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DECIDABILITY OF CLASSES OF FINITE ALGEBRAS WITH A DISTINGUISHED SUBSET CLOSED UNDER A DISCRIMINATOR CLONE

A b s t r a c t. We show that if T is the smallest discriminator clone on a set A, then the first order theory of finite powers of a finite algebra $\mathbf{A}$ with a distinguished subset closed under T is decidable. If $\mathbf{A}$ is a primal algebra and C is any discriminator clone on A, then the first order theory of finite algebras from $\mathcal{V}(\mathbf{A})$ with a distinguished subset closed under C is decidable. In particular, the first order theory of algebras from $\mathcal{V}(\mathbf{A})$ with a distinguished subalgebra is decidable.

Let C be a clone of operations on a two element set. In [2] we have proved that the first order theory of finite Boolean algebras with distinguish subset closed under C is decidable if and only if C is a clone containing the ternary discriminator function. Decidability of the theories referred to in the abstract is a consequence of the fact, that finite discriminator subsets of products of algebras have a 'nice' ordered structure, especially in case of Boolean algebras [3].

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In order to get the above results we translate each sentence in the language of algebras with an additional unary predicate into the language of Boolean pairs and use the fact that the first order theory of finite Boolean pairs is decidable. The idea of our proof is based on the one of Ershov, who found a translation from the language of Boolean powers into the language of Boolean algebras. In [6] Werner uses the Ershov-style translation to prove that the first order theory of filtered Boolean powers is decidable. He translates each sentence in the language of these structures into the language of Boolean algebras with quantification over filters.

We start with some notational conventions.

Definition 1. If $\mathbf{B}$ is a Boolean algebra and $G \subseteq B$, then by B^a we denote a set of atoms of $\mathbf{B}$ and by $Sg^{\mathbf{B}}(G)$ we denote the subuniverse of $\mathbf{B}$ generated by G. We shall denote by $\mathbf{SU}(X)$ the Boolean algebra of all subsets of X. For $X' \subseteq X$, a subset D of A^X and $a, b \in A^X$ we introduce the following technical notation:

1) $[\![a = b]\!] = \{i \in X : a(i) = b(i)\}$,
2) $E_D = \{[\![x = y]\!] : x, y \in D\}$,
3) $\mathbf{B}_D = \langle Sg^{\mathbf{SU}(\mathrm{X})}(E_D), \cap, \cup, \neg, \emptyset, X\rangle$,
4) $D{\restriction_{X'}} = \{x{\restriction_{X'}} : x \in D\}$.

We say that D has the patchwork property if the following condition holds

$$a, b \in D \;\;\&\;\; X' \in E_D \Longrightarrow \exists c \in D\; X' \subseteq [\![c = a]\!] \;\;\&\;\; \neg X' \subseteq [\![c = b]\!]\,,$$

where $\neg X' = X \setminus X'$.

Definition 2. Let $\pi = \langle X_1, \ldots, X_k\rangle$ be a sequence of elements from $SU(X)$. We say that π is a decomposition of X if $X_1 \cup \ldots \cup X_k = X$ and for every $1 \leq i < j \leq k$ we have $X_i \cap X_j = \emptyset$. Evidently every partition of X is a decomposition of X.

If $d_1, \ldots, d_k \in A^X$ and π is a k-element decomposition of X, then

$$d_1\restriction_{X_1} \cup \ldots \cup d_k\restriction_{X_k}$$

denotes the function $d \in A^X$ such that $d(j) = d_i(j)$ if $j \in X_i$, for every $1 \leq i \leq k$. If $D \subseteq A^X$, then by $Pg(D, \pi)$ we denote the set of all elements $d \in A^X$ satisfying the following condition: for every $1 \leq i \leq k$ there exists $y \in D$ such that $d\restriction_{X_i} = y\restriction_{X_i}$.

The ternary discriminator function on a set A is the function $t : A^3 \to A$ defined by $t(x, y, z) = x$ if $x \neq y$, and $t(x, y, z) = z$ if $x = y$. By T we denote the clone of functions generated by t i.e., the smallest clone of operations on A containing the ternary discriminator function t.

Definition 3. Let $\mathbf{A}$ be an algebra, C be a clone of operations on the underlying set of $\mathbf{A}$. The notation $A^* \leq_C A^X$ means that $A^* \subseteq A^X$ and A^* is closed under operations from C.

We define the following classes of structures

1. $\mathcal{P}_{\text{fin}}(\mathbf{A}, C) = \{\langle \mathbf{A}^X, A^* \rangle : X \text{ is finite } \ \& \ A^* \leq_C A^X\}$,
2. $\mathcal{V}_{\text{fin}}(\mathbf{A}, C) = \{\langle \mathbf{D}, D^* \rangle : \mathbf{D} \in \mathcal{V}_{\text{fin}}(\mathbf{A}) \ \& \ D^* \leq_C D\}$,
3. $\mathcal{VP}_{\text{fin}}(\mathbf{A}) = \{\langle \mathbf{D}, D^* \rangle : \mathbf{D} \in \mathcal{V}_{\text{fin}}(\mathbf{A}) \ \& \ D^* \text{ is a subuniverse of } \mathbf{D}\}$.

Definition 4. Let $\langle \mathbf{A}^X, D \rangle$ be a structure from $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$. If $G \subseteq D$, then we say that G is a base of D iff $D = Pg(G, B^a_D)$.

Proposition 5. *Let* $\mathbf{A}$ *be an algebra and* $D \subseteq \mathbf{A}^X$. *Then we have* $D \leq_T A^X$ *iff* D *has the patchwork property.*

Proof. Let $a, b \in D$ and $Y \in E_D$. We pick $c, d \in D$ such that $Y = [\![c = d]\!]$. If D is closed under t, then $e = t(t(c, d, a), t(c, d, b), b)$ belongs to D. One can easily check that $Y \subseteq [\![e = a]\!] \ \& \ \neg Y \subseteq [\![e = b]\!]$. So, D has the patchwork property. Let $a, b, c \in D$. If D has the patchwork property, then we can find an element $d \in D$ such that $[\![a = b]\!] \subseteq [\![d = c]\!] \ \& \ [\![a \neq b]\!] \subseteq [\![d = a]\!]$. It's enough to note that $t(a, b, c) = d$. Hence D is closed under the ternary discriminator, which was to be proved.

Assumption: For Propositions 6–14 we assume that $\mathbf{A}$ is a fixed finite algebra.

Let $\langle \mathbf{A}^X, D\rangle$ be a structure from $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T)$. We establish several structural facts about this structure.

Proposition 6. *Let* $\bigcap E_D = X^*$. *We have:*

1. D has the patchwork property,
2. $\mathbf{E}_D = \langle E_D, \cap, \cup, \neg^*, X^*, X\rangle$, where $\neg^*(x) = \neg x \cup X^*$, is a subalgebra of the interval Boolean algebra $\mathbf{SU}(X)|X^* = \langle [X^*, X], \cap, \cup, \neg^*, X^*, X\rangle$.

Proof. (1) is an easy consequence of Proposition 5. For (2) we fix $a \in D$ and show that for any $b, c \in D$ there is an element $d \in D$, which satisfies the following conditions:

(1) $[\![a = b]\!] \cap [\![a = c]\!] = [\![a = d]\!]$,

(2) $[\![a = b]\!] \cup [\![a = c]\!] = [\![a = d]\!]$,

(3) $[\![a = b]\!] \cup [\![a \neq c]\!] = [\![a = d]\!]]$,

(4) $[\![b = c]\!] = [\![a = d]\!]$.

This part of the proof is similar as in [6]. Using *the patchwork property* we proceed as follows:

ad(1) We pick $d \in D$ such that $[\![a = b]\!] \subseteq [\![c = d]\!]$ and $[\![a \neq b]\!] \subseteq [\![b = d]\!]$.

ad(2) We pick $d \in D$ such that $[\![a = b]\!] \subseteq [\![a = d]\!]$ and $[\![a \neq b]\!] \subseteq [\![c = d]\!]$.

ad(3) We pick $e \in D$ such that $[\![a = b]\!] \cup [\![a = c]\!] = [\![a = e]\!]$ and then we pick $d \in D$ such that $[\![a = e]\!] \subseteq [\![b = d]\!]$ and $[\![a \neq e]\!] \subseteq [\![a = d]\!]$.

ad(4) We pick $e \in D$ such that $[\![a = b]\!] \cap [\![a = c]\!] = [\![a = e]\!]$ and then we pick $d \in D$ such that $[\![b = c]\!] \subseteq [\![a = d]\!]$ and $[\![b \neq c]\!] \subseteq [\![e = d]\!]$.

From (4) it follows that for every fixed element $a \in D$ we have that $E_D = \{[\![a = x]\!] : x \in D\}$. From (1), (2) we know that E_D is closed under $\cup$ and $\cap$. Since X is finite, then $X^* = \bigcap E_D = \bigcap\{[\![a = x]\!] : x \in A\}$ is the least element of E_D and $X = [\![a = a]\!]$ is the greatest element of E_D. From

(3) we can conclude that $X^* \cup [\![a \neq b]\!]$ is a complement of $[\![a = b]\!]$ in the interval $[X^*, X]$ because we have:

$$X^* \cup [\![a \neq b]\!] \cup [\![a = b]\!] = X,$$
$$(X^* \cup [\![a \neq b]\!]) \cap [\![a = b]\!] = X^* \cap [\![a = b]\!] = X^*.$$

This completes the proof.

The next proposition can be interpreted as a special version of the patchwork property.

Proposition 7. *Let* $B_D^a = \{X_1, ..., X_k\}$ *and* $d_1, ..., d_k$ *belong to* D. *Then the element* $d = d_1\restriction_{X_1} \cup ... \cup d_k\restriction_{X_k}$ *also belongs to* D.

Proof. Let us denote $\bigcap E_D$ by X^*. First we show an easy fact that if $Y \in B_D$, then $Y \cup X^*$ belongs to E_D. If $Y \in E_D$, then the claim is obvious. If $Y \notin E_D$, then according to Proposition 6 we can represent Y as a finite sum of elements of the form Y_i or $\neg Y_i$ or $Y_i \cap \neg Y_j$, where $Y_i, Y_j \in E_D$ (see [5], Proposition 4.4). From Proposition 6 we know that $(Y_i \cap \neg Y_j) \cup X^* = (Y_i \cup X^*) \cap \neg^* Y_j \in E_D$ and $\neg Y_i \cup X^* = \neg^* Y_i \in E_D$. So, we obtain that $Y \cup X^*$ belongs to E_D. Moreover, one can easily check that if $X^* \neq \emptyset$, then $X^* \in B_D^a$. We know, from Proposition 5, that D has the patchwork property. Let $B_D^a = \{X_1, \ldots, X_k\}$ and $d_1, \ldots, d_k \in D$. So, we can pick some elements $c_1, ..., c_{k-2}, c \in D$ satisfying the following conditions:

$$X_2 \cup X^* \subseteq [\![c_1 = d_2]\!] \ \ \& \ \ \neg(X_2 \cup X^*) \subseteq [\![c_1 = d_1]\!],$$
$$X_3 \cup X^* \subseteq [\![c_2 = d_3]\!] \ \ \& \ \ \neg(X_3 \cup X^*) \subseteq [\![c_2 = c_1]\!],$$
$$\vdots$$
$$X_{k-1} \cup X^* \subseteq [\![c_{k-2} = d_{k-1}]\!] \ \ \& \ \ \neg(X_{k-1} \cup X^*) \subseteq [\![c_{k-2} = c_{k-3}]\!],$$
$$X_k \cup X^* \subseteq [\![c = d_k]\!] \ \ \& \ \ \neg(X_k \cup X^*) \subseteq [\![c = c_{k-2}]\!].$$

Now, if $X_i = X^*$, then $c\restriction_{X_i} = d_k\restriction_{X_i} = d_i\restriction_{X_i}$ because X^* is a minimal element of E_D. If $X_i \neq X^*$, then $X_i \cap (X_j \cup X^*) = \emptyset$ for every $1 \leq i \neq j \leq k$. One can easily check that if $i = 1$, then $c\restriction_{X_1} = c_{k-2}\restriction_{X_1} = ... = c_1\restriction_{X_1} = d_1\restriction_{X_1}$.

If $1 < i < k$, then $c\restriction_{X_i} = c_{k-2}\restriction_{X_i} = ... = c_{k-(k-i+1)}\restriction_{X_i} = c_{i-1}\restriction_{X_i} = d_i\restriction_{X_i}$. If $i = k$, then $c\restriction_{X_k} = d_k\restriction_{X_k}$. Thus $c = d_1\restriction_{X_1} \cup ... \cup d_k\restriction_{X_k} = d$ and $d \in D$. The proposition is proved.

Remark 8. From Proposition 7 we know that $Pg(D, B_D^a) \subseteq D$. Since for every $d \in D$ we have $d = d\restriction_{X_1} \cup ... \cup d\restriction_{X_k}$, then $D = Pg(D, B_D^a)$.

Proposition 9. *If $Y \in B_D^a$, then $|D\restriction_Y| \leq |A|$.*

Proof. Suppose, contrary to our claim, that $|D\restriction_Y| > |A|$. Let $j \in Y$. Then there exist $d_1\restriction_Y, d_2\restriction_Y \in D\restriction_Y$ and $i \in Y$ such that $j \in [\![d_1 = d_2]\!]$ and $i \notin [\![d_1 = d_2]\!]$. Hence $0 < [\![d_1 = d_2]\!] \cap Y < Y$ and $[\![d_1 = d_2]\!] \cap Y \in B_D$, because Y and $[\![d_1 = d_2]\!]$ belong to B_D. This contradicts the assumption that $Y \in B_D^a$.

Proposition 10. *There exists $G \subseteq D$ such that G is a base of D and $|G| \leq |A|$. Moreover, $E_G \subseteq B_D$.*

Proof. Let us suppose that $|A| = m$, $B_D^a = \{X_1, \ldots, X_k\}$ and $D\restriction_{X_i} = \{d_{i_1}, \ldots, d_{i_{r_i}}\}$, where $1 \leq i \leq k$. From Proposition 9 we know that $r_i \leq m$. We define $g_1, \ldots, g_m \in A^X$ by putting $g_j = g_j\restriction_{X_1} \cup ... \cup g_j\restriction_{X_k}$, where

$$g_j\restriction_{X_i} = \begin{cases} d_{i_j}, & \text{if } |D\restriction_{X_i}| \geq j; \\ d_{i_1}, & \text{otherwise.} \end{cases}$$

Now, if d is an element of D, then by definition we have $d\restriction_{X_i} \in D\restriction_{X_i}$, for every $1 \leq i \leq k$. Thus there exists $j \leq m$ such that $d\restriction_{X_i} = d_{i_j} = g_j\restriction_{X_i}$. Hence $d = g_{j_1}\restriction_{X_1} \cup ... \cup g_{j_k}\restriction_{X_k}$, where $\{j_1, ..., j_k\} \subseteq \{1, ..., m\}$. Then $D \subseteq Pg(G, B_D^a)$, where $G = \{g_1, \ldots, g_m\}$. From Proposition 7 it follows that $g_1, \ldots, g_m \in D$ and $Pg(G, B_D^a) \subseteq D$. We can conclude that G is a base of D and $|G| \leq |A|$. Since $G \subseteq D$, then $E_G \subseteq E_D \subseteq B_D$. The proof is now complete.

Proposition 11. *Let $G \subseteq A^X$, π be a partition of X and $G^* = Pg(G, \pi)$. If $E_G \subseteq Sg^{\mathbf{SU}(\mathrm{X})}(\pi)$, then G^* is closed under the ternary discriminator. Moreover, $B_{G^*} \subseteq Sg^{\mathbf{SU}(\mathrm{X})}(\pi)$ and $\bigcap E_G = \bigcap E_{G^*}$.*

Proof. Let us suppose that $\pi = \{X_1, ..., X_k\}$ and $E_G \subseteq Sg^{\mathbf{SU}(\mathrm{X})}(\pi)$. Of course we have $G^* \subseteq A^X$. According to Proposition 5 it suffices to show that G^* has the patchwork property. By Definition 2, every element d of G^* can be presented in the following form

$$d = g_{i_1}\restriction_{X_1} \cup ... \cup g_{i_k}\restriction_{X_k},$$

where $g_{i_j} \in G$, for every $1 \leq j \leq k$. Let $d_1, d_2 \in G^*$ and $Y = [\![d_3 = d_4]\!]$ for some $d_3, d_4 \in G^*$. We have to show that the element $d = d_1\restriction_Y \cup d_2\restriction_{\neg Y}$ also belongs to G^*. If $Y = \emptyset$, then $d = d_2$ and it belongs to G^*. Let us suppose that $\emptyset < Y \leq X$. First we prove that for every $1 \leq i \leq k$ we have $X_i \subseteq Y$ or $Y \cap X_i = \emptyset$. Assume, contrary to this claim, that there are $1 \leq i \leq k$ and $j, j' \in X$ such that $j \in Y \cap X_i$ and $j' \in X_i$ and $j' \notin Y$. By the definition of $Pg(G, \pi)$ we know that there exist $g_1, g_2 \in G$ such that $d_3\restriction_{X_i} = g_1\restriction_{X_i}$ and $d_4\restriction_{X_i} = g_2\restriction_{X_i}$. Thus we obtain that $j, j' \in X_i$, $j \in [\![g_1 = g_2]\!]$ and $j' \notin [\![g_1 = g_2]\!]$. Hence $[\![g_1 = g_2]\!] \notin Sg^{\mathbf{SU(X)}}(\pi)$, which means that $E_G \nsubseteq Sg^{\mathbf{SU(X)}}(\pi)$. We obtain a contradiction and the claim is proved. Now, we can conclude that $E_{G^*} \subseteq Sg^{\mathbf{SU(X)}}(\pi)$ and of course $B_{G^*} \subseteq Sg^{\mathbf{SU(X)}}(\pi)$. So, without loss of generality we can assume that $Y = X_1 \cup \ldots \cup X_s$, where $1 \leq s \leq k$. Let $d_i\restriction_{X_j} = g_{i_j}\restriction_{X_j}$, where $i \in \{1, 2\}, 1 \leq j \leq k$ and $g_{i_j} \in G$. Now, $d = d_1\restriction_Y \cup d_2\restriction_{\neg Y} = g_{1_1}\restriction_{X_1} \cup ... \cup g_{1_s}\restriction_{X_s} \cup g_{2_{s+1}}\restriction_{X_{s+1}} \cup \ldots \cup g_{2_k}\restriction_{X_k}$. Hence $d \in Pg(G, \pi)$. It remains to show that $\bigcap E_G = \bigcap E_{G^*}$. Since $G \subseteq G^*$ then $\bigcap E_{G^*} \subseteq \bigcap E_G$. Let us assume that $\bigcap E_{G^*} \subset \bigcap E_G$. Then there exists $i \in X$ such that $i \in \bigcap E_G$ and $i \notin \bigcap E_{G^*}$. So, there exist $g_1^*, g_2^* \in G^*$ such that $i \notin [\![g_1^* = g_2^*]\!]$. Let us pick $1 \leq j \leq k$ such that $i \in X_j$. Since $g_1^*, g_2^* \in G^*$ then there exist $g_1, g_2 \in G$ such that $g_1^*\restriction_{X_j} = g_1\restriction_{X_j}$ and $g_2^*\restriction_{X_j} = g_2\restriction_{X_j}$. Hence $i \notin [\![g_1 = g_2]\!]$ and $i \notin \bigcap E_G$, which contradicts our assumption. The proposition is proved.

Definition 12. Let $G \subseteq A^X$ and $\mathbf{B}_1$ be a subalgebra of $\mathbf{SU}(X)$. We say that G matches $\mathbf{B}_1$ iff

$$\mathbf{B}_{Pg(G, B_1^a)} = \mathbf{B}_1.$$

Remark 13. Note that if G is a base of D, then $Pg(G, B^a_D) = D$. Hence $\mathbf{B}_{Pg(G,B^a_D)} = \mathbf{B}_D$ and G matches $\mathbf{B}_D$.

Proposition 14. *Let $G \subseteq A^X$ and $\mathbf{B}_1$ be a subalgebra of $\mathbf{SU}(X)$. G matches $\mathbf{B}_1$ iff the following conditions are satisfied:*

1. $E_G \subseteq B_1$,
2. if $\bigcap E_G \neq \emptyset$, then $\bigcap E_G \in B^a_1$,
3. if $X' \in B^a_1$ and $X' \neq \bigcap E_G$, then there are $g_1, g_2 \in G$ such that $[\![g_1 = g_2]\!] \cap X' = \emptyset$.

Proof. Let us denote $Pg(G, B^a_1)$ by G^*.

($\Rightarrow$) If G matches $\mathbf{B}_1$, then $B_{G^*} = B_1$. Since $E_G \subseteq E_{G^*} \subseteq B_{G^*}$, then $E_G \subseteq B_1$. From Proposition 11 it follows that $\bigcap E_G = \bigcap E_{G^*}$. If $\bigcap E_{G^*} \neq \emptyset$, then it is an atom of $\mathbf{B}_{G^*}$ and the second condition is satisfied. Now, let us suppose that $X' \in B^a_1, X' \neq \bigcap E_G$ and for every $g_1, g_2 \in G$ we have $[\![g_1 = g_2]\!] \cap X' \neq \emptyset$. Since $[\![g_1 = g_2]\!] \in B_{G^*}$, then for every $g_1, g_2 \in G$ we have $[\![g_1 = g_2]\!] \cap X' = X'$. Hence $X' \subseteq \bigcap E_G$. If $\bigcap E_G = \emptyset$, then we obtain a contradiction. If $\bigcap E_G \neq \emptyset$, then $\bigcap E_G$ is an atom of $\mathbf{B}_{G^*}$. Hence $X' = \bigcap E_G$, which also contradicts our assumption. We can conclude that the third condition is also satisfied.

($\Leftarrow$) Let us assume that the conditions $1-3$ are satisfied. We have to show that G matches $\mathbf{B}_1$. From the first condition and Proposition 11 we know that $B_{G^*} \subseteq B_1$ and $\bigcap E_G = \bigcap E_{G^*}$.

Case 1: $\bigcap E_G \neq \emptyset$. It follows from the second condition that $\bigcap E_G$ is an atom of $\mathbf{B}_1$. Hence $\bigcap E_{G^*}$ is an atom of $\mathbf{B}_1$. Let $X' \in B^a_1$ and $X' \neq \bigcap E_{G^*}$. From the third condition we can conclude that there are $g_1, g_2 \in G$ such that $[\![g_1 = g_2]\!] \cap X' = \emptyset$. Therefore there exist $g^*_1, g^*_2 \in G^*$ such that $[\![g^*_1 = g^*_2]\!] = \neg X'$. Then $\neg X' \in E_{G^*}$ and $X' \in B_{G^*}$. We obtain that $B_1 \subseteq B_{G^*}$. Hence $\mathbf{B}_1 = \mathbf{B}_{G^*}$.

Case 2: $\bigcap E_G = \emptyset$. If $X' \in B^a_1$ then $X' \neq \bigcap E_G$. From the third condition we obtain in the similar manner as above that X' belongs to B_{G^*} and $\mathbf{B}_{G^*} = \mathbf{B}_1$.

Definition 15. By $\mathcal{BP}^*_{\text{fin}}$ we denote the following class of structures

$$\mathcal{BP}^*_{\text{fin}} = \{\langle \mathbf{SU}(X), B\rangle : \mid X \mid < \omega \text{ and } B \text{ is a subuniverse of } \mathbf{SU}(X)\}.$$

Evidently, $\mathcal{I}(\mathcal{BP}^*_{\text{fin}}) = \mathcal{BP}_{\text{fin}}$, where $\mathcal{BP}_{\text{fin}}$ denotes the class of finite Boolean pairs.

Let $\mathbf{A}$ be an algebra. For $\langle \mathbf{SU}(X), B\rangle \in \mathcal{BP}^*_{\text{fin}}$ and $G \subseteq A^X$ we can define a structure $\langle \mathbf{A}^X, Pg(G, B^a)\rangle$.

Let $\mathcal{K}(\mathbf{A})$ denote the class of the all structures $\langle \mathbf{A}^X, Pg(G, B^a)\rangle$ such that

1. $\langle \mathbf{SU}(X), B\rangle \in \mathcal{BP}^*_{\text{fin}}$,
2. $G \subseteq A^X$,
3. $\mid G \mid \leq \mid A \mid$,
4. G matches $\mathbf{B}$.

Lemma 16. *If* $\mathbf{A}$ *is a finite algebra, then*

$$\mathcal{P}_{\text{fin}}(\mathbf{A}, T) = \mathcal{K}(\mathbf{A}).$$

Proof. Assume that $\langle \mathbf{A}^X, D\rangle \in \mathcal{P}_{\text{fin}}(\mathbf{A}, T)$. By Definition 1 we have that $\langle \mathbf{SU}(X), B_D\rangle \in \mathcal{BP}^*_{\text{fin}}$. Moreover, from Proposition 10 we know that there exists $G \subseteq A^X$ such that $\mid G \mid \leq \mid A \mid$ and G is a base of D. From Remark 13 we know that G matches $\mathbf{B}_D$. Hence $\langle \mathbf{A}^X, D\rangle = \langle \mathbf{A}^X, Pg(G, B^a_D)\rangle$ and it belongs to $\mathcal{K}(\mathbf{A})$. Conversely, let $\langle \mathbf{A}^X, Pg(G, B^a)\rangle$ belong to $\mathcal{K}(\mathbf{A})$. Then $\langle \mathbf{SU}(X), B\rangle$ belongs to $\in \mathcal{BP}^*_{\text{fin}}$, $G \subseteq A^X$ and G matches $\mathbf{B}$. Hence from Proposition 14 we know that $E_G \subseteq B$. Moreover it follows from Proposition 11 that $Pg(G, B^a) \subseteq A^X$ and it is closed under the ternary discriminator. We can conclude that $\langle \mathbf{A}^X, Pg(G, B^a)\rangle$ belongs to $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$, which completes the proof of the lemma.

Definition 17. Let $\mathbf{A}$ be a finite algebra of type $\mathcal{T}$ and $A = \{a_1, \ldots, a_m\}$.■ By L we denote the first order language of $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$ i.e., the only non-logical constants of L are the all function symbols belonging to $\mathcal{T}$ and an unary relation symbol D related to a subset closed under the ternary discriminator. We denote variables of L by x_i, $0 \leq i < \omega$. Moreover, we can assume that the atomic formulas of L are expressions of the form $f(x_{i_1}, \ldots, x_{i_k}) = x_{i_0}$ or of the form $D(x_{i_0})$, where $f(x_{i_1}, \ldots, x_{i_k})$ is a term of L and x_{i_0} is a single variable. The other formulas of L can be obtained from the atomic formulas using the symbols $\neg, \&, \exists$.

Let L_0 denote the first order language of $\mathcal{BP}_{\text{fin}}$ i.e., the only nonlogical symbols of L_0 are $\cap, \cup, \neg, 0, 1, B'$, where B' is an unary relation symbol related to a subuniverse of a Boolean algebra. We denote variables of L_0 by $u_{i,j}$, $0 \leq i, j < \omega$. However, we find it convenient to distinguish some variables of L_0 and denote them by $p_{i,j}$. They can be written in the form of an $m \times m$ - matrix:

$$\begin{matrix} p_{1,1} & p_{1,2} & \cdots & p_{1,m-1} & p_{1,m} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,m-1} & p_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{m,1} & p_{m,2} & \cdots & p_{m-1,m-1} & p_{m,m} \end{matrix}$$

We define an Ershov-style translation of L into L_0. To each formula $\phi(x_1, \ldots, x_k) \in L$ we assign a formula

$$\widehat{\phi}(u_{1,1}, \ldots, u_{1,m}, u_{2,1}, \ldots, u_{2,m}, \ldots u_{m,m}, p_{1,1}, \ldots, p_{m,m})$$

from L_0.

Note that if formula ϕ contains k variables $x_0, \ldots, x_k$ and no ralation symbols, then formula $\widehat{\phi}$ contains $k \times m$ variables $u_{0,1}, \ldots, u_{0,m}, \ldots, u_{k,1}, \ldots, u_{k,m}$. If formula ϕ contains k variables $x_0, \ldots, x_k$ and a relation symbol, then formula $\widehat{\phi}$ contains $k \times m + m^2$ variables $u_{0,1}, \ldots, u_{0,m}, \ldots, u_{k,1}, u_{k,2} \ldots, u_{k,m}, p_{1,1}, \ldots, p_{m,m}$.

The assignment is as follows:

1. if $\phi \equiv f(x_1, \ldots, x_k) = x_0$, then

    (a) if $k = 0$ and $f = a_i$ is an constant then

$$\widehat{\phi} \equiv u_{0,i} = 1,$$

(b) if $k \neq 0$ then

$$\widehat{\phi} \equiv \bigwedge_{\mathbf{A} \models f(a_{i_1},\ldots,a_{i_k})=a_{i_0}} \neg(u_{1,i_1} \cap \ldots \cap u_{k,i_k}) \cup u_{0,i_0} = 1,$$

2. if $\phi \equiv D(x_0)$, then

$$\begin{array}{rl} \widehat{\phi} \equiv \forall_b \, At'(b) \rightarrow [& (u_{0,1} \cap b = p_{1,1} \cap b \;\; \& \;\; \ldots \;\; \& \;\; u_{0,m} \cap b = p_{1,m} \cap b) \vee \\ & (u_{0,1} \cap b = p_{2,1} \cap b \;\; \& \;\; \ldots \;\; \& \;\; u_{0,m} \cap b = p_{2,m} \cap b) \vee \\ & \vdots \\ & (u_{0,1} \cap b = p_{m,1} \cap b \;\; \& \;\; \ldots \;\; \& \;\; u_{0,m} \cap b = p_{m,m} \cap b)], \end{array}$$

where $At'(b) \equiv$ b is an atom of $\mathbf{B}'$.

3. if $\phi \equiv (\psi_1 \;\; \& \;\; \psi_2)$, then $\widehat{\phi} = \widehat{\psi_1} \;\; \& \;\; \widehat{\psi_2}$,
4. if $\phi \equiv \neg\psi$, then $\widehat{\phi} = \neg\widehat{\psi}$,
5. if $\phi \equiv \exists_{x_0}\psi(x_0)$, then

$$\widehat{\phi} \equiv \exists_{u_{0,1}} \ldots \exists_{u_{0,m}} \, (\widehat{\psi}(u_{0,1}, \ldots, u_{0,m}) \;\; \& \;\; u_{0,1} \cup \ldots \cup u_{0,m} = 1 \;\; \& \bigwedge_{1 \leq i < j \leq m} (u_{0,i} \cap u_{0,j} = 0))$$

If there is no danger of confusion we don't distinguish notationally between a symbol and its interpretation.

Definition 18. Let $\mathbf{A}$ be a finite algebra and $A = \{a_1, ..., a_m\}$. If $c_i \in A^X$, then for every $1 \leq j \leq m$ we set that $c_{i,j}$ denotes $[\![c_i = \widehat{a}_j]\!]$, where $\widehat{a}_j \in A^X$ is the constant map with the value a_j. If π is an m-element decomposition of X, then by $\bar{a}\restriction_\pi$ we denote the following element of A^X:

$$\bar{a}\restriction_\pi = \widehat{a}_1\restriction_{\pi(1)} \cup \ldots \cup \widehat{a}_m\restriction_{\pi(m)} .$$

We say that $\bar{a}\restriction_\pi$ is an element of A^X defined by a decomposition π. Conversely, if $c_i \in A^X$ then we say that $\pi^{c_i} = \langle c_{i,1}, \ldots, c_{i,m} \rangle$ is a decomposition of X defined by an element c_i.

Lemma 19. *Let* $\mathbf{A}$ *be a finite algebra,* $\langle \mathbf{A}^X, D\rangle$ *be a given structure belonging to* $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$ *and let* $\{g_1, ..., g_m\} \subseteq A^X$ *be a base of* D*, where* $m = \mid A \mid$*. For any formula* $\phi(x_1, ..., x_k) \in L$ *and any* $c_1, \ldots, c_k \in A^X$ *we have*

$$\langle \mathbf{A}^X, D\rangle \models \phi(x_1/c_1, ..., x_k/c_k)$$

iff

$$\langle \mathbf{SU}(X), B_D\rangle \models \widehat{\phi}(\ldots, u_{i,j}/ [\![c_i = \widehat{a}_j]\!], \ldots, p_{i,j}/ [\![g_i = \widehat{a}_j]\!], \ldots).$$

Proof. Let $\mathbf{A}^* = \langle \mathbf{A}^X, D\rangle$ and $\mathbf{B}^* = \langle \mathbf{SU}(X), B_D\rangle$. We proceed by induction over the complexity of ϕ.

1. $\phi \equiv f(c_1, \ldots, c_k) = c_0$,

$$\widehat{\phi} \equiv \bigwedge_{\mathbf{A} \models f(a_{i_1}, \ldots, a_{i_k}) = a_{i_0}} \neg(c_{1,i_1} \cap \ldots \cap c_{k,i_k}) \cup c_{0,i_0} = 1.$$

$$\begin{aligned}
\mathbf{B}^* \models \widehat{\phi} \quad &\text{iff} \quad f(a_{i_1}, \ldots, a_{i_k}) = a_{i_0} \Rightarrow \neg(c_{1,i_1} \cap \ldots \cap c_{k,i_k}) \cup c_{0,i_0} = X \\
&\text{iff} \quad f(a_{i_1}, \ldots, a_{i_k}) = a_{i_0} \Rightarrow c_{1,i_1} \cap \ldots \cap c_{k,i_k} \leq c_{0,i_0} \\
&\text{iff} \quad f(a_{i_1}, \ldots, a_{i_k}) = a_{i_0} \;\;\&\;\; (\forall_{1 \leq r \leq k}\, c_r(j) = a_{i_r}) \Rightarrow c_0(j) = a_{i_0} \\
&\text{iff} \quad \forall_{j \in X}\, f(c_1, \ldots, c_k)(j) = c_0(j) \\
&\text{iff} \quad \mathbf{A}^* \models f(c_1, \ldots, c_k) = c_0 \\
&\text{iff} \quad \mathbf{A}^* \models \phi
\end{aligned}$$

2. $\phi \equiv D(c_0)$,

$$\begin{aligned}
\widehat{\phi} \equiv \forall_b\, At'(b) \rightarrow [&(c_{0,1} \cap b = g_{1,1} \cap b \;\;\&\;\; \ldots \;\;\&\;\; c_{0,m} \cap b = g_{1,m} \cap b) \vee \\
&(c_{0,1} \cap b = g_{2,1} \cap b \;\;\&\;\; \ldots \;\;\&\;\; c_{0,m} \cap b = g_{2,m} \cap b) \vee \\
&\vdots \\
&(c_{0,1} \cap b = g_{m,1} \cap b \;\;\&\;\; \ldots \;\;\&\;\; c_{0,m} \cap b = g_{m,m} \cap b)].
\end{aligned}$$

Let us remind that $D = Pg(\{g_1, \ldots, g_m\}, B^a_D)$.

$$\begin{aligned}
\mathbf{B}^* \models \widehat{\phi} \text{ iff } \quad & b \text{ is an atom of } \mathbf{B}_D \Rightarrow \\
& \exists_{1\leq j\leq m} c_{0,1} \cap b = g_{j,1} \cap b \ \& \ \ldots \ \& \ c_{0,m} \cap b = g_{j,m} \cap b \\
\text{iff } \quad & b \text{ is an atom of } \mathbf{B}_D \Rightarrow \exists_{1\leq j\leq m} c_0{\restriction_b} = g_j{\restriction_b} \\
\text{iff } \quad & c_0 = g_{i_1}{\restriction_{b_1}} \cup \ldots \cup g_{i_k}{\restriction_{b_k}}, \text{where} \\
& \{b_1, \ldots, b_k\} = B^a_D \ \text{ and } \ \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\} \\
\text{iff } \quad & c_0 \in D \\
\text{iff } \quad & \mathbf{A}^* \models D(c_0)
\end{aligned}$$

3. $\phi \equiv (\psi_1 \ \& \ \psi_2)$, $\widehat{\phi} = \widehat{\psi_1} \ \& \ \widehat{\psi_2}$,

$$\begin{aligned}
\mathbf{B}^* \models \widehat{\phi} \text{ iff } \quad & \mathbf{B}^* \models \widehat{\psi_1} \text{ and } \mathbf{B}^* \models \widehat{\psi_2} \\
\text{iff } \quad & \mathbf{A}^* \models \psi_1 \text{ and } \mathbf{A}^* \models \psi_2 \\
\text{iff } \quad & \mathbf{A}^* \models \psi_1 \ \& \ \psi_2 \\
\text{iff } \quad & \mathbf{A}^* \models \phi
\end{aligned}$$

4. $\phi \equiv \neg\psi$, $\widehat{\phi} = \neg\widehat{\psi}$,

$$\mathbf{B}^* \models \widehat{\phi} \text{ iff } \mathbf{B}^* \models \neg\widehat{\psi} \text{ iff } \mathbf{B}^* \not\models \widehat{\psi} \text{ iff } \mathbf{A}^* \not\models \psi \text{ iff } \mathbf{A}^* \models \phi.$$

5. $\phi \equiv \exists_{x_0}\psi(x_0)$,

$$\widehat{\phi} \equiv \exists_{u_{0,1}} \ldots \exists_{u_{0,m}} (\widehat{\psi}(u_{0,1}, \ldots, u_{0,m}) \ \& \ u_{0,1} \cup \ldots \cup u_{0,m} = 1 \ \& \\ \bigwedge_{1\leq i<j\leq m} u_{0,i} \cap u_{0,j} = 0)$$

$$\begin{aligned}
\mathbf{B}^* \models \widehat{\phi} \text{ iff } & \text{ there are } c_{0,1}, \ldots, c_{0,m} \text{ such that it is a decomposition of } X \\
& \text{and } \mathbf{B}^* \models \widehat{\psi}(c_{0,1}, \ldots, c_{0,m}) \\
\text{iff } & \text{ there is } c_0 \in A^X \text{ such that } c_{0,i} = [\![c_0 = \widehat{a}_i]\!] \text{ and } \mathbf{A}^* \models \psi(c_0) \\
& (\text{ simply take } c_0 = \widehat{a}_1\restriction_{c_{0,1}} \cup \ldots \cup \widehat{a}_m\restriction_{c_{0,m}}) \\
\text{iff } & \mathbf{A}^* \models \exists_{x_0} \psi(x_0).
\end{aligned}$$

This finishes the proof.

Definition 20. Let L_0 be the language of $\mathcal{BP}_{\text{fin}}$. By $\bar{p}_i$ we denote a sequence of variables $(p_{i,1}, \ldots, p_{i,m})$. We define the following auxiliary terms and formulas:

- $E^{\cap}_G(\bar{p}_1, \ldots, \bar{p}_m) = \bigcup_{1 \le j \le m} p_{1,j} \cap \ldots \cap p_{m,j}$,
- $[\![\bar{p}_i = \bar{p}_k]\!] = \bigcup_{1 \le j \le m} p_{i,j} \cap p_{k,j}$,
- $\Psi_1(\bar{p}_1, \ldots, \bar{p}_m) \equiv \bigwedge_{1 \le i \le m} \left(p_{i,1} \cup \ldots \cup p_{i,m} = 1 \;\; \& \bigwedge_{0 \le j < k \le m} p_{i,j} \cap p_{i,k} = 0 \right)$,
- $\Psi_2(\bar{p}_1, \ldots, \bar{p}_m) \equiv \bigwedge_{1 \le i < k \le m} \left(\bigcup_{1 \le j \le m} p_{i,j} \cap p_{k,j} \in B' \right)$,
- $\Psi_3(\bar{p}_1, \ldots, \bar{p}_m) \equiv E^{\cap}_G(\bar{p}_1, \ldots, \bar{p}_m) \neq \emptyset \to E^{\cap}_G(\bar{p}_1, \ldots, \bar{p}_m)$ is an atom of $\mathbf{B}'$,
- $\Psi_4(\bar{p}_1, \ldots, \bar{p}_m) \equiv$ if b is an atom of $\mathbf{B}'$ and $b \neq E^{\cap}_G(\bar{p}_1, \ldots, \bar{p}_m)$, then

$$\bigvee_{1 \le i < k \le m} [\![\bar{p}_i = \bar{p}_k]\!] \cap b = \emptyset.$$

Remark 21. Suppose that $\{a_1, \ldots, a_m\}$ is the universe of an algebra $\mathbf{A}$ and $\langle \mathbf{SU}(X), B' \rangle \in \mathcal{BP}_{\text{fin}}$. For every $1 \le i \le m$ let π_i be a m-element sequence of elements of $SU(X)$. If $\langle \mathbf{SU}(X), B' \rangle \models \Psi_1(\pi_1, \ldots, \pi_m)$, then for every $1 \leq i \leq m$ we have that π_i is a decomposition of X and defines an element $g_i = \bar{a}\restriction_{\pi_i}$ belonging to A^X. We denote the set of these elements by G. Using Proposition 14 one can easily check that G matches $\mathbf{B}'$ iff

$$\langle \mathbf{SU}(X), B' \rangle \models \Psi_2(\bar{p}_1, \ldots, \bar{p}_m) \;\&\; \Psi_3(\bar{p}_1, \ldots, \bar{p}_m) \;\&\; \Psi_4(\bar{p}_1, \ldots, \bar{p}_m).$$

We are now in a position to prove the main theorem of this paper.

Theorem. *Let* $\mathbf{A}$ *be a finite algebra and* T *be the smallest discriminator clone on* A. *Then the theory of* $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$ *is decidable.*

Proof. Let L be the language of $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$. For any sentence $\phi \in L$ we construct a sentence ϕ^* in the language of Boolean pairs L_0 as follows:

$$\phi^* \equiv \forall_{\bar{p}_1} \ldots \forall_{\bar{p}_m} \left(\left(\bigwedge_{1 \leq i \leq 4} \Psi_i(\bar{p}_1, \ldots, \bar{p}_m) \right) \to \widehat{\phi} \right).$$

Now, we proceed to show that

$$\mathcal{P}_{\text{fin}}(\mathbf{A}, T) \models \phi \text{ if and only if } \mathcal{BP}^*_{\text{fin}} \models \phi^*.$$

Since $\mathcal{I}(\mathcal{BP}^*_{\text{fin}}) = \mathcal{BP}_{\text{fin}}$ and the theory of $\mathcal{BP}_{\text{fin}}$ is decidable, we will conclude that the theory of $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$ is also decidable.

($\Rightarrow$) Let's assume that $\mathcal{P}_{\text{fin}}(\mathbf{A}, T) \models \phi$ and $\mathcal{BP}^*_{\text{fin}} \not\models \phi^*$. If so, there exists $\mathbf{B}^* \in \mathcal{BP}^*_{\text{fin}}$ such that $\mathbf{B}^* \not\models \phi^*$. Let $\mathbf{B}^* = \langle \mathbf{SU}(X), B' \rangle$. If $\mathbf{B}^* \not\models \phi^*$, then there are $b_{1,1}, \ldots, b_{m,m} \in SU(X)$ such that:

(i) $\mathbf{SU}(X) \models \Psi_1(\bar{b}_1, \ldots, \bar{b}_m)$,

(ii) $\langle \mathbf{SU}(X), B' \rangle \models \Psi_2(\bar{b}_1, \ldots, \bar{b}_m) \;\&\; \Psi_3(\bar{b}_1, \ldots, \bar{b}_m) \;\&\; \Psi_4(\bar{b}_1, \ldots, \bar{b}_m)$,

(iii) $\langle \mathbf{SU}(X), B' \rangle \not\models \widehat{\phi}(b_{1,1}, \ldots, b_{m,m})$.

From (i) it follows that for every $1 \leq i \leq m$ we have that $b_{i,1}, \ldots, b_{i,m}$ is a decomposition of X. Let $G = \{g_1, \ldots, g_m\}$ be a set of elements of A^X defined by these decompositions. Hence for every $1 \leq i, j \leq m$ we have that $g_{i,j} = [\![g_i = \widehat{a}_j]\!] = b_{i,j}$. From (ii), Proposition 14 and Remark 21 we know that G matches $\mathbf{B}'$ and $\bigcap E_G \subseteq B'$. Let $D = Pg(G, B'^a)$. From Proposition 11 it follows that D is closed under the ternary discriminator. Hence $\langle \mathbf{A}^X, D\rangle$ belongs to $\mathcal{P}_{\text{fin}}(\mathbf{A}, T)$ and $\langle \mathbf{A}^X, D\rangle \models \phi$. From Lemma 19 it follows that $\langle \mathbf{SU}(X), B_D\rangle \models \widehat{\phi}(g_{1,1}, \ldots, g_{m,m})$. Since G matches $\mathbf{B}'$, then $\mathbf{B}_D = \mathbf{B}'$. Hence $\langle \mathbf{SU}(X), B'\rangle \models \widehat{\phi}(g_{1,1}, \ldots, g_{m,m})$, which contradicts (iii).

($\Leftarrow$) Let's assume that $\mathcal{BP}^*_{\text{fin}} \models \phi^*$ and $\mathcal{P}_{\text{fin}}(\mathbf{A}, T) \not\models \phi$. Then there is a structure $\mathbf{A}^* \in \mathcal{P}_{\text{fin}}(\mathbf{A}, T)$ such that $\mathbf{A}^* \not\models \phi$. Let $\mathbf{A}^* = \langle \mathbf{A}^X, D\rangle$. According to Proposition 10 let $G = \{g_1, \ldots, g_m\}$ be a base of D, where $m =\mid A \mid$. By $\mathbf{B}^*$ we denote the structure $\langle \mathbf{SU}(X), B_D\rangle$. Of course $\mathbf{B}^*$ belongs to $\mathcal{BP}^*_{\text{fin}}$. It follows from Lemma 19 that $\mathbf{B}^* \not\models \widehat{\phi}(g_{1,1}, \ldots, g_{m,m})$. For every $1 \leq i \leq m$ a sequence $g_{i,1}, \ldots, g_{i,m}$ is a decomposition of X defined by an element g_i. So, $\mathbf{B}^* \models \Psi_1(g_{1,1}, \ldots, g_{m,m})$. From Remark 13 we know that G matches $\mathbf{B}_D$. From Lemma 14 and Remark 21 we can conclude that

$$\mathbf{B}^* \models \Psi_2(g_{1,1}, \ldots, g_{m,m}) \;\;\&\;\; \Psi_3(g_{1,1}, \ldots, g_{m,m}) \;\;\&\;\; \Psi_4(g_{1,1}, \ldots, g_{m,m}).$$

Consequently, $\mathbf{B}^* \not\models \phi^*$ and $\mathcal{BP}^*_{\text{fin}} \not\models \phi^*$, which contradicts our assumption. This completes the proof of the Theorem.

Corollary 1. *If* $\mathbf{A}$ *is a finite primal algebra, then the first order theory of* $\mathcal{V}_{\text{fin}}(\mathbf{A}, T)$ *is decidable.*

Proof. Let $\mathbf{A}$ be a finite primal algebra. Using the well-known facts (see [1], Corollary 6.10, Corollary 10.2, Corollary 10.8) one can easy observe that in this case $V(\mathbf{A})$ is arithmetical and $\mathbf{A}$ is simple with no subalgebras except itself. Hence every subdirectly irreducible algebra in $\mathcal{V}(\mathbf{A})$ is isomorphic to $\mathbf{A}$. Then $\mathcal{V}(\mathbf{A}) = \mathcal{IP}_S(\mathbf{A})$. Since $\mathcal{V}(\mathbf{A})$ is a congruence-permutable variety and $\mathbf{A}$ is simple, then we have $\mathcal{V}_{\text{fin}}(\mathbf{A}) = \mathcal{I}(\{\mathbf{A}^X :\mid X \mid< \omega\})$. In this case we obtain that $\mathcal{I}(\mathcal{P}_{\text{fin}}(\mathbf{A}, T)) = \mathcal{V}_{\text{fin}}(\mathbf{A}, T)$, which proves the Corollary.

Corollary 2. *Let* $\mathbf{A}$ *be a finite algebra,* $\tau(\mathbf{A})$ *be a set of term operations of* $\mathbf{A}$ *and* F *be a finite subset of* $\tau(\mathbf{A})$. *If* T^* *is a clone generated by* $T \cup F$, *then the first order theory of* $\mathcal{P}_{\text{fin}}(\mathbf{A}, T^*)$ *is decidable.*

Proof. Let $F = \{f_1, \ldots, f_k\}$ and $F \subseteq \tau(\mathbf{A})$. Then for every $1 \leq i \leq k$ we have that f_i is definable in terms of basic function symbols of $\mathbf{A}$ and hence for every f_i we can define a sentence Θ_{f_i} in the language L as follows:

$$\Theta_{f_i} \equiv \forall_{x_1} \ldots \forall_{x_{\sigma(i)}} (D(x_1) \ \& \ \ldots \ \& \ D(x_{\sigma(i)}) \rightarrow D(f_i(x_1, \ldots, x_{\sigma(i)}))),$$

where $\sigma(i)$ denotes an arity of f_i. Observe that $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T^*) \subseteq \mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T)$ and $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T^*)$ is finitely axiomatizable by $\{\Theta_{f_1}, \ldots, \Theta_{f_k}\}$. For any sentence $\phi \in L$ we define a sentence $\phi' \equiv (\Theta_{f_1} \ \& \ \ldots \ \& \ \Theta_{f_k}) \rightarrow \phi$. One can easy check that $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T) \models \phi'$ iff $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T^*) \models \phi$. If so, we can conclude that the theory of $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T^*)$ is decidable, because of decidability of $\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T)$.

Corollary 3. *If* $\mathbf{A}$ *is a finite, primal algebra and* T' *is a discriminator clone on* A*, then the first order theory of* $\mathcal{V}_{\mathrm{fin}}(\mathbf{A}, T')$ *is decidable. In particular,* $\mathcal{VP}_{\mathrm{fin}}(\mathbf{A})$ *is decidable.*

Proof. If $\mathbf{A}$ is a primal algebra and T' is a discriminator clone, then T' is generated by a finite subset of $\tau(\mathbf{A})$ [4]. Since in this case $\mathcal{I}(\mathcal{P}_{\mathrm{fin}}(\mathbf{A}, T')) = \mathcal{V}_{\mathrm{fin}}(\mathbf{A}, T')$ we obtain that the first order theory of $\mathcal{V}_{\mathrm{fin}}(\mathbf{A}, T')$ $\blacksquare$ is decidable. If T' is the clone of the all operations on A, then $\mathcal{V}_{\mathrm{fin}}(\mathbf{A}, T') = \mathcal{VP}_{\mathrm{fin}}(\mathbf{A})$. Hence $\mathcal{VP}_{\mathrm{fin}}(\mathbf{A})$ is decidable.

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