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CUT ELIMINATION IN NONCOMMUTATIVE SUBSTRUCTURAL LOGICS*

A b s t r a c t. The present paper is concerned with the cut eliminability for some sequent systems of noncommutative substructural logics, i.e. substructural logics without exchange rule. Sequent systems of several extensions of noncommutative logics **FL** and **LBB′I**, which is sometimes called $\mathbf{T}_{\rightarrow} - \mathbf{W}$, will be introduced. Then, the cut elimination theorem and the decision problem for them will be discussed in comparison with their commutative extensions.

1. Introduction

Usually, standard logics like the classical and the intuitionistic logics can be formulated by sequent calculi with so-called structural rules, i.e. exchange, contraction and weakening rules. In recent years, logics without some of these structural rules, which are now called substructural logics,

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have been studied extensively. Relevant logic, **BCK** logic, linear logic and Lambek's calculus of syntactic categories are examples of substructural logics. But they have been studied rather separately and sometimes their results are even overlapping. So, it will be necessary to try to discuss them in a common framework.

In the present paper, we will introduce sequent calculi of various noncommutative substructural logics, i.e. substructural logics without exchange rule, and will show when the cut elimination theorem holds for them, comparing them with the cut elimination in commutative logics. Then, it will be shown that it is hard to get a cut-free system for a noncommutatives logic with contraction and even if we get it, unlike commutative case, its decidability will not be its immediate consequence. As for related topics, see e.g. [9], [12], [6], [4], [13], [14], [1] and also [2].

We will discuss mainly noncommutative extensions of two basic systems **FL** and $\mathbf{BB'I}$ in our paper. The system **FL** is defined to be a sequent calculus obtained from the sequent calculus **LJ** of the intuitionistic predicate logic by deleting all structural rules. It is called the full Lambek logic in [10] and [11]. On the other hand, $\mathbf{BB'I}$ is the implicational logic, which is formulated as a Hilbert-type system by the axiom schemes **B**, $\mathbf{B'}$ and **I** with modus ponens as a single rule of inference. Sometimes, it is called $\mathbf{T}_{\rightarrow} - \mathbf{W}$ by relevant logicians.

We will follow the terminologies in [10]. Therefore, for instance, $\mathbf{FL_c}$ and $\mathbf{FL_w}$ denote **FL** with contraction and with weakening, respectively.

The present paper will be organized as follows. In the next section, we will consider the full Lambek logic **FL** and its noncommutative extensions. It is well known that the cut elimination theorem holds in **FL** and $\mathbf{FL_w}$. On the other hand, we will show that the cut elimination theorem holds neither in **FL** with contraction rule nor in **FL** with contraction and weakening rules. In fact, it will be shown that even the cut elimination theorem for their implicational fragments doesn't hold. Next, we will introduce a new rule called *global contraction rule*, which is a stronger form of the contraction rule. But, it is equivalent to the contraction rule if the logic under consideration has the exchange rule. We will show the cut elimination for **FL** with global contraction rule.

In Section 3, we will discuss the noncommutative extensions of **FL** with *knotted structural rules*, i.e. $(n \rightsquigarrow k)$ rules, which are weaker forms of the contraction rule when $n > k$. In their joint paper [5], Hori, Ono and Schellinx proved that the cut elimination theorem holds for $\mathbf{FL_e}$ with $(n \rightsquigarrow 1)$ rule when $n > 0$, by replacing the original cut rule by *multi-cut*. In noncommutative case, the lack of exchange rule makes the cut elimination harder. In fact, it will be shown that the cut elimination theorem doesn't hold for any of **FL** with $(n \rightsquigarrow 1)$ rule when $n > 0$. On the other hand, it will be proved that the cut elimination theorem holds for **FL** with *global* $(n \rightsquigarrow k)$ rule if and only if $k = 1$.

In Section 4, we will introduce the Gentzen-type sequent calculus $\mathbf{LBB'I}$ of the implicational logic $\mathbf{BB'I}$, following [2] and [7]. Then we will discuss the cut eliminability of some of its noncommutative, implicational extensions with knotted structural rules.

2. Noncommutative extensions of full Lambek logic

Following [11], we will introduce **FL** (*full Lambek logic*), which is the sequent calculus obtained from the sequent calculus **LJ** for the intuitionistic predicate logic by deleting all the structural rules. The language of **FL** consists of logical constans $1, 0, \top$ and $\bot$, logical connectives $\supset, \wedge, \vee$ and $*$ (*multiplicative conjunction* or *fusion*) and two quantifiers $\forall$ and $\exists$. Sequents (in **FL**) are defined as expressions of the form $A_1, \ldots, A_n \rightarrow B$, where A_i's and B are (possibly empty) formulas.

We will use Greek capital letters Γ, Δ, Σ, etc. for finite sequences of formulas separated by commas. Sometimes the sequence consisting of n occurrences of a formula A, i.e. $\overbrace{A, A, \ldots, A}^{n}$, will be denoted by A^n. (When $n = 0$, A^n denotes the empty sequence.) More generally, Γ^n denotes the sequence of formulas, which consists of repetitions of Γ by n times. To abbreviate parenthese in formulas, we use the convention that $\supset$ associates to the right. Also, we will abbreviate a formula $\overbrace{A \supset A \supset A \ldots A}^{n} \supset B$ to $A^n \supset B$. Now, we will give a precise definition of **FL**.

Definition 1. The sequent calculus **FL** consists of the following initial sequents

1. $A \to A$
2. $\Gamma, \bot, \Delta \to C$
3. $\Gamma \to \top$
4. $\to 1$
5. $0 \to$,

and the following rules of inference;

cut rule:

$$\frac{\Gamma \to A \quad \Delta, A, \Sigma \to C}{\Delta, \Gamma, \Sigma \to C}$$

rules for logical constants:

$$\frac{\Gamma, \Delta \to C}{\Gamma, 1, \Delta \to C}\,(1w) \qquad \frac{\Gamma \to}{\Gamma \to 0}\,(0w)$$

rules for logical connectives:

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B}\,(\to\supset) \qquad \frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta, A \supset B, \Gamma, \Sigma \to C}\,(\supset\to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}\,(\to \vee 1) \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}\,(\to \vee 2)$$

$$\frac{\Gamma, A, \Delta \to C \quad \Gamma, B, \Delta \to C}{\Gamma, A \vee B, \Delta \to C}\,(\vee \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B} (\to \wedge)$$

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} (\wedge 1 \to) \qquad \frac{\Gamma, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C} (\wedge 2 \to)$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A * B} (\to *) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A * B, \Delta \to C} (* \to)$$

rules for quantifiers:

$$\frac{\Gamma \to A(t)}{\Gamma \to \exists z A(z)} (\to \exists) \qquad \frac{\Gamma, A(x), \Delta \to C}{\Gamma, \exists z A(z), \Delta \to C} (\exists \to)$$

$$\frac{\Gamma \to A(x)}{\Gamma \to \forall z A(z)} (\to \forall) \qquad \frac{\Gamma, A(t), \Delta \to C}{\Gamma, \forall z A(z), \Delta \to C} (\forall \to)$$

Here, t is any term, and x is any variable satisfying the eigenvariable condition, that is, x does not occur in the lower sequent of $(\exists \to)$ and $(\to \forall)$.

Sequent calculi $\mathbf{FL_w}$ and $\mathbf{FL_c}$ are defined to be the systems obtained from **FL** by adding the following weakening rules and contraction rule, respectively;

$$\frac{\Gamma, \Sigma \to C}{\Gamma, A, \Sigma \to C} (weakening \to) \qquad \frac{\Gamma \to}{\Gamma \to C} (\to weakening)$$

$$\frac{\Gamma, A, A, \Sigma \to C.}{\Gamma, A, \Sigma \to C} (contraction)$$

In the same way, we define $\mathbf{FL_{w,c}}$ to be the system which is obtained from **FL** by adding both weakening and contraction rules . We can get the sequent calculus $\mathbf{FL_{e,w,c}}$, which is equivalent to **LJ**, from $\mathbf{FL_{w,c}}$ by adding the following exchange rule;

$$\frac{\Gamma, B, A, \Sigma \to C}{\Gamma, A, B, \Sigma \to C}\,(exchange)$$

Note that the exchange rule is an admissible rule in any fragment of $\mathbf{FL_{w,c}}$ if it has the rules for the implication and also either the rules for fusion or for (additive) conjunction. Except for $\mathbf{FL_c}$ and $\mathbf{FL_{w,c}}$, the cut elimination theorem holds for any extension of **FL** which is obtained by adding some of these structural rules (see e.g. [6], [8], and [12]).

Next, we will discuss the cut elimination in noncommutative extensions of **FL** with contraction rule. In commutative case, the contraction rule causes some complications in proving the cut elimination theorem. To avoid them, every application of the cut rule is replaced by the *mix rule* and then the mix elimination theorem will be proved. In noncommutative case, the situation become more complicated because of the lack of the exchange rule. In fact, we have the following. (The cut elimination theorem for $\mathbf{FL_c}$ has been erroneously reported in some literatures, e.g. [4] and [13].)

Theorem 1. *The cut elimination theorem does not hold for* $\mathbf{FL_c}$ *and* $\mathbf{FL_{w,c}}$. *In fact, it doesn't hold even for their implicational fragments.*

Proof. The following sequent is provable but is not provable without cut rule (in both $\mathbf{FL_c}$ and $\mathbf{FL_{w,c}}$).

(1) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, t \supset r, q \supset t \longrightarrow s.$

To see this, consider the following sequent, which is provable without cut:

(2) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, q \supset r, q \supset r \longrightarrow s.$

By applying the contraction rule, we have the following.

(3) $p \supset ((q \supset r) \supset s), (q \supset r) \supset p, q \supset r \longrightarrow s.$

Then, taking a cut for $t \supset r, q \supset t \longrightarrow q \supset r$ with (3), we have (1). But, this cut is not interchangeable with the contraction rule just in the above. In fact, the fact that (1) is not provable in $\mathbf{FL}_{w,c}$ without cut can be shown as follows. Suppose otherwise. Then there is a cut-free proof $\mathbf{P}$ of (1) in $\mathbf{FL}_{w,c}$. Clearly, (1) is not an initial sequent and $\mathbf{P}$ must contain at least one application of the rule for the logical connective $\supset$. Let I be its lowest application. Then, I must be ($\supset\rightarrow$). Moreover, the lower sequent of I is of the following form;

$$(4) \quad (p \supset ((q \supset r) \supset s))^i, ((q \supset r) \supset p)^j, (t \supset r)^m, (q \supset t)^n \rightarrow s^k,$$

where k is either 0 or 1, since (1) is obtained from the lower sequent by applying only weakening and contraction rules. Then we can show that any of i, j, m, n and k is positive. In fact, if not then (4) is not a tautology and *a fortiori* is not provable in $\mathbf{FL}_{w,c}$ without cut. For instance, when $j = 0$, the classical valuation v defined by

$$v(q) = v(r) = v(t) = \mathbf{t},\ v(p) = v(s) = \mathbf{f}$$

gives the sequent (4) the value $\mathbf{f}$.

So, the principal formula of I is one of $p \supset ((q \supset r) \supset s)$, $(q \supset r) \supset p$, $t \supset r$ and $q \supset t$. Suppose that it is $q \supset t$. Then, because of the form of the rule ($\supset\rightarrow$), the left upper sequent of I must be of the form $(q \supset t)^{n'} \rightarrow q$ for some $n' \geq 0$. But it is obvious that this sequent is not a tautology. Thus, $q \supset t$ cannot be the principal formula of I. Similarly, we can show that $t \supset r$ is not the principal formula. So, suppose that the principal formula is $(q \supset r) \supset p$. In this case, the left upper sequent is of the form;

$$((q \supset r) \supset p)^{j'}, (t \supset r)^{m'}, (q \supset t)^{n'} \rightarrow q \supset r$$

with some j', m', n'. Then, it is easy to see that this sequent is provable in $\mathbf{FL}_{w,c}$ (without cut) if and only if both m' and n' are positive. (Note that when n' is positive, $m' = m$.) Then, because of the form of ($\supset\rightarrow$), the right upper sequent must be of the form;

$$(p \supset ((q \supset r) \supset s))^i, ((q \supset r) \supset p)^{j''}, p, (q \supset t)^{n''} \rightarrow s.$$

Now, let u be the classical valuation defined by

$$u(p) = u(q) = u(t) = \mathbf{t},\ u(r) = u(s) = \mathbf{f}.$$

Then, the above sequent takes the value $\mathbf{f}$ under u. This is a contradiction. Thus, $(q \supset r) \supset p$ cannot be the principal formula of I. Similarly, we can show that the formula $p \supset ((q \supset r) \supset s)$ cannot be the principal formula. Therefore, we are led to a contradiction. Thus, (1) is not provable in $\mathbf{FL}_{w,c}$ without cut.

Note that when we have the exchange rule, (1) follows from the following sequent, which is provable without cut, by using the exchange and the contraction.

$$p \supset ((q \supset r) \supset s), (q \supset r) \supset p, t \supset r, q \supset t, t \supset r, q \supset t \longrightarrow s.$$

It is interesting to find a "cut-free" system which is equivalent to $\mathbf{FL_c}$. Here, we will introduce a stronger contraction, called *global contraction*, by which not only two formulas, but even two sequences of formulas can be contracted.

Definition 2. The global contraction is a rule of inference of the following from:

$$\frac{\Gamma, \Pi, \Pi, \Delta \to C}{\Gamma, \Pi, \Delta \to C}$$

It is obvious that the global contraction rule is equal to the usual contraction in any system with the exchange rule. Let $\mathbf{FL_{gc}}$ be the system obtained from $\mathbf{FL}$ by adding the global contraction. We will first note the following;

Lemma 1. *Any fragment of* $\mathbf{FL_{gc}}$ *is equivalent to that of* $\mathbf{FL_c}$ *as long as the language contains the fusion.*

Proof. It is clear that contraction rule is just a special form of global contraction rule. On the other hand, we can replace an application of the global contraction rule by several applications of rule for fusion, contraction rule and cut rule as follows:

$$
\dfrac{\dfrac{\vdots}{\Pi \to \Pi^*}\,(\to *) \qquad \dfrac{\dfrac{\dfrac{\Gamma, \Pi, \Pi, \Delta \to C}{\text{some applications of } (* \to)}}{\Gamma, \Pi^*, \Pi^*, \Delta \to C}}{\Gamma, \Pi^*, \Delta \to C}\,(contraction)}{\Gamma, \Pi, \Delta \to C}\,(cut)
$$

Here, Π^* denotes the formula $A_1 * \ldots * A_k$ if Π is a sequence of formulas $A_1, \ldots, A_k$.

Next we have the following theorem:

Theorem 2. *The cut elimination theorem holds for* $\mathbf{FL_{gc}}$.

Proof. First we will replace each application of cut by the *multi-cut* rule introduced in [5]. Then the proof goes similarly to the proof of the cut elimination theorem for $\mathbf{BCI}_1^n$, i.e. **BCI** with the $(n \rightsquigarrow 1)$ rule discussed in the next section. (See [5].)

Similarly, we can show the cut elimination theorem for $\mathbf{FL_{w,gc}}$, i.e. $\mathbf{FL_{gc}}$ with weakening, which is an alternative sequent calculus for the intuitionistic logic. Although $\mathbf{FL_{gc}}$ enjoys the cut elimination theorem, Kripke's method does not seem to work well in proving its decidability. In fact, the decision problem of propositional $\mathbf{FL_{gc}}$ remains open.

3. Knotted structural rules in noncommutative logics

In the present section, we will discuss noncommutative extensions of **FL** with the *knotted* structural rule, $(n \rightsquigarrow k)$ rule. Following [5], the $(n \rightsquigarrow k)$ rule is defined as follows:

Definition 3. Let (n, k) be any pair of natural numbers n and k such that $n \neq k$ and $k > 0$. Then, $(n \rightsquigarrow k)$ rule is a rule of inference of the form:

$$\frac{\Gamma, \overbrace{A, \ldots, A}^{n}, \Delta \to B}{\Gamma, \underbrace{A, \ldots, A}_{k}, \Delta \to B}$$

It is obvious that the $(0 \rightsquigarrow 1)$ and $(2 \rightsquigarrow 1)$ rules are exactly the weakening and contraction rules. In general, $(n \rightsquigarrow k)$ rule is a restricted form of the weakening rule when $n < k$ and of the contraction rule when $n > k$. Note that there is a remark on a rule related to $(k + 1 \rightsquigarrow k)$ in [3]. In the following, $\mathbf{FL}_{(\mathbf{n} \rightsquigarrow \mathbf{k})}$ will denote the system obtained from **FL** by adding the $(n \rightsquigarrow k)$ rule. Similarly, $\mathbf{FL}_{\mathbf{w},(\mathbf{n} \rightsquigarrow \mathbf{k})}$ will denote the system obtained from $\mathbf{FL}_{\mathbf{w}}$ by adding the $(n \rightsquigarrow k)$ rule. Notice that when we dicuss systems $\mathbf{FL}_{\mathbf{w},(\mathbf{n} \rightsquigarrow \mathbf{k})}$, it is enough for us to consider $\mathbf{FL}_{\mathbf{w},(\mathbf{k+1} \rightsquigarrow \mathbf{k})}$ only.

It is proved in [5]that the cut elimination theorem holds for $\mathbf{FL}_{\mathbf{e},(\mathbf{n} \rightsquigarrow \mathbf{1})}$ for each $n \geq 2$. Here, because of the lack of the exchange rule, we have a weaker result as shown in the following.

Theorem 3. *The cut elimination theorem holds for* $\mathbf{FL}_{(\mathbf{n} \rightsquigarrow \mathbf{k})}$ *if and only if* $n = 0$ *and* $k = 1$.

Proof. We remark first that $\mathbf{FL}_{(\mathbf{0} \rightsquigarrow \mathbf{1})}$ is just $\mathbf{FL}_{\mathbf{w}}$, whose cut elimination theorem is proved in [5].So, it remains to show the only-if part. In fact, for $n > 0$,

$$p \supset (q \supset r)^{n-1} \supset s, (q \supset r) \supset p, (t \supset r, q \supset t)^{k} \to s$$

and for $n = 0$ and $k \neq 1$,

$$(q \supset r) \supset p, (t \supset r, q \supset t)^{k+1} \to p$$

will give us counterexamples. They are generalizations of the counterexamples given in Section 2.

Notice that the counterexamples in the proof of the above theorem will also work well for $\mathbf{FL_{w,(k+1 \rightsquigarrow k)}}$ when $k > 1$. Thus we have the following:

Theorem 4. *The cut elimination theorem does not hold for* $\mathbf{FL_{w,(k+1 \rightsquigarrow k)}}$.

Next, we will generalize the idea of global contraction in section 2 and introduce rules called global $(n \rightsquigarrow k)$ rule, which is a stronger form of the $(n \rightsquigarrow k)$ rule.

Definition 4. Let (n,k) be any pair of natural numbers n and k such that $n \neq k$ and $k > 0$. Then, global $(n \rightsquigarrow k)$ rule is a rule of inference of the following form:

$$\frac{\Gamma, \overbrace{\Pi, \ldots, \Pi}^{n}, \Sigma \rightarrow B}{\Gamma, \underbrace{\Pi, \ldots, \Pi}_{k}, \Sigma \rightarrow B}$$

In the following, $\mathbf{FL_{g(n \rightsquigarrow k)}}$ (and $\mathbf{FL_{w,g(n \rightsquigarrow k)}}$) will denote the systems obtained from **FL** (and $\mathbf{FL_w}$, respectively) by adding the global $(n \rightsquigarrow k)$ rule. By the same argument as the proof of Lemma 1 in Section 2 we can show that any fragment of $\mathbf{FL_{g(n \rightsquigarrow k)}}$ which contains rules for fusion is equivalent to that of $\mathbf{FL_{(n \rightsquigarrow k)}}$.

Next, using multi-cut rule, we can show that the cut elimination theorem holds for $\mathbf{FL_{g(n \rightsquigarrow 1)}}$, similarly to the proof of the cut elimination theorem for $\mathbf{FL_{gc}}$. On the other hand, as the following examples show, the cut elimination theorem holds for neither $\mathbf{FL_{g(n \rightsquigarrow k)}}$ nor $\mathbf{FL_{w,g(k+1 \rightsquigarrow k)}}$ when $k > 1$. For $n \neq 0$,

$p \supset (q \supset r)^{n-1} \supset s, (q \supset r) \supset p, (t \supset r, q \supset t)^{k-1}, t \supset r, u \supset t, q \supset u \rightarrow s$

and for $n = 0$

$$(q \supset r) \supset p, (t \supset r, q \supset t)^{k}, t \supset r, u \supset t, q \supset u \rightarrow p.$$

Thus we have the following theorem. (Compare this with the results in commutative case in [5].)

Theorem 5. *The cut elimination theorem holds for* $\mathbf{FL_{g(n \rightsquigarrow k)}}$ *or* $\mathbf{FL_{w,g(k+1 \rightsquigarrow k)}}$ *if and only if* $k = 1$.

4. Noncommutative extensions of the implicational logic BB′I

Next we will discuss extensions of the implicational logic $\mathbf{BB'I}$, which sometimes is called $\mathbf{T}_{\rightarrow} - \mathbf{W}$. First, let us consider the logic $\mathbf{BI}$ which is formulated as a Hilbert-style system having the following two axiom schemes

$$\mathbf{B} : (A \supset B) \supset ((C \supset A) \supset (C \supset B))$$
$$\mathbf{I} : A \supset A$$

and modus ponens as its single rule of inference. The logic $\mathbf{BB'I}$ is an extension of $\mathbf{BI}$, which is obtained by adding the following axiom:

$$\mathbf{B'} : (C \supset A) \supset ((A \supset B) \supset (C \supset B))$$

Now, let us consider the following rule, which we call *modus ponens 2*:

$$\frac{C \supset (A \supset B) \quad A}{C \supset B}$$

Then, it is easy to see that $\mathbf{BI}$ with modus ponens 2 becomes equivalent to the implicational fragment $\mathbf{FL}_{\supset}$ of $\mathbf{FL}$. More precisely, for any formula A, A is provable in $\mathbf{BI}$ with modus ponens 2 if and only if the sequent $\rightarrow A$ is provable in $\mathbf{FL}_{\supset}$. Note here that modus ponens 2 can be replaced by the rule

$$\frac{A}{(A \supset B) \supset B}$$

and that if we add moreover the axiom

$$\mathbf{C} : (A \supset (B \supset C)) \supset (B \supset (A \supset C)),$$

which corresponds to the exchange rule in sequent calculi to $\mathbf{BI}$, then modus ponens 2 becomes an admissible rule in it.

Thus, both $\mathbf{FL}_{\supset}$ and $\mathbf{BB'I}$ are extensions of $\mathbf{BI}$, which are subsystems of $\mathbf{FL}_{e\supset}$, i.e. **BCI**. Moreover they are incomparable. In fact, it is easy to see that $\mathbf{B}'$ is not provable in $\mathbf{FL}_{\supset}$. On the other hand, $((p \supset p) \supset q) \supset q$ is provable in $\mathbf{FL}_{\supset}$, but not in $\mathbf{BB'I}$.

Now we will introduce the Gentzen-type sequent calculus $\mathbf{LBB'I}$ of $\mathbf{BB'I}$ logic, following [2] and [7].

Definition 5. The sequent calculus $\mathbf{LBB'I}$ consists of the following initial sequent;

$$p \to p \qquad \text{for any propositional variable } p.$$

and the following rules of inference;

$$\frac{\Gamma \to A \quad \Delta, A, \Sigma \to B,}{\Delta \circ \Gamma, \Sigma \to B}\,(cut)$$

where either Γ is nonempty or Δ is empty,

$$\frac{\Gamma, A \to B,}{\Gamma \to A \supset B}\,(\to\supset)$$

$$\frac{\Gamma \to A_1 \quad \Delta, A_2, \Lambda \to B.}{\Delta \circ (A_1 \supset A_2 \circ \Gamma), \Lambda \to B}\,(\supset\to)$$

where Γ is not empty.

Here we will give some explanations for the notational convention of "$\circ$", which is sometimes called as a *guarded merge*. Suppose that Γ is a sequence of formula $A_1, \ldots, A_n$ and Δ is a sequence (of sequences of formulas) $\Delta_1, \ldots, \Delta_n$. Then, $\Delta \circ \Gamma$ denotes any sequence of formula of the form $\Delta_1, A_1, \Delta_2, A_2, \ldots, \Delta_n, A_n$, where some Δ_i may be empty. In other words, $\Delta \circ \Gamma$ denotes any sequence of formulas, obtained from Δ and Γ by merging them into one, which however must keep the original orders and whose rightmost formula must be the rightmost formula of Γ.

We will consider extensions of **LBB′I** which are obtained by adding structural rules. Let **LBB′IK** and **LBB′IW** be sequent calculi obtained from **LBB′I** by adding the weakening rule and the contraction rule. (Here, we will follow the traditional notation in combinatory logic and hence **W** denotes *the contraction rule* but not the weakening.) The latter is sometimes called $\mathbf{T}_{\rightarrow}$. Then the following can be shown (see e.g. [2] and [7]).

Theorem 6. *The cut elimination theorem holds for* **LBB′I** *and* **LBB′IK**. *Thus, they are decidable.*

Let **LBB′IKW** be the sequent calculus obtained from **LBB′I** by adding both the weakening and the contraction rules. As is well-known, **LBB′IKW** is equivalent to the implication fragment of **LJ**. In section 2, we have shown that the cut elimination theorem doesn't hold for either $\mathbf{FL_c}$ or $\mathbf{FL_{w,c}}$, each of which has the contraction but not the exchange. In contrast with this, we can show the following. (See also [2].)

Theorem 7. *The cut elimination theorem holds for* **LBB′IW** *and* **LBB′IKW**.

Sketch of Proof. To prove the theorem, we will introduce a new rule, called *multi-cut**. Here, a *multi-cut** rule is a rule of inference of the following form:

$$\frac{\Gamma \rightarrow A \quad \Delta, A^m, \Sigma \rightarrow B}{\Delta \circ \{m\Gamma\}, \Sigma \rightarrow B}\,(\textit{multi-cut}^*)$$

where Δ must be empty when Γ is empty. Here, suppose $\Gamma \equiv A_1, A_2, \ldots, A_k$ and $\Delta \equiv \Delta_1, \Delta_2, \ldots, \Delta_k$ then $\Delta \circ \{m\Gamma\}$ denotes any sequence of formula of the form $\Delta_1, A_1^m, \Delta_2, A_2^m, \ldots, \Delta_k, A_k^m$. The formula A in the above is called the multi-cut* formula of the above multi-cut* rule.

Note that the multi-cut* can be derived by some m repetitions of application of cut-rule. On the other hand when $m = 1$ the multi-cut* rule is just the cut rule. Then we can prove the multi-cut* elimination theorem by using the standard technique.

Here, because of the lack of the exchange rule, Kripke's method doesn't seem to work well for proving the decidability. In fact, the decision problem of **BB′IW** is a longstanding open problem.

Next, consider the extensions of **LBB′I** with knotted structural rules. Let $\mathbf{LBB'I_{(n\rightsquigarrow k)}}$ be the sequent calculus obtained from **LBB′I** by adding the $(n \rightsquigarrow k)$ rule. Also, $\mathbf{LBB'IK_{(n\rightsquigarrow k)}}$ is the sequent calculus obtained from $\mathbf{LBB'I_{(n\rightsquigarrow k)}}$ by adding the weakening rule. It is clear that $\mathbf{LBB'I_{(0\rightsquigarrow 1)}}$ is equivalent to **LBB′IK**, which was already shown to enjoy the cut elimination theorem. For $n > 0$, using the multi-cut* rule, we can prove that the cut elimination theorem holds for $\mathbf{LBB'I_{(n\rightsquigarrow 1)}}$, similarly to the proof of the theorem just in the above.

Lemma 2. *The cut elimination theorem holds in* $\mathbf{LBB'I_{(n\rightsquigarrow 1)}}$.

On the other hand, we have also the following:

Lemma 3. *The cut elimination theorem does not hold for* $\mathbf{LBB'I_{(n\rightsquigarrow k)}}$ *when* $k > 1$.

Proof. The following sequent $S(p, q)$ is provable but is not provable without cut rule;

$$p^n \supset p, q^n \supset p, (q, p)^{k-1}, q \to p.$$

From the above two lemmas we get the following theorem:

Theorem 8. *The cut elimination theorem holds for* $\mathbf{LBB'I_{(n\rightsquigarrow k)}}$ *if and only if* $k = 1$.

Note that $\mathbf{LBB'IK_{(2\rightsquigarrow 1)}}$ is equivalent to **LBB′IKW**, which enjoys the cut elimination theorem. On the other hand, we can show that the above sequent $S(p, q)$ will work well also as an example which shows that the cut elimination theorem fails in $\mathbf{LBB'IK_{(k+1\rightsquigarrow k)}}$ when $k > 1$. Thus we have the following result, which is similar to the result in comutative case (see Theorem 4 of [5]):

Theorem 9. *The cut elimination theorem holds for* $\mathbf{LBB'IK_{(k+1 \rightsquigarrow k)}}$ *if and only if* $k = 1$.

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