 2 cannot be reduced by the method used in the proof of Theorem 1 in [2] (cf. the relation of exact likeness of two finite sequences of elements of domains of interpretations). However, we have presented here a method which improves their estimations.

Observe that for the sentences containing no individual constants, a corollary of the “only if” part of (1) is:

**Corollary 3.** If a sentence \(\varphi\) is satisfiable, then \(\varphi\) is also true in some interpretation whose domain contains at most \(2^k \cdot n^+\) members, \(k\) being the number of predicate letters and \(n^+\) being the number of variables in non-tautological identities which are subformulas of \(\varphi\), or when non-tautological identities do not occur in \(\varphi\), \(n^+ = 1\).

Corollary 3 we can obtain also from Theorem 1 in [2] and from the following analogue of Theorem 4 in [2] (which we do not prove):

For any monadic sentence \(\sigma\) there exists a monadic sentence \(\sigma'\) which has the properties:

(i) \(\sigma\) and \(\sigma'\) are logically equivalent;

---

5 For pure monadic formulas, a corollary of Theorem 4 is: a formula \(\varphi\) is logically valid iff \(\varphi\) is true in every interpretation whose domain contains at most \(2^k\) members.

For a formula \(\varphi(x_1, \ldots, x_r, a_1, \ldots, a_q)\) — containing \(r \geq n^+\) variables and \(q \geq l\) individual constants — we write \(\varphi(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+q})\) for the formula obtained by replacing all occurrences of \(a_i\) in \(\varphi(x_1, \ldots, x_r, a_1, \ldots, a_q)\) by \(x_{r+i}\). If \(x_i \neq x_j\) for \(i, j = r + 1, \ldots, r + q\) and \(i \neq j\), then: \(\varphi(x_1, \ldots, x_r, a_1, \ldots, a_q)\) is logically valid iff \(\varphi(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+q})\) is logically valid. Thence, by Theorem 1, we obtain: \(\varphi(x_1, \ldots, x_r, a_1, \ldots, a_q)\) is logically valid iff \(\varphi(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+q})\) is true in every interpretation, whose domain contains at most \(2^k \cdot (q + r)\) members. Let us notice that \(2^k \cdot (q + r) \geq l + 2^k \cdot n^+\).
(ii) $\sigma$ and $\sigma'$ contain exactly the same predicate letters (possibly none);

(iii) $\sigma'$ has as many variables as $\sigma$ in non-tautological identities or, when non-tautological identities do not occur in $\sigma$, then $\sigma'$ has exactly one variable.

References


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